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Numerical Approximations of Abel's Integral Equations Using Haar Wavelet

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ABSTRACT: This study proposes a numerical technique for solving Abel's integral equations through Haar wavelet approximations. The proposed method transforms the original integral equation into an equivalent system of algebraic equations, which significantly simplifies the computational procedure and enables efficient numerical implementation. By using Haar wavelet bases, the method achieves accurate approximations with relatively low computational effort. Numerical examples are presented to demonstrate the precision, stability, and effectiveness of the proposed technique for different test problems. The obtained numerical results show excellent agreement with the corresponding exact solutions, confirming the reliability of the method. A comparative analysis is carried out to indicate the advantages of the present approach in terms of accuracy, simplicity, and computational efficiency when compared with exact and existing numerical solutions. The study highlights the wide applicability of the proposed method for solving Abel-type integral equations. Due to its efficiency and robustness, the technique is suitable for various applications in applied mathematics, science, and engineering.

Keywords: Abel's integral equations, Haar wavelets, collocation points.

Contents

1. Introduction

Volterra-type integral equations play a fundamental role in numerous areas of applied mathematics, physics, and engineering. They naturally arise in problems related to heat conduction, population dynamics, chemical kinetics, viscoelasticity, and various models of diffusion and transport phenomena. Since the kernel of a Volterra equation often exhibits weak or strong singularities, analytical solutions are rarely obtainable, highlighting the need for robust and efficient numerical methods [5,6,7].

Classical numerical methods—such as collocation, finite-difference, and Galerkin techniques—have been widely employed; however, they may experience loss of accuracy or demand fine discretizations when handling singular kernels or rapidly varying solutions. To address these challenges, wavelet-based techniques have emerged as a powerful alternative, owing to their capacity to represent functions with localized features and to facilitate multi-resolution analysis.

The Abel integral equation, a special case of the first-kind Volterra equation with a weakly singular kernel, holds significant importance across various scientific and engineering disciplines. Its applications extend to microscopy, seismology, semiconductor technology, scattering theory, heat conduction, metallurgy, fluid dynamics, chemical reaction kinetics, plasma diagnostics, X-ray radiography, physical electronics, and nuclear physics.

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In 1823, Abel while generalizing the tautochrone problem, derived the equation

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt = f(x), \tag{1.1}$$

where $f(x)$ is a known function and $y(x)$ is an unknown function to be determined. This equation represents a special case of a linear Volterra integral equation of the first kind. Over the past few years, several numerical methods have been developed to solve Eq. (1) among the many available wavelet families, the Haar wavelet is particularly attractive because of its compact support, orthogonality, and simple piecewise-constant form. These properties allow for the straightforward construction of operational matrices and facilitate the conversion of integral equations into systems of linear algebraic equations, resulting in substantial computational savings [8,9,10,11]. Furthermore, hybrid approaches that combine Haar wavelets with other basis functions have shown enhanced accuracy and convergence rates, particularly for weakly singular or quasi-singular integral equations [4,12].

Alongside the development of wavelet methods, quadrature techniques—such as Gaussian and Gauss Legendre rules have been refined to handle integrals with complex weight functions and singular behavior. For instance, generalized Gaussian quadrature on arbitrary domains and specialized Gauss–Legendre schemes have enabled efficient and accurate evaluation of weighted integrals [1,2]. More recently, hybrid wavelet strategies—such as those employing Chebyshev or block pulse wavelets—have been successfully applied to nonlinear quasi-singular integral equations, further expanding the scope of wavelet-based numerical analysis [3].

Motivated by these developments, the present work focuses on the application of the Haar wavelet method to Abel's integral equations. The main idea is to approximate both the unknown solution and the forcing function using a finite Haar basis expansion. By leveraging the operational matrices associated with Haar functions, the original integral equation is systematically transformed into a finite-dimensional system of linear algebraic equations, which can be efficiently solved using standard linear algebra routines. This approach preserves the localization and multi-resolution advantages of wavelets while ensuring a straightforward implementation.

The remainder of this paper is organized as follows. Section 2 introduces the formulation of Haar wavelets and the associated function-approximation framework. Section 3 describes the proposed solution methodology. Section 4 presents numerical experiments and assesses the accuracy of the approach through illustrative examples. Finally, Section 5 offers concluding remarks.

2. Haar Wavelet Preliminaries

The Haar wavelet $H(t)$ is defined on the real line by

$$H(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For integers $n \ge 1$ and $j = 0, 1, \ldots, 2^n - 1$, define

$$h_n(t) = 2^{j/2}\, H(2^j t - k)\big|_{[0,1]},$$

where $k = 0, 1, \ldots, 2^j - 1$ is the translation index. Set $h_0(t) = 1$ for all t. The sequence $\{h_n\}_{n=0}^{\infty}$ forms a complete orthonormal basis of $L^2[0,1]$, and for any $f \in L^2[0,1]$

$$f(t) = \sum_{k=0}^{\infty} \langle f, h_k \rangle h_k(t), \qquad \langle f, h_k \rangle = \int_0^1 f(x)\, h_k(x)\, dx.$$

Function Approximation

A function $u(t)$ defined on $[0,1]$ can be expanded as

$$u(t) = \sum_{k=0}^{\infty} u_k h_k(t), \qquad u_k = \langle u, h_k \rangle. \tag{2.1}$$

For numerical work we retain only the first k terms (with k a power of 2):

$$u(t) \approx u_k(t) = \sum_{j=0}^{k-1} u_j h_j(t). \tag{2.2}$$

This may be written compactly as

$$u(t) \approx \mathbf{U}^{\mathsf{T}}\mathbf{H}(t), \tag{2.3}$$

where

$$\mathbf{U} = [u_0, u_1, \dots, u_{k-1}]^{\mathsf{T}}, \qquad \mathbf{H}(t) = [h_0(t), h_1(t), \dots, h_{k-1}(t)]^{\mathsf{T}}.$$

Solution of the Singular Volterra Equation

Consider the singular Volterra integral equation

$$y(x) = \int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt.$$

We approximate

$$y(x) \approx \mathbf{Y}^{\mathsf{T}}\mathbf{H}(x), \qquad f(x) \approx \mathbf{F}^{\mathsf{T}}\mathbf{H}(x),$$

where $\mathbf{F}$ is known and $\mathbf{Y}$ is an unknown $k \times 1$ coefficient vector.

From the integral representation,

$$\mathbf{Y}^{\mathsf{T}}\mathbf{H}(x) = \mathbf{F}^{\mathsf{T}} \int_0^x \frac{\mathbf{H}(t)}{\sqrt{x-t}}\,dt.$$

Define the $k \times k$ matrix A by

$$A\mathbf{H}(x) = \int_0^x \frac{\mathbf{H}(t)}{\sqrt{x-t}}\,dt.$$

Then

$$\mathbf{Y}^{\mathsf{T}}\mathbf{H}(x) = \mathbf{F}^{\mathsf{T}}\mathbf{H}(x) + \mathbf{Y}^{\mathsf{T}}A\mathbf{H}(x), \tag{2.4}$$

which simplifies to the linear system

$$\mathbf{Y}^{\mathsf{T}} = \mathbf{F}^{\mathsf{T}} + \mathbf{Y}^{\mathsf{T}}A. \tag{2.5}$$

Computation of Coefficients

The coefficient vector $\mathbf{Y}$ satisfies

$$\mathbf{Y}^{\mathsf{T}}(I - A) = \mathbf{F}^{\mathsf{T}}.$$

When $(I - A)$ is nonsingular, the solution is

$$\mathbf{Y}^{\mathsf{T}} = \mathbf{F}^{\mathsf{T}}(I - A)^{-1}.$$

Thus the approximate solution of the original Volterra equation is

$$y(x) = \mathbf{Y}^{\mathsf{T}}\mathbf{H}(x).$$

3. Method of Solution

This section presents the solutions of Abel integral equations using the Haar wavelet method.

Theorem 3.1. *Consider the Abel integral equation of the second kind [14]*

$$f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \le x \le 1.$$

where $f(x) = x$.

Proof. **Haar approximation and collocation points**

The interval $[0,1]$ is divided into four equal subintervals:

$$I_1 = [0, 0.25), \quad I_2 = [0.25, 0.5), \quad I_3 = [0.5, 0.75), \quad I_4 = [0.75, 1).$$

The unknown function $y(t)$ is approximated by Haar functions as

$$y(t) \approx c_1\chi_{I_1}(t) + c_2\chi_{I_2}(t) + c_3\chi_{I_3}(t) + c_4\chi_{I_4}(t),$$

where c_1, c_2, c_3, and c_4 are unknown coefficients.

The kernel integrals are defined by

$$K_{ik} = \int_{I_k \cap [0,x_i]} \frac{h_k(t)}{\sqrt{x_i - t}}\, dt.$$

The collocation points are chosen as the midpoints of the subintervals:

$$x_1 = \frac{1}{8}, \quad x_2 = \frac{3}{8}, \quad x_3 = \frac{5}{8}, \quad x_4 = \frac{7}{8}.$$

Computation of $f(x_i)$

Since $f(x) = x$, we obtain

$$f(x_1) = 0.125, \quad f(x_2) = 0.375, \quad f(x_3) = 0.625, \quad f(x_4) = 0.875.$$

Evaluation of the integrals K_{ik}

For Haar functions, the integrals take the form

$$K_{ik} = \begin{cases} 2(\sqrt{x_i - a_k} - \sqrt{x_i - b_k}), & k < i, \\ 2\sqrt{x_i - a_k}, & k = i, \\ 0, & k > i, \end{cases}$$

where $[a_k, b_k)$ denotes the support of the Haar functions.

Construction of the system matrix

Define

$$A_{ik} = \begin{cases} K_{ik}, & k < i, \\ 1 + K_{ii}, & k = i, \\ 0, & k > i. \end{cases}$$

The resulting system is

$$\sum_{k=1}^{4} A_{ik} c_k = f(x_i), \quad i = 1, 2, 3, 4.$$

The corresponding system matrix is

$$A = \begin{bmatrix} 0.70710678118 & 0 & 0 & 0 \\ 0.51763809020 & 0.70710678118 & 0 & 0 \\ 0.35639395869 & 0.5176380902 & 0.70710678118 & 0 \\ 0.28968986330 & 0.3563939586 & 0.51763809020 & 0.707106781187 \end{bmatrix}.$$

Linear system

The system can be written as $Kc = f$, where

$$f \approx \begin{bmatrix} 0.125 \\ 0.375 \\ 0.625 \\ 0.875 \end{bmatrix}.$$

Solution by forward substitution
Solving the system using forward substitution yields the Haar approximation

$$y_{\text{Haar}}(t) = \begin{cases} c_1 \approx 0.1767766953, & t \in [0, 0.25), \\ c_2 \approx 0.4009205580, & t \in [0.25, 0.5), \\ c_3 \approx 0.5012907607, & t \in [0.5, 0.75), \\ c_4 \approx 0.5959732425, & t \in [0.75, 1]. \end{cases}$$

□

Theorem 3.2. *Consider the Abel integral equation of the form [14]*

$$f(x) = \int_0^x \frac{y(t)}{\sqrt{x-t}}\, dt, \quad 0 \le x \le 1,$$

where

$$f(x) = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3).$$

Proof. The unknown function $y(t)$ is approximated using four Haar (box) basis functions on the interval $[0, 1]$:

$$\chi_k(t) = \begin{cases} 1, & t \in \left[\frac{k-1}{4}, \frac{k}{4}\right), \\ 0, & \text{otherwise}, \end{cases} \qquad k = 1, 2, 3, 4.$$

Accordingly, the Haar wavelet approximation of $y(t)$ is given by

$$y(t) \approx \sum_{k=1}^{4} c_k \chi_k(t),$$

where c_1, c_2, c_3, and c_4 are unknown coefficients.

The collocation points are chosen as the midpoints of the four equal subintervals:

$$x_1 = \frac{1}{8}, \quad x_2 = \frac{3}{8}, \quad x_3 = \frac{5}{8}, \quad x_4 = \frac{7}{8}.$$

Collocation equations
At each collocation point x_i, the integral equation is enforced in the form

$$\sum_{k=1}^{4} c_k \int_0^{x_i} \frac{\chi_k(t)}{\sqrt{x_i - t}}\, dt = f(x_i).$$

Let the support of $\chi_k(t)$ be denoted by $[a_k, b_k) = \left[\frac{k-1}{4}, \frac{k}{4}\right)$. If $a_k \ge x_i$, the corresponding integral vanishes. Otherwise, when the support overlaps the interval $[0, x_i]$, we define

$$A_{ik} = \int_{a_k}^{\min(b_k, x_i)} \frac{dt}{\sqrt{x_i - t}}.$$

Evaluating the antiderivative

$$\int (x_i - t)^{-\frac{1}{2}}\, dt = -2\sqrt{x_i - t},$$

we obtain, for $a < b \le x_i$,

$$\int_a^b \frac{dt}{\sqrt{x_i - t}} = 2(\sqrt{x_i - a} - \sqrt{x_i - b}).$$

Hence, in compact form,

$$A_{ik} = \begin{cases} 2(\sqrt{x_i - a_k} - \sqrt{x_i - \min(b_k, x_i)}), & a_k < x_i, \\ 0, & a_k \geq x_i. \end{cases}$$

The right-hand side vector is defined by

$$r_i = f(x_i).$$

Computation of matrix entries
The supports of the Haar functions are

$$[a_1, b_1) = [0, 0.25), \quad [a_2, b_2) = [0.25, 0.5), \quad [a_3, b_3) = [0.5, 0.75), \quad [a_4, b_4) = [0.75, 1).$$

Only the nonzero entries of the matrix are evaluated. The resulting system matrix is

$$A = \begin{bmatrix} 0.7071067812 & 0 & 0 & 0 \\ 0.5176380902 & 0.7071067812 & 0 & 0 \\ 0.3563939587 & 0.5176380902 & 0.7071067812 & 0 \\ 0.2896898633 & 0.3563939587 & 0.5176380902 & 0.7071067812 \end{bmatrix}.$$

Computation of the right-hand side vector
Using $r_i = f(x_i)$, we obtain

$$r \approx \begin{bmatrix} 0.70184557 \\ 1.162414106 \\ 1.428201294 \\ 1.679848264 \end{bmatrix}.$$

Solution by forward substitution
Solving the system $Ac = r$ using forward substitution yields

$$c \approx \begin{bmatrix} 0.9925595238 \\ 0.9172977928 \\ 0.8480060076 \\ 0.8859117826 \end{bmatrix}.$$

The corresponding Haar wavelet approximation is

$$y_{\text{Haar}}(t) = \begin{cases} c_1 \approx 0.9925595238, & t \in [0, 0.25), \\ c_2 \approx 0.9172977928, & t \in [0.25, 0.5), \\ c_3 \approx 0.8480060076, & t \in [0.5, 0.75), \\ c_4 \approx 0.8859117826, & t \in [0.75, 1]. \end{cases}$$

□

The numerical results corresponding to this example are discussed in Section 4.

Theorem 3.3. *Consider the Abel integral equation of the second kind [14]*

$$4y(x) = \frac{4}{\sqrt{x+1}} - \arcsin\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}\, dt, \quad 0 \leq x \leq 1.$$

Proof. Using the identity

$$\frac{\pi}{2} - \arcsin(a) = \arccos(a),$$

the equation can be rewritten as

$$4y(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt = f(x),$$

where

$$f(x) = \frac{4}{\sqrt{x+1}} + \arccos\left(\frac{1-x}{1+x}\right).$$

Haar approximation and collocation points
The interval $[0,1]$ is divided into four equal subintervals:

$$I_1 = [0, 0.25), \quad I_2 = [0.25, 0.5), \quad I_3 = [0.5, 0.75), \quad I_4 = [0.75, 1).$$

The unknown function $y(t)$ is approximated as

$$y(t) \approx c_1\chi_{I_1}(t) + c_2\chi_{I_2}(t) + c_3\chi_{I_3}(t) + c_4\chi_{I_4}(t),$$

where $\chi_{I_k}(t)$ denotes the characteristic function of the interval I_k. The unknown coefficients are c_1, c_2, c_3, and c_4.

The collocation points are chosen as the midpoints of the subintervals:

$$x_1 = \frac{1}{8} = 0.125, \quad x_2 = \frac{3}{8} = 0.375, \quad x_3 = \frac{5}{8} = 0.625, \quad x_4 = \frac{7}{8} = 0.875.$$

Formulation of the discrete system
At each collocation point x_i, the equation is enforced as

$$4c_i + \sum_{k=1}^{4} c_k \int_{a_k}^{\min(b_k, x_i)} \frac{dt}{\sqrt{x_i - t}} = f(x_i),$$

where $[a_k, b_k)$ is the support of the k-th Haar function.

Define

$$A_{ik} = \int_{a_k}^{\min(b_k, x_i)} \frac{dt}{\sqrt{x_i - t}} = \begin{cases} 2\Big(\sqrt{x_i - a_k} - \sqrt{x_i - \min(b_k, x_i)}\Big), & a_k < x_i, \\ 0, & a_k \geq x_i. \end{cases}$$

Thus, the resulting linear system is

$$\sum_{k=1}^{4} (4\,\delta_{ik} + A_{ik})\, c_k = f(x_i),$$

where δ_{ik} is the Kronecker delta.

Let M be defined by $M_{ii} = 4 + A_{ii}$ and $M_{ik} = A_{ik}$ for $i \neq k$. Then

$$Mc = f.$$

Evaluation of matrix entries

The resulting matrix $A = [A_{ik}]$ is

$$A = \begin{bmatrix} 0.707106781187 & 0 & 0 & 0 \\ 0.517638090205 & 0.707106781187 & 0 & 0 \\ 0.356393958693 & 0.517638090205 & 0.707106781187 & 0 \\ 0.289689863303 & 0.356393958693 & 0.517638090205 & 0.707106781187 \end{bmatrix}.$$

The diagonal entries of M are

$$M_{ii} = 4 + A_{ii} = 4.7071067811.$$

Computation of the right-hand side vector
Using

$$f(x) = \frac{4}{\sqrt{x+1}} + \arccos\left(\frac{1-x}{1+x}\right),$$

we obtain

$$f \approx \begin{bmatrix} 4.450909985 \\ 4.510896454 \\ 4.475786311 \\ 4.425267148 \end{bmatrix}.$$

Solution by forward substitution
Solving the system $Mc = f$ yields

$$c_1 = 0.9455723424, \quad c_2 = 0.8541726111, \quad c_3 = 0.7853311882, \quad c_4 = 0.7308959177.$$

Hence, the Haar wavelet approximation of the solution is

$$y_{\text{Haar}}(x) = \begin{cases} 0.9455723424, & x \in [0, 0.25), \\ 0.8541726111, & x \in [0.25, 0.5), \\ 0.7853311882, & x \in [0.5, 0.75), \\ 0.7308959177, & x \in [0.75, 1]. \end{cases}$$

□

4. Results and Discussion

The proposed Haar wavelet-based approach is applied to Examples 1, 2 and 3 to obtain approximate solutions of the Abel integral equation. The method converts the given integral equation into a system of algebraic equations, which is subsequently solved to compute the approximate values of $y(x)$. The accuracy and effectiveness of the proposed method are demonstrated by comparing the approximate solution $y_{\text{approx}}(x)$ with the exact solution $y_{\text{exact}}(x)$. The corresponding numerical results are presented in tabular form.

x	$y_{\text{exact}}(x)$	$y_{\text{approx}}(x)$	$\lvert Error \rvert$
0.0	0.00000000	0.00000000	0.00000000
0.1	0.20131685	0.20136717	0.00005032
0.2	0.28470502	0.28474060	0.00003559
0.3	0.34869101	0.34872007	0.00002906
0.4	0.40263370	0.40265886	0.00002516
0.5	0.45015816	0.45018067	0.00002251
0.6	0.49312356	0.49314140	0.00001785
0.7	0.53263432	0.53265334	0.00001902
0.8	0.56946091	0.56947923	0.00001879
0.9	0.60395055	0.60396732	0.00001678
1.0	0.63661977	0.63663569	0.00001591

Table 1: Numerical comparative results of Example 1

For comparison, Methods [16] and [17] represent Hermite and Chebyshev wavelet-based numerical schemes, respectively.

Table 2: Comparative analysis of Example 1 with existing results

x	Exact solution	Present Method	Method [14] $(k = 1, M = 10)$	Method [16] $(k = 1, M = 8)$	Method [17] $(m = 16)$
0.1	0.201317	0.20136717	0.200842	0.200128	0.200460
0.2	0.284705	0.28474060	0.284667	0.286092	0.297987
0.3	0.348691	0.34872007	0.348628	0.347394	0.337588
0.4	0.402634	0.40265886	0.402609	0.404161	0.405769
0.5	0.450158	0.45018067	0.450129	0.449568	0.464014
0.6	0.493124	0.49314140	0.493113	0.492704	0.490550
0.7	0.532634	0.53265334	0.532607	0.532315	0.539721
0.8	0.569410	0.56947923	0.569440	0.569156	0.562698
0.9	0.603951	0.60396732	0.603690	0.603742	0.606044

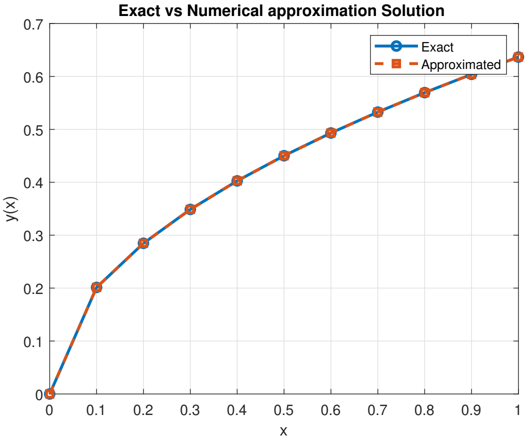

Figure 1: Comparison of numerical and exact solutions of the Abel integral equation for Example 1 using the Haar wavelet method.

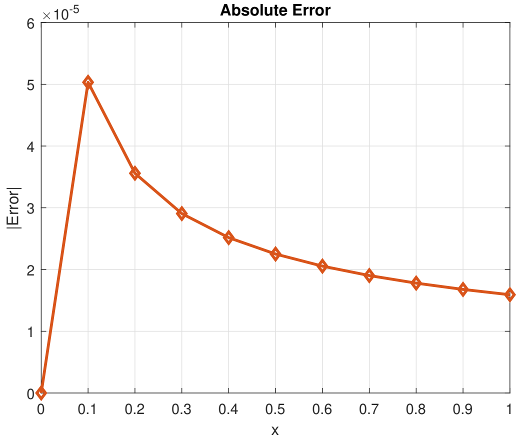

Figure 2: Absolute error distribution for Example 1.

Table 3: Numerical comparative results of Example 2

x	$y_{\text{exact}}(x)$	$y_{\text{approx}}(x)$	$\lvert\text{Error}\rvert$
0.0	1.00000000	1.00000000	0.00000000
0.1	0.99100000	0.99099150	0.00000850
0.2	0.96800000	0.96798600	0.00001400
0.3	0.93700000	0.93698350	0.00001650
0.4	0.90400000	0.90398400	0.00001600
0.5	0.87500000	0.87498750	0.00001250
0.6	0.85600000	0.85594900	0.00005100
0.7	0.85300000	0.85300350	0.00000350
0.8	0.87200000	0.87201600	0.00001600
0.9	0.91900000	0.91903151	0.00003151
1.0	1.00000000	1.00005001	0.00005001

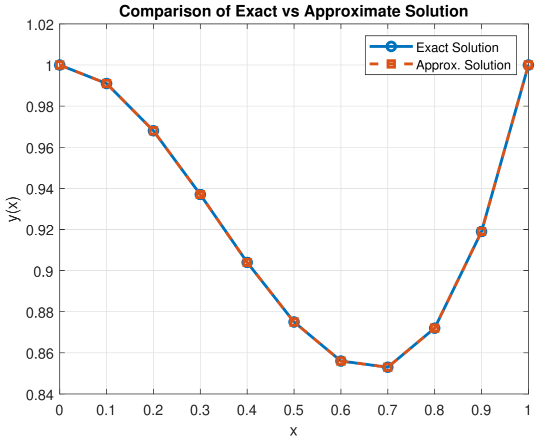

Figure 3: Comparison of exact and Haar wavelet numerical solutions for Example 2 of the Abel integral equation.

Figure 4: Absolute error distribution for Example 2.

x	$y_{\text{exact}}(x)$	$y_{\text{approx}}(x)$	$\lvert\text{Error}\rvert$
0.0	1.00000000	1.00000000	0.00000000
0.1	0.95346259	1.05389665	0.10043406
0.2	0.91287093	1.00023409	0.08736316
0.3	0.87705802	0.95606820	0.07901018
0.4	0.84515425	0.91765235	0.07249810
0.5	0.81649658	0.88360392	0.06713434
0.6	0.79056942	0.85315233	0.06258292
0.7	0.76696499	0.82561020	0.05864521
0.8	0.74535599	0.80056465	0.05519045
0.9	0.72547625	0.77760277	0.05212652
1.0	0.70710678	0.75649230	0.04938552

Table 4: Numerical results of Example 3

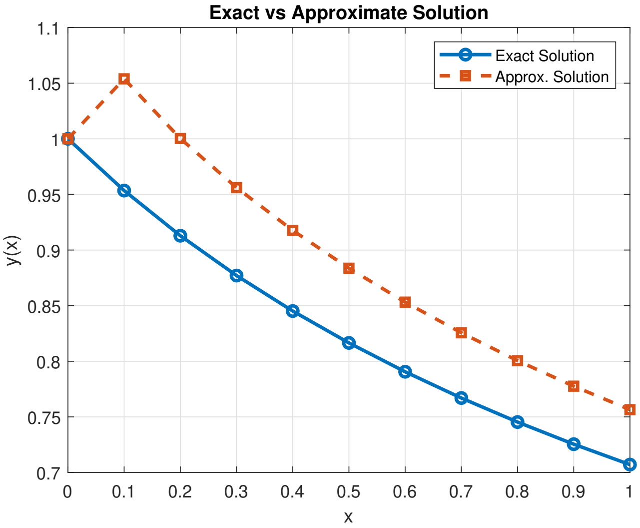

Figure 5: Comparison of numerical and exact solutions for Example 3.

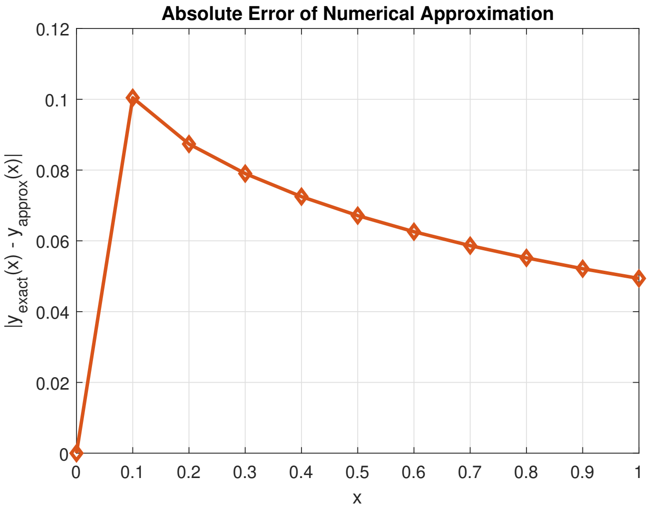

Figure 6: Absolute error distribution for Example 3.

Remark 4.1. Example 1 was previously investigated in [13]. However, the methodology and numerical analysis presented therein were limited. In the present study, the example has been revisited using the Haar wavelet framework, and a more detailed computational procedure along with comprehensive numerical results has been provided.

Remark 4.2. For Example 3, the Haar wavelet method exhibits comparatively lower accuracy. In this work, computations were carried out up to the resolution level $j = 4$. Increasing the resolution level would further enhance the accuracy of the numerical solution.

5. Computational Complexity and Convergence Analysis

The efficiency of the proposed Haar wavelet method is closely related to the number of basis functions used in the approximation. Let the resolution level be denoted by j, so that the total number of Haar basis functions is $k = 2^j$. Owing to the Volterra structure of Abel's integral equation, the discretization process leads to a lower triangular system of algebraic equations. This structural property allows the resulting linear system to be solved efficiently using forward substitution, requiring a computational effort of order $O(k^2)$.

The assembly of the system matrix involves the evaluation of integrals with weakly singular kernels over the compact supports of Haar basis functions. Due to the piecewise-constant nature of Haar wavelets, these integrals admit closed-form expressions and do not require numerical quadrature. As a result, the pre-processing and matrix construction steps incur relatively low computational cost. Compared with methods based on higher-order polynomial, spline, or smooth wavelet bases, the proposed approach remains computationally simple while maintaining satisfactory accuracy.

From the convergence perspective, Haar wavelet approximations are known to converge in the $L^2[0, 1]$ sense for functions belonging to this space. When applied to Abel-type integral equations with weakly singular kernels, the convergence behavior depends largely on the regularity of the exact solution. Since Haar wavelets yield piecewise-constant approximations, the overall convergence is generally of first order. Nevertheless, increasing the resolution level j enhances the approximation quality by refining the partition of the interval, thereby reducing the local truncation error.

The numerical experiments presented in this study clearly indicate that the approximation error decreases as the number of Haar basis functions increases. In particular, the first two examples achieve accurate results even at relatively low resolution levels, reflecting the smoothness of their exact solutions. In contrast, Example 3 exhibits slower convergence, which can be attributed to the reduced smoothness of the solution. This observation is consistent with the theoretical convergence characteristics of Haar wavelet methods applied to integral equations with weak singularities.

6. Conclusion

The proposed Haar wavelet-based numerical method proves to be an efficient and reliable tool for solving Abel's integral equations. By transforming the problem into a system of algebraic equations, it ensures computational simplicity while providing highly accurate results. The presented examples and comparative analyses demonstrate its robustness, versatility, and potential for extension to a wider class of integral equations in applied mathematics and engineering.

The present study can be naturally extended in several directions. One possible extension is the use of higher-order wavelet bases, which may further improve the accuracy and convergence rate of the proposed method for smoother solutions. Another promising direction is the application of the Haar wavelet collocation framework to fractional Abel integral equations, where the kernel involves fractional-order singularities. In addition, the methodology can be adapted to more general classes of weakly singular integral equations arising in applied sciences and engineering, which will be explored in future work.

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