Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.79351

Quartic Points on $\mathcal{C}_a : y^7 = x^a(x-1)^a$

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ABSTRACT: The goal of this work is to give a parametrization of the set of quartic points on the family of quotients of the Fermat curve of degree 7 of affine equation

$$\mathcal{C}_a : y^7 = x^a (x-1)^a$$

where $a \in \{1,2,3\}$. We use the Mordell-Weil group, the Riemann-Roch spaces and birational morphisms to give this parametrization on $\mathcal{C}_a$.

Key Words: Fermat quotients curve, birational morphism, quartic points.

Contents

1. Introduction

Arithmetic questions concerning the number of algebraic points of degree d on a smooth plane curve over the number field $\mathbb{Q}$ lead to geometric questions about this curve.
Faltings's theorem states that a curve $\mathcal{C}$ of genus $g \geq 2$ defined over $\mathbb{Q}$ has only a finite $\mathbb{Q}$-rational points. Let K be an algebraic extension of $\mathbb{Q}$ of finite degree, the determination of the K-rational points is very difficult. But it is more easier to determine the algebraic points on all extensions K of degree d over $\mathbb{Q}$. This means to determine the set of algebraic points of degree at most d over $\mathbb{Q}$.
Currently, one of the most important problems concerning the classification of algebraic points on a curve $\mathcal{C}$ is the determination of the set of the algebraic points of degree at most d denoted $\mathcal{C}^{(d)}(\mathbb{Q})$ because there is no general algorithm for determining this set or deciding if it is empty.
However, our work consists of studying special cases of curves where we give a parametrization of quartic points on $\mathcal{C}$ over $\mathbb{Q}$. It seems that a necessary condition is that the Mordell-Weil group $J(\mathbb{Q})$ of $\mathcal{C}$ is finite.
Consider the family curves $\mathcal{C}_a$ $(a \in \{1, 2, 3\})$ with affine equation

$$\mathcal{C}_a : y^7 = x^a (x-1)^a$$

This family $\mathcal{C}_a$ has genus $g = 3$ and is a subfamily of the quotients of Fermat curves with affine equation

$$\mathcal{C}_{a,b}(p) : y^p = x^a (x-1)^b$$

2010 *Mathematics Subject Classification:* 14H45, 14H50, 11D41.
Submitted October 05, 2025. Published December 20, 2025

where $1 \leq a, b, a+b \leq p-1$ and $p \geq 5$ is a positive prime.
The curves $\mathcal{C}_{a,b}(p)$ are called quotients of the Fermat curve and are frequently mentioned in the literature (see [1], [2], [4] and [5]).
Gross and Rohrlich, in [4], proved that $J_a(\mathbb{Q})$ of the family curves $\mathcal{C}_a$ has rank zero and given by

$$J_a(\mathbb{Q}) = \frac{\mathbb{Z}}{7\mathbb{Z}}.$$

They also showed that the set $\mathcal{C}_{a,b}(p)(\mathbb{Q})$ is formed by the following three rational points

$$P_0 = (0,0), \quad P_1 = (1,0), \quad P_\infty = (1,0).$$

In [5], Sall determined the set $\mathcal{C}_1^{(3)}(\mathbb{Q})$ of cubic points on the curve $\mathcal{C}_1$.
Let denote by $\mathcal{G}_a^{(d)}$ the set of algebraic points of degree d on $\mathcal{C}_a$, i.e.

$$\mathcal{G}_a^{(d)} = \left\{Q \in \mathcal{C}_a\left(\overline{\mathbb{Q}}\right) \;\middle|\; [\mathbb{Q}(Q):\mathbb{Q}] = d\right\}.$$

The main objective of this work is to determine the set of quartic points $\mathcal{G}_a^{(4)}$ on $\mathcal{C}_a$ for all $a \in \{1,\ 2,\ 3\}$.
The main result is provided by the following theorem:

Theorem 1.1 *The quartic points on the curve* $\mathcal{C}_a : y^7 = x^a(x-1)^a$ *with* $a \in \{1,2,3\}$ *is given by the set* $\mathcal{G}_a^4 = \mathcal{G}_{4,1} \cup \mathcal{G}_{4,2}$ *with*

$$\mathcal{G}_{4,1} = \left\{\left(\frac{1}{2} \pm \sqrt{y^7 + \frac{1}{4}},\ y^a\right) \;\middle|\; y = \frac{1}{2}\alpha \pm \sqrt{\beta + \frac{1}{4}\alpha^2},\ \alpha \in \mathbb{Q},\ \beta \in \mathbb{Q}^*\right\}$$

$$\mathcal{G}_{4,2} = \left\{\left(\gamma + \lambda y^3, y^a\right) \;\middle|\; y^4 - \lambda^2 y^3 + (-1)^\gamma \lambda = 0,\ \lambda \in \mathbb{Q}^*,\ \gamma \in \{0,1\}\right\}$$

2. Preliminaries

2.1. Algebraic extension

A complex number $\lambda \in \mathbb{C}$ is algebraic if there exists a non-zero polynomial $P \in \mathbb{Q}[X]$ with $P(\lambda) = 0$.
The algebraic closure of $\mathbb{Q}$ is the

$$\overline{\mathbb{Q}} = \{\lambda \in \mathbb{C} \mid \lambda \text{ algebraic}\}.$$

$\overline{\mathbb{Q}}$ is the algebraic number fields.
Let $\theta \in \overline{\mathbb{Q}}$. then the smallest subfield $\overline{\mathbb{Q}}$ that contains $\mathbb{Q}$ and θ is commonly denoted $\mathbb{Q}(\theta)$. In this case $\mathbb{Q}(\theta)$ is an algebraic extension of $\mathbb{Q}$ of finite degree over $\mathbb{Q}$. We note that :

$$deg(\theta) = [\mathbb{Q}(\theta)) : \mathbb{Q}].$$

Definition 2.1 Let $\mathcal{H}$ be an algebraic plane curve defined on $\mathbb{Q}$. The degree of an algebraic point $Q \in \mathcal{H}$ is the degree of its field of definition over $\mathbb{Q}$.
In other words, if we denote by $deg(Q)$ the degree of Q over $\mathbb{Q}$, then

$$deg(Q) = [\mathbb{Q}(Q) : \mathbb{Q}] = d.$$

Q is called algebraic point of degree d over $\mathbb{Q}$.

More specifically:

- If $deg(Q) = 1$, then Q is a rational point.
- If $deg(Q) = 2$, then Q is a quadratic point.
- If $deg(Q) = 3$, then Q is a cubic point.
- If $deg(Q) = 4$, then Q is a quartic point.
- If $deg(Q) = 5$, then Q is a quintic point.

2.2. Divisors and rational functions on curve

Let $\mathcal{H}$ be a smooth plane curve defined on a number field K.

Definition 2.2 A divisor D of $\mathcal{H}$ is a formal finite sum of distinct points of $\mathcal{H}$:

$$D = \sum_{Q \in \mathcal{H}} n_Q Q$$

where the $n_Q \in \mathbb{Z}$ are almost all zero. The degree of D is the sum defined by:

$$deg\left(\sum_{Q \in \mathcal{H}} n_Q Q\right) = \sum_{Q \in \mathcal{H}} n_Q deg\,(Q)\,.$$

The set of divisors is an abelian group, where the law of the group is the formal addition of points. This group is denoted $Div(\mathcal{H})$.

Definition 2.3 A rational function on an algebraic curve $\mathcal{H}$ is a function $\psi : \mathcal{H} \longrightarrow \mathbb{P}^1$, defined by polynomials, which has only a finite number of poles.

Let $K(\mathcal{H})$ denote the field of all rational functions on $\mathcal{H}$ defined on K, then there is a natural morphism $K(\mathcal{H})^* \longrightarrow Div(\mathcal{H})$ that associates to a rational function ψ its divisor

$$div(\psi) = \sum_{P \in \mathcal{H}} ord_P(\psi) P$$

where $ord_P(\psi)$ is the order of vanishing of ψ at P. It is a standard fact in the theory of algebraic curves that if ψ is a non-zero rational function, then the number of poles of ψ equals the number of zeros of ψ (see [3]).

Proposition 2.1 *(see [3]) Let ψ and φ be two rational functions of $K(\mathcal{H})$, then:*

1. $div\,(\psi\varphi) = div\,(\psi) + div\,(\varphi)$;
2. $div\left(\dfrac{\psi}{\varphi}\right) = div\,(\psi) - div\,(\varphi)$.

Definition 2.4 Let $\mathcal{H}$ be a smooth curve and let $D \in Div(\mathcal{H})$. It is associated with the set of functions:

$$\mathcal{L}(D) = \{\psi \in K(\mathcal{H})^* \mid div(\psi) + D \geq 0\} \cup \{0\}\,.$$

$\mathcal{L}(D)$ is called linear system. It is a finite-dimensional vector space over K. We denote by $l(D)$ the K-dimension of $\mathcal{L}(D)$.

The following theorem classifies the curves according to their genus. It is called Riemann-Roch theorem.

Theorem 2.1 *(see [3]) Let $\mathcal{H}$ be a smooth curve. Then there exists a divisor $K_{\mathcal{H}}$ called the canonical divisor and an integer $g \geq 0$ called the genus of $\mathcal{H}$ such that for any divisor $D \in Div(\mathcal{H})$ we have :*

$$l(D) = deg(D) + 1 - g + l(K_{\mathcal{H}} - D).$$

In particular, using the previous notation, we have :

(i) $l(K_{\mathcal{H}}) = g$ and $deg(K_{\mathcal{H}}) = 2g - 2$.

(ii) If $deg(D) > 2g - 2$, then $l(D) = deg(D) + 1 - g$.

2.3. Jacobian map

Let $J(\mathcal{H})$ be the jacobian of the curve $\mathcal{H}$ and P_∞ be a K-rational point of $\mathcal{H}$, then we defined the map

$$\begin{array}{cccc} j: & \mathcal{H} & \longrightarrow & J(\mathcal{H}) \\ & P & \longmapsto & [P-P_\infty] \end{array}$$

where $j(P)=[P-P_\infty]$ is the class of $P-P_\infty$.
Let $Div\,(\mathcal{H})$ be the group of all divisors on $\mathcal{H}$ and $Div^0\,(\mathcal{H})$ denote the subgroup of divisors of degree 0. The map j extends by linearity to $Div^0\,(\mathcal{H})$ and then we note:

$$\begin{array}{cccc} j_0: & Div^0(\mathcal{H}) & \longrightarrow & J(\mathcal{H}) \\ & D & \longmapsto & [D-deg(D)P_\infty] \end{array}$$

j_0 is called the Abel-Jacobi map.
We have the following classical theorem :

Theorem 2.2 *(Abel-Jacobi) (See [3]). The map j_0 is surjective and its kernel consists of the divisors of functions on $\mathcal{H}$. In other words, the kernel of j_0 is formed by the divisors of rational functions.*

2.4. Normalization of the curve $\mathcal{C}_a$

We know that in general for any curve

$$\mathcal{C}_a : y^7 = x^a\,(x-1)^a\,,$$

we can describe the associated curve

$$\mathcal{V}_a : y^7 = x^a\,(x-1)^a \quad \text{with} \quad x(x-1)\neq 0.$$

The curve $\mathcal{V}_a$ is a smooth projective curve. There are then three points P_0, P_1 and P_∞ such that

$$\mathcal{C}_a - \mathcal{V}_a = \{P_0,\, P_1,\, P_\infty\}\,.$$

Let $\mathcal{H}_a$ be the projective equation of $\mathcal{C}_a$:

$$\mathcal{H}_a : Y^7 = Z^{7-2a}X^a\,(X-Z)^a$$

The curve $\mathcal{H}_a$ is the Zariski closure of $\mathcal{V}_a\subset\mathbb{A}^2\subset\mathbb{P}^2$ which is smooth except (perhaps) at three points

$$P_0'=(0,0,1),\qquad P_1'=(1,0,1),\qquad P_\infty'=(1,0,0).$$

More precisely P_∞', P_0' and P_1' are singular unless $a=1$.
Let ν_a be the normalization map defined by

$$\nu_a : \mathcal{C}_a \longrightarrow \mathcal{H}_a$$

Then ν_a is bijective and we have :

$$\nu_a^{-1}\,(P_\infty') = P_\infty,\quad \nu_a^{-1}\,(P_0') = P_0,\quad \nu_a^{-1}\,(P_1') = P_1.$$

The curves $\mathcal{C}_a$ are birationally equivalent to the curve $\mathcal{C}_1$. Thus we have the lemma:

Lemma 2.1 *For any $a\in\{1,2,3\}$, the curve $\mathcal{C}_a$ is birationally equivalent to the curve $\mathcal{C}_1$.*

Proof. Let the morphism φ_a be defined by

$$\begin{array}{cccc} \varphi_a: & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_a \\ & (x,y) & \longmapsto & (x,y^a) \end{array}$$

$$\begin{array}{rcl} (x,y^a)\in\mathcal{C}_a & \Longleftrightarrow & (y^a)^7 - x^a\,(x-1)^a = 0 \\ & \Longleftrightarrow & (y^7)^a - (x(x-1))^a = 0 \\ & \Longleftrightarrow & (y^7 - x(x-1))\left(\sum\limits_{0\le k\le a-1} x^{a-1-k}(x-1)^{a-1-k}y^{7k}\right) = 0 \\ & \Longleftrightarrow & y^7 - x(x-1) = 0 \\ & \Longleftrightarrow & (x,y)\in\mathcal{C}_1. \end{array}$$

2.5. Geometric lemmas

Let x and y be the functions defined on $\mathcal{C}_a$ by $x = \dfrac{X}{Z}$ and $y = \dfrac{Y}{Z}$.

Lemma 2.2 *For all $a \in \{1,2,3\}$, we have :*

a) $div(x) = 7P_0 - 7P_\infty$;

b) $div(x-1) = 7P_1 - 7P_\infty$;

c) $div(y) = aP_0 + aP_1 - 2aP_\infty$;

d) $7j(P_0) = 7j(P_1) = 0$;

e) $j(P_0) + j(P_1) = 0$.

f) The Mordell-Weil group of $\mathcal{C}_a$ is given by:

$$J_a(\mathbb{Q}) \cong \frac{\mathbb{Z}}{7\mathbb{Z}} = \left\{ mj(P_1) \;\middle|\; 0 \le m \le 6 \right\} = \left\{ mj(P_0) \;\middle|\; 0 \le m \le 6 \right\}.$$

Proof. See [4,5].

If m is a non-zero natural number then mP_∞ is a divisor on $\mathcal{C}_a$. We have the lemma:

Lemma 2.3 *A $\mathbb{Q}$-basis of $\mathcal{L}(mP_\infty)$ is given by*
i) For $0 \le m \le 6$, we have :

1. $\mathcal{L}(P_\infty) = \langle 1 \rangle$
2. $\mathcal{L}(2P_\infty) = \langle 1,\ y \rangle$
3. $\mathcal{L}(3P_\infty) = \langle 1,\ y \rangle$
4. $\mathcal{L}(4P_\infty) = \left\langle 1,\ y,\ y^2 \right\rangle$
5. $\mathcal{L}(5P_\infty) = \left\langle 1,\ y,\ y^2 \right\rangle$
6. $\mathcal{L}(6P_\infty) = \left\langle 1,\ y,\ y^2,\ y^3 \right\rangle$

ii) For $m \ge 7$, a $\mathbb{Q}$-basis of $\mathcal{L}(mP_\infty)$ is given by :

$$\mathcal{B}_m = \left\{ y^i \mid i \in \mathbb{N} \text{ and } 0 \le i \le \left[\frac{m}{2}\right] \right\} \cup \left\{ xy^j \mid j \in \mathbb{N} \text{ and } 0 \le j \le \left[\frac{m-7}{2}\right] \right\}.$$

Proof. The rational functions in $\mathcal{B}_m$ are of the form $P_{rs}(x,y) = x^r y^s$ and verify

$$div\left(P_{rs}(x,y)\right) = div\left(x^r y^s\right) = rdiv(x) + sdiv(y) \Longrightarrow 0 \le 7r + 2s \le m \ \text{ with } \ r,s \in \mathbb{N}$$

We have two cases:
Case 1: $1 \le m \le 6$. We have

$$7r + 2s \le m \text{ with } r,s \in \mathbb{N} \Longrightarrow r = 0 \text{ and } 0 \le s \le m.$$

So we obtain i).
Case 2: $m \ge 7$. It is clear that $m \ge 2g - 1 = 5$.
a) According to the Riemann-Roch theorem, $dim\left(\mathcal{L}(mP_\infty)\right) = m - g + 1 = m - 2$. It remains to show that the cardinal $\#(\mathcal{B}_m) = m - g + 1$.

1. Suppose m is even, and let $m = 2h$. Then we have

$$i \leq \left[\frac{m}{2}\right] = h \quad and \quad j \leq \left[\frac{m-7}{2}\right] = h - g - 1.$$

So we get $\mathcal{B}_m = \{1,\, y,\, \ldots,\, y^h\} \cup \{x,\, xy,\, \ldots,\, xy^{h-g-1}\}$. Therefore $\#\,(\mathcal{B}_m) = h + 1 + (h - g - 1 + 1) = m + 1 - g = dim\,(\mathcal{L}(mP_\infty))$.

2. Suppose m is odd, and let $m = 2h + 1$. Then we have

$$i \leq \left[\frac{m}{2}\right] = h \quad and \quad j \leq \left[\frac{m-7}{2}\right] = h - g.$$

So we get $\mathcal{B}_m = \{1,\, y,\, \ldots,\, y^h\} \cup \{x,\, xy,\, \ldots,\, xy^{h-g}\}$. Hence $\#\,(\mathcal{B}_m) = h + 1 + (h - g + 1) = m + 1 - g = dim\,(\mathcal{L}(mP_\infty))$.

b) Let us show that the family $\mathcal{B}_m$ is free.

$P_{rs}(x,y)$	1	y	y^2	$\ldots$	$y^{[\frac{m}{2}]}$	x	xy	xy^2	$\ldots$	$xy^{[\frac{m-7}{2}]}$
Number of poles	0	2	4	$\ldots$	$2[\frac{m}{2}]$	7	9	11	$\ldots$	$7+2[\frac{m-7}{2}]$

The multiplicities of the poles of the elements of $\mathcal{B}_m$ are all different. Therefore, the family is free. So $\mathcal{B}_m$ is a basis of $\mathcal{L}\,(mP_\infty)$.

3. Proof of the Main Result

According to Lemma 2.1, the curves $\mathcal{C}_a$ are birationally equivalent. If $(x, y) \in \mathcal{C}_1$, then $(x, y^a) \in \mathcal{C}_a$. It suffices to prove the theorem on the curve $\mathcal{C}_1$.
Let $R \in \mathcal{C}\left(\overline{\mathbb{Q}}\right)$ be an algebraic point of degree 4 over the field $\mathbb{Q}$ i.e. $[\mathbb{Q}(R) : \mathbb{Q}] = 4$ and R_1, R_2, R_3 and R_4 the Galois conjugates of R. By the Lemma 2.2, we have

$$[R_1 + R_2 + R_3 + R_4 - 4P_\infty] \in J_1\,(\mathbb{Q})\,.$$

According to the Lemma 2.2 we have :

$$[R_1 + R_2 + R_3 + R_4 - 4P_\infty] = -m\,[P_0 - P_\infty] \quad \text{with} \quad 0 \leq m \leq 6.$$

The linearity of Abel Jacobi's application gives

$$[R_1 + R_2 + R_3 + R_4 + mP_0 - (4 + m)P_\infty] = 0 \quad \text{with} \quad 0 \leq m \leq 6.$$

According to the Theorem 2.2, there exists a rational function ψ such that

$$div(\psi) = R_1 + R_2 + R_3 + R_4 + mP_0 - (4 + m)P_\infty.$$

So we obtain :

$$\psi \in \mathcal{L}((4 + m)P_\infty) \quad \text{and} \quad ord_{P_0}\,(\psi) = m.$$

For the following

$$k = \left[\frac{m+4}{2}\right] \quad \text{et} \quad l = \left[\frac{m-3}{2}\right].$$

According to the Lemma 2.3, the expression of ψ is of the form

$$\psi\,(x, y) = \sum_{i=0}^{k} a_i y^i + x \sum_{j=0}^{l} b_j y^j \quad \text{where} \quad a_i, b_j \in \mathbb{Q}.$$

At points R_i, we must have $\psi\,(x, y) = 0$ for all d (with $1 \leq i \leq d$) and at point P_0 for all m (with $1 \leq m \leq 6$).

Remark 3.1 According to the Lemma 2.2, we can also replace P_0 by P_1 in the formula, and this is what we do in the proof because

$$j(P_0) + j(P_1) = 0 \Longrightarrow \langle\, j(P_0)\,\rangle = \langle\, j(P_1)\,\rangle.$$

Depending on the values of m, the reasoning is subdivided into the following cases:

3.1. Case $m = 0$

The function $\psi \in \mathcal{L}(4P_\infty)$, so

$$\psi(x,y) = a_0 + a_1 y + a_2 y^2 = 0 \quad (a_2 \neq 0)$$

The equation can be written as

$$y^2 - \alpha y - \beta = 0$$

Hence y is given by

$$y = \frac{1}{2}\alpha \pm \sqrt{\beta + \frac{1}{4}\alpha^2},\ \alpha \in \mathbb{Q},\ \beta \in \mathbb{Q}^*$$

At points R_i, we must have $(x,y) \in \mathcal{C}_1$. Thus, we find

$$x = \frac{1}{2}\alpha \pm \sqrt{y^7 + \frac{1}{4}}$$

A first family of points of degree 4 is

$$\mathcal{G}_{4,1} = \left\{ \left(\frac{1}{2}\alpha \pm \sqrt{y^7 + \frac{1}{4}},\ y^a \right) \middle| \ y = \frac{1}{2}\alpha \pm \sqrt{\beta + \frac{1}{4}\alpha^2},\ \alpha \in \mathbb{Q},\ \beta \in \mathbb{Q}^* \right\}$$

3.2. Cases $m \in \{1,6\}$

The function $\psi \in \mathcal{L}(5P_\infty)$.
Or $\mathcal{L}(5P_\infty) = \mathcal{L}(4P_\infty)$, one of the points $R_i (1 \leq i \leq 4)$ is egal to P_0.
Therefore $[\mathbb{Q}(x,y) : \mathbb{Q}] < 4$. We have a contradiction.

3.3. Cases $m \in \{2,5\}$

The function $\psi \in \mathcal{L}(6P_\infty)$ and $ord_{P_0}(\psi) = 2$. So

$$\psi(x,y) = a_2 y^2 + a_3 y^3 = y^2(a_2 + a_3 y) = 0.$$

Hence $[\mathbb{Q}(x,y) : \mathbb{Q}] < 4$. We have a contradiction.

3.4. Cases $m \in \{3,4\}$

For $m = 3$, the function $\psi \in \mathcal{L}(7P_\infty)$ and $ord_{P_0}(\psi) = 3$ (or $ord_{P_1}(\psi) = 3$), so

$$\psi(x,y) = a_3 y^3 + b_1(x-1) = 0.$$

The value of x can be written as:

$$x = 1 + \lambda y^3 \quad \text{with} \quad \lambda \in \mathbb{Q}^*.$$

The affine equation of the curve gives

$$y^4 - \lambda^2 y^3 - \lambda = 0.$$

We obtain the family $\mathcal{G}_{4,2,1}$ of quartic points

$$y^4 - \lambda^2 y^3 - \lambda = 0.$$

$$\mathcal{G}_{4,2,1} = \left\{ \left(1 + \lambda y^3, y^a\right) \;\middle|\; y^4 - \lambda^2 y^3 - 1\lambda = 0,\ \lambda \in \mathbb{Q}^* \right\}$$

For $m = 4$, the function $\psi \in \mathcal{L}(7P_\infty)$ and $ord_{P_0}(\psi) = 3$, so

$$\psi(x, y) = a_3 y^3 + b_1 x = 0.$$

The value of x can be written as:

$$x = \lambda y^3 \quad \text{with} \quad \lambda \in \mathbb{Q}^*.$$

The affine equation of the curve gives

$$y^4 - \lambda^2 y^3 + \lambda = 0.$$

We obtain the family of quartic points

$$\mathcal{G}_{4,2,0} = \left\{ \left(\lambda y^3, y^a\right) \;\middle|\; y^4 - \lambda^2 y^3 + \lambda = 0,\ \lambda \in \mathbb{Q}^* \right\}$$

Finally the set of quartic points is given by $\mathcal{G}_{4,2} = \mathcal{G}_{4,2,1} \cup \mathcal{G}_{4,2,0}$.

$$\mathcal{G}_{4,2} = \left\{ \left(\gamma + \lambda y^3, y^a\right) \;\middle|\; y^4 - \lambda^2 y^3 + (-1)^\gamma \lambda = 0,\ \lambda \in \mathbb{Q}^*,\ \gamma \in \{0, 1\} \right\}$$

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