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Applications of the Cauchy-Schwarz Inequality for the Numerical Radius

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ABSTRACT: The main goal of this article is to establish several new norm and numerical radius inequalities for operators based on the angle between two vectors in Hilbert space. These enhancements and extensions are achieved through the use of the polar and Cartesian decompositions of operators. In particular, it is proved that, if $X \in \mathscr{B}(\mathscr{H})$ has the polar decomposition $X = U|X|$ and $\mu(\psi) = \frac{1}{4}(2 + \cos\psi \cot\psi \log(\frac{1+\sin\psi}{1-\sin\psi}))$, then

$$\omega^{2r}(X) \leq \mu^{2r}(\theta) \left\| \frac{1}{p} f^{2pr}(|X|) + \frac{1}{q} g^{2qr}(|X^*|) \right\|,$$

where $\theta_{X,x} = \angle_{f(|X|)x, g(|X|)U^*x}$, either $0 \leq \theta < \theta_{X,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{X,x} < \theta \leq \pi$ for all unit vectors $x \in \mathscr{H}$, f, g are nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$, $r \geq 1$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Keywords: Angle, numerical radius, operator norm, cartesian decompositions, Cauchy-Schwarz inequality.

Contents

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ present the C^*-algebra containing all bounded linear operators that act on a nontrivial complex Hilbert space $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. For $X \in \mathscr{B}(\mathscr{H})$, the symbol X^* denotes the adjoint of X, and $|X| = (X^*X)^{\frac{1}{2}}$. Every operator $X \in \mathscr{B}(\mathscr{H})$ has the polar decomposition $X = U|X|$, where U is a partial isometry, and admits the Cartesian decomposition $X = S + iT$, in which S and T are self-adjoint operators. Recall that the numerical radius and the operator norm are defined as follow:

$$\omega(X) = \sup_{\|x\|=1} |\langle Xx, x \rangle| \quad \text{and} \quad \|X\| = \sup_{\|x\|=1} \|Xx\|.$$

It is well known that $\omega(\cdot)$ is a norm on $\mathscr{B}(\mathscr{H})$, see [9]. This norm is equivalent to the operator norm. In fact, the following inequalities hold

$$\frac{1}{2}\|X\| \leq \omega(X) \leq \|X\|.$$

For more information, we refer to [1,2,6,3,5,4,7], as well as their respective references.

Let $X \in \mathscr{B}(\mathscr{H})$. In the work presented in [15], it has been firmly established that

$$\frac{1}{4}\||X|^2 + |X^*|^2\| \leq \omega^2(X) \leq \frac{1}{2}\||X|^2 + |X^*|^2\|. \tag{1.1}$$

The second inequality in (1.1) is simply a specific instance of a broader result outlined in [11], which states that

$$\omega^{2r}(X) \leq \frac{1}{2}\||X|^{2r} + |X^*|^{2r}\| \tag{1.2}$$

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for all $r \geq 1$. Furthermore, numerous refinements and extensions of this recent inequality have been developed, underscoring its significance in the field.

EL-Haddad et al. in their work [11], demonstrated that if $X, Y \in \mathscr{B}(\mathscr{H})$, with $0 \leq \alpha \leq 1$ and $r \geq 1$, then the following inequality holds:

$$\|X+Y\|^r \leq 2^{r-2}\left(\||X|^{2\alpha}+|Y|^{2\alpha}\|+\||X^*|^{2(1-\alpha)}+|Y^*|^{2(1-\alpha)}\|\right).$$

Furthermore, they proved that:

$$\omega^r(X+Y) \leq 2^{r-2}\left\||X|^{2\alpha r}+|Y|^{2\alpha r}+|X^*|^{2(1-\alpha)r}+|Y^*|^{2(1-\alpha)r}\right\|$$

for $X, Y \in \mathscr{B}(\mathscr{H})$, with $0 \leq \alpha \leq 1$ and $r \geq 1$. Moreover, in the work of [19], the authors showed that if $X \in \mathscr{B}(\mathscr{H})$ can be expressed in its Cartesian decomposition $X = S + iT$, then the following inequality is applicable:

$$\omega(X) \leq \frac{1}{2}\||S|^{2\alpha r}+|S|^{2(1-\alpha)r}+|T|^{2\alpha r}+|T|^{2(1-\alpha)r}\| \tag{1.3}$$

This holds for any $0 \leq \alpha \leq 1$. For related inequalities, we refer to [8,10,13,16,18,20,22,23,25], as well as their respective references.

Our results will effectively utilize the angle $\angle_{x,y}$ between two vectors $x, y \in \mathbb{C}^n$ or $x, y \in \mathscr{H}$. For such vectors, we apply the Cauchy-Schwarz inequality, which asserts that $|\langle x, y\rangle| \leq \|x\|\|y\|$. This foundational principle allows us to confidently define the angle between the non-zero vectors x and y as follows: $\angle_{x,y} = \cos^{-1}\left(\frac{|\langle x,y\rangle|}{\|x\|\|y\|}\right)$. In [24, Theorem 2.3], Sababheh et al. establish that for any operator $X \in \mathscr{B}(\mathscr{H})$ possessing the polar decomposition $X = U|X|$, and for any vectors $x, y \in \mathscr{H}$, the following inequality holds:

$$|\langle Xx, y\rangle| \leq \mu(\theta_{X,x,y})\sqrt{\langle |X|^{2\alpha}x, x\rangle\langle |X^*|^{2(1-\alpha)}y, y\rangle}. \tag{1.4}$$

where $0 \leq \alpha \leq 1$. This result provides a clear framework for estimating the inner product involving the operator X such that $\mu(\psi) := \frac{1}{4}(2+\cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$ and $\theta_{X,x,y} = \angle_{|X|^\alpha x, |X|^{1-\alpha}U^*y}$.
The function $\mu(\psi) := \frac{1}{4}(2+\cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$ is defined on $\mathbb{R} - \{n\pi\}_{n\in\mathbb{Z}}$. Since $\lim_{\psi\to n\pi}\mu(\psi) = 1$, we define $\mu(n\pi) := 1$. It was also shown that the function μ is decreasing on the interval $[0, \frac{\pi}{2}]$ and is increasing on $[\frac{\pi}{2}, \pi]$, and the inequalities $\frac{1}{2} \leq \mu(\psi) \leq 1$ hold for all $\psi \geq 0$. In fact, the following inequalities hold
(i) If $0 \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$, then $\mu(\theta_2) \leq \mu(\theta_1)$;
(ii) If $\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \pi$, then $\mu(\theta_1) \leq \mu(\theta_2)$.

In this paper, we discuss enhancements to the operator norm and the numerical radius related to the addition and multiplication of two operators on a Hilbert space. We achieve these improvements by applying the Cauchy-Schwarz inequality to the angle between two vectors x and y, as well as utilizing the concepts of the polar decomposition and the Cartesian decomposition. Furthermore, we present enhancements to some existing results.

2. Main Results

In this section, we present a version of the Cauchy-Schwarz inequality that is expressed in terms of the angle between the vectors x and y. By using this version, we offer improvements on some inequalities related to the numerical radius.

To present our main findings, we require the following lemmas.

Lemma 2.1 *[24, Corollary 2.1] Let $x, y \in \mathscr{H}$. Then*

$$|\langle x, y\rangle| \leq \mu(\theta_{x,y})\|x\|\|y\|, \tag{2.1}$$

where $\theta_{x,y} = \angle_{x,y}$ and $\mu(\psi) = \frac{1}{4}(2+\cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$.

The following lemma is a simple consequence of the classical Jensen and Young inequalities (see [12]).

Lemma 2.2 *[14] Let $a, b \geq 0$ and $p, q \geq 1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $r \geq 1$,*

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \leq \left(\frac{a^{rp}}{p}+\frac{b^{rq}}{q}\right)^{\frac{1}{r}}.$$

Lemma 2.3 *[17] If $X \in \mathscr{B}(\mathscr{H})$ is positive and $x \in \mathscr{H}$ is a unit vector, then*

$$\langle Xx, x\rangle^r \leq \langle X^r x, x\rangle \quad \text{for all } r \geq 1.$$

The following lemma shows an extension of the inequality (1.4), see also [3].

Lemma 2.4 *Let $X \in \mathscr{B}(\mathscr{H})$ have the polar decomposition $X = U|X|$ and $x, y \in \mathscr{H}$. If f, g are nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$, then*

$$|\langle Xx, y\rangle| \leq \mu(\theta_{X,x,y})\sqrt{\langle f^2(|X|)x, x\rangle\langle g^2(|X^*|)y, y\rangle}, \tag{2.2}$$

where $\theta_{X,x,y} = \angle_{f(|X|)x, g(|X|)U^*y}$ and $\mu(\psi) := \frac{1}{4}(2+\cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$.

Proof: Assume $X \in \mathscr{B}(\mathscr{H})$ has the polar decomposition $X = U|X|$. We have

$$\begin{aligned}
|\langle Xx, y\rangle| &= |\langle U|X|x, y\rangle| \\
&= |\langle Ug(|X|)f(|X|)x, y\rangle| \\
&= |\langle f(|X|)x, g(|X|)U^*y\rangle| \\
&\leq \mu(\theta_{X,x,y})\|f(|X|)x\|\,\|g(|X|)U^*y\| \quad (\text{by } (2.1)) \\
&= \mu(\theta_{X,x,y})\sqrt{\langle f(|X|)x, f(|X|)x\rangle\langle g(|X|)U^*y, g(|X|)U^*y\rangle} \\
&= \mu(\theta_{X,x,y})\sqrt{\langle f^2(|X|)x, x\rangle\langle Ug^2(|X|)U^*y, y\rangle} \\
&= \mu(\theta_{X,x,y})\sqrt{\langle f^2(|X|)x, x\rangle\langle g^2(|X^*|)y, y\rangle},
\end{aligned}$$

as required. □

First, we obtain a refinement and a generalization of the recent inequalities as follows.

Theorem 2.5 *Let $X \in \mathscr{B}(\mathscr{H})$ have the polar decomposition $X = U|X|$, $x \in \mathscr{H}$, and $\mu(\psi) = \frac{1}{4}(2 + \cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$, and let f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. If either $0 \leq \theta < \theta_{X,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{X,x} < \theta \leq \pi$ for all unit vectors $x \in \mathscr{H}$, then*

$$\omega^{2r}(X) \leq \mu^{2r}(\theta)\left\|\frac{1}{p}f^{2pr}(|X|)+\frac{1}{q}g^{2qr}(|X^*|)\right\|, \tag{2.3}$$

where $\theta_{X,x} = \angle_{f(|X|)x, g(|X|)U^*x}$, $r \geq 1$, $p, q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Assume $X \in \mathscr{B}(\mathscr{H})$ has the polar decomposition $X = U|X|$ and $x \in \mathscr{H}$. Applying the inequality (2.2), we deduce that

$$\begin{aligned}
|\langle Xx, x\rangle|^{2r} &\leq \mu^{2r}(\theta_{X,x})(\langle f^2(|X|)x, x\rangle\langle g^2(|X^*|)x, x\rangle)^r \\
&\leq \mu^{2r}(\theta_{X,x})(\frac{1}{p}\langle f^2(|X|)x, x\rangle^p + \frac{1}{q}\langle g^2(|X^*|)x, x\rangle^q)^r \quad (\text{by Lemma } 2.2) \\
&\leq \mu^{2r}(\theta_{X,x})((\frac{1}{p}\langle f^2(|X|)x, x\rangle^{pr} + \frac{1}{q}\langle g^2(|X^*|)x, x\rangle^{qr})^{\frac{1}{r}})^r \quad (\text{by Lemma } 2.2) \\
&= \mu(\theta_{X,x})^{2r}(\frac{1}{p}\langle f^2(|X|)x, x\rangle^{pr} + \frac{1}{q}\langle g^2(|X^*|)x, x\rangle^{qr}) \\
&\leq \mu^{2r}(\theta_{X,x})(\frac{1}{p}\langle f^{2pr}(|X|)x, x\rangle + \frac{1}{q}\langle g^{2qr}(|X^*|)x, x\rangle) \quad (\text{by Lemma } 2.3) \\
&= \mu^{2r}(\theta_{X,x})\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle,
\end{aligned}$$

where $\theta_{X,x} = \angle_{f(|X|)x, g(|X|)U^*x}$, $r \geq 1$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Hence,

$$|\langle Xx, x\rangle|^{2r} \leq \mu^{2r}(\theta_{X,x})\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle. \tag{2.4}$$

Therefore, if $0 \leq \theta < \theta_{X,x} \leq \frac{\pi}{2}$ for all unit vectors $x \in \mathscr{H}$, then

$$\begin{aligned} |\langle Xx, x\rangle|^{2r} &\leq \mu^{2r}(\theta_{X,x})\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle \\ &\leq \mu^{2r}(\theta)\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle \quad \text{(by the property (i) of } \mu). \end{aligned}$$

Moreover, if $\frac{\pi}{2} \leq \ \theta_{X,x} < \theta \leq \pi$ for all unit vectors $x \in \mathscr{H}$, we have

$$\begin{aligned} |\langle Xx, x\rangle|^{2r} &\leq \mu^{2r}(\theta_{X,x})\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle \\ &\leq \mu^{2r}(\theta)\langle(\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|))x, x\rangle \quad \text{(by the property (ii) of } \mu). \end{aligned}$$

Now, by taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequalities, we get the desired results. □

Remark 2.6 *It follows from $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$ and the inequality (2.5), that we have*

$$\begin{aligned} \omega^{2r}(X) &\leq \mu^{2r}(\theta)\left\|\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|)\right\| \\ &\leq \left\|\frac{1}{p}f^{2pr}(|X|) + \frac{1}{q}g^{2qr}(|X^*|)\right\|, \end{aligned}$$

which is a refinement and a generalization of the inequality (1.2).

By setting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ for $0 \leq \alpha \leq 1$, we achieve the following result.

Corollary 2.7 *[21] Let $X \in \mathscr{B}(\mathscr{H})$ have the polar decomposition $X = U|X|$, $x \in \mathscr{H}$, and $\mu(\psi) = \frac{1}{4}(2 + \cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$. If either $0 \leq \theta < \theta_{X,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{X,x} < \theta \leq \pi$ for all unit vectors $x \in \mathscr{H}$, then*

$$\omega^{2r}(X) \leq \mu^{2r}(\theta)\left\|\frac{1}{p}|X|^{2\alpha pr} + \frac{1}{q}|X^*|^{2(1-\alpha)qr}\right\|, \tag{2.5}$$

where $\theta_{X,x} = \angle_{|X|^\alpha x, |X|^{1-\alpha}U^*x}$, $0 \leq \alpha \leq 1$, $r \geq 1$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

To show that the set of angles θ is non-empty for the cases where either $0 \leq \theta < \theta_{X,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{X,x} < \theta \leq \pi$, as stated in Theorem 2.5, we consider the following example, which is also discussed in [21].

Example 2.8 *Consider the matrix $X = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \in \mathbf{M}_n(\mathbb{R})$. Then, we have the polar the composition of X as follows:*

$$X = U|X| = \begin{bmatrix} 0.7673 & 0.6392 \\ -0.6425 & 0.7681 \end{bmatrix}\begin{bmatrix} 2.0523 & -0.9032 \\ -0.9032 & 5.7582 \end{bmatrix}.$$

For any unit vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we can calculate the angle $\theta_{X,x}$ as follows:

$$\begin{aligned} \cos(\theta_{X,x}) &= \frac{\langle |X|^{\frac{1}{2}}x, |X|^{\frac{1}{2}}U^*x\rangle}{\||X|^{\frac{1}{2}}x\|\||X|^{\frac{1}{2}}U^*x\|} \\ &= \frac{(1.413x_1 - 0.238x_2)(0.932x_1 - 1.091x_2) + (-0.238x_1 + 2.388x_2)(1.344x_1 + 1.987x_2)}{\sqrt{(1.413x_1 - 0.238x_2)^2 + (-0.238x_1 + 2.388x_2)^2}\sqrt{(0.932x_1 - 1.091x_2)^2 + (1.362x_1 + 1.987x_2)^2}}. \end{aligned}$$

Therefore,

$$0.404289 \lesssim \cos(\theta_{T,x}) \lesssim 0.921583,$$

whence

$$22.841^\circ \lesssim \theta_{T,x} \lesssim 66.122^\circ.$$

Now, we can consider $0 \leq \theta \leq 22.841^\circ$. *Moreover, for the matrix* $X = \begin{bmatrix} -2 & 0 \\ \frac{1}{2} & -2 \end{bmatrix} \in \mathbf{M}_n(\mathbb{R})$, *we can calculate the angle* $\theta_{X,x}$, *as follows*

$$\begin{aligned}\cos(\theta_{X,x}) &= \frac{\langle |X|^{\frac{1}{2}}x, |X|^{\frac{1}{2}}U^*x\rangle}{\||X|^{\frac{1}{2}}x\|\||X|^{\frac{1}{2}}U^*x\|} \\ &= \frac{(1.429x_1 - 0.087x_2)(-1.409x_1 + 0.264x_2) + (-0.087x_1 + 1.405x_2)(-0.088x_1 + -1.403x_2)}{\sqrt{(1.429x_1 - 0.087x_2)^2 + (-0.087x_1 + 1.405x_2)^2}\sqrt{(-1.409x_1 + 0.264x_2)^2 + (-0.088x_1 + -1.403x_2)^2}}.\end{aligned}$$

Therefore,

$$-0.951078 \lesssim \cos(\theta_{X,x}) \lesssim -0.968691,$$

whence

$$162.004^\circ \lesssim \theta_{X,x} \lesssim 165.625^\circ.$$

So, we can consider $165.625^\circ \leq \theta \leq \pi$.

Theorem 2.9 *Let* $X \in \mathscr{B}(\mathscr{H})$ *have the Cartesian decomposition* $X = S + iT$, $x \in \mathscr{H}$, $S = U|S|$, $T = V|T|$ *be the polar decompositions of* S *and* T, $\theta_{S,x} = \angle_{f(|S|)x,g(|S|)U^*x}$, $\theta_{T,x} = \angle_{f(|T|)x,g(|T|)V^*x}$, *and let* $\mu(\psi) = \frac{1}{4}(2 + \cos\psi \cot\psi \log(\frac{1+\sin\psi}{1-\sin\psi}))$ *and* f, g *be nonnegative continuous functions on* $[0, +\infty)$ *satisfying the relation* $f(t)g(t) = t$ $(t \in [0, +\infty))$. *Then*

(i) If $0 \leq \theta' < \theta_{S,x} \leq \frac{\pi}{2}$ *and* $0 \leq \theta'' < \theta_{T,x} \leq \frac{\pi}{2}$ *for all unit vectors* $x \in \mathscr{H}$, *then*

$$\omega(X) \leq \frac{\mu(\theta)}{2} \left\| f^2(|S|) + g^2(|T|) + f^2(|T|) + g^2(|S|) \right\|, \tag{2.6}$$

where $\theta = \min\{\theta', \theta''\}$;

(ii) If $\frac{\pi}{2} \leq \theta_{S,x} < \theta' \leq \pi$ *and* $\frac{\pi}{2} \leq \theta_{T,x} < \theta'' \leq \pi$ *for all unit vectors* $x \in \mathscr{H}$, *then*

$$\omega(X) \leq \frac{\mu(\theta)}{2} \left\| f^2(|S|) + g^2(|T|) + f^2(|T|) + g^2(|S|) \right\|, \tag{2.7}$$

where $\theta = \max\{\theta', \theta''\}$.

Proof: Assume $X \in \mathscr{B}(\mathscr{H})$ has the Cartesian decomposition $X = S + iT$, $\theta_{S,x,y} = \angle_{f(|S|)x,g(|S|)U^*y}$ and $\theta_{T,x,y} = \angle_{f(|T|)x,g(|T|)V^*y}$. We have

$$\begin{aligned}&|\langle Xx, y\rangle| \\ &= |\langle (S + iT)x, y\rangle| \\ &\leq |\langle Sx, y\rangle| + |\langle Tx, y\rangle| \quad \text{(by the triangle inequality)} \\ &\leq \mu(\theta_{S,x,y})\sqrt{\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle} + \mu(\theta_{T,x,y})\sqrt{\langle f^2(|T|)x, x\rangle\langle g^2(|T^*|)y, y\rangle} \\ &\qquad \text{(by the inequality (2.2))} \\ &= \mu(\theta_{S,x,y})\sqrt{\langle f^2(|S|)x, x\rangle\langle g^2(|S|)y, y\rangle} + \mu(\theta_{T,x,y})\sqrt{\langle f^2(|T|)x, x\rangle\langle g^2(|T|)y, y\rangle} \\ &\qquad \text{(since } S \text{ and } T \text{ are self-adjoint)} \\ &\leq \mu_\theta \times \sqrt{\langle f^2(|S|)x, x\rangle\langle g^2(|S|)y, y\rangle} + \mu_\theta \times \sqrt{\langle f^2(|T|)x, x\rangle\langle g^2(|T|)y, y\rangle} \\ &\qquad \text{(by } \mu_\theta = \max\{\mu(\theta_{S,x,y}), \mu(\theta_{T,x,y})\}\text{)} \\ &= \mu_\theta \times \sqrt{\langle f^2(|S|)x, x\rangle\langle g^2(|S|)y, y\rangle} + \sqrt{\langle f^2(|T|)x, x\rangle\langle g^2(|T|)y, y\rangle} \\ &\leq \mu_\theta \times \sqrt{\langle f^2(|S|)x, x\rangle + \langle f^2(|T|)x, x\rangle}\sqrt{\langle g^2(|S|)y, y\rangle + \langle g^2(|T|)y, y\rangle} \\ &\qquad \text{(by the Cauchy-Schwarz inequality),}\end{aligned}$$

whence

$$|\langle Xx,y\rangle| \le \mu_\theta \times \sqrt{\langle f^2(|S|)x,x\rangle + \langle f^2(|T|)x,x\rangle}\sqrt{\langle g^2(|S|)y,y\rangle + \langle g^2(|T|)y,y\rangle},$$

where $\mu_\theta = \max\{\mu(\theta_{S,x,y}), \mu(\theta_{T,x,y})\}$. Hence

$$\begin{aligned} |\langle Tx,x\rangle| &= |\langle (S+iT)x,x\rangle| \\ &\le \mu_\theta \times \sqrt{\langle (f^2(|S|)+f^2(|T|))x,x\rangle}\sqrt{\langle (g^2(|S|)+g^2(|T|))x,x\rangle} \\ &\le \frac{\mu_\theta}{2}\left(\langle (f^2(|S|)+f^2(|T|))x,x\rangle + \langle (g^2(|S|)+g^2(|T|))x,x\rangle\right) \\ &\qquad \text{(by the arithmetic-geometric mean inequality)} \\ &= \frac{\mu_\theta}{2}\langle (f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|))x,x\rangle \\ &\le \frac{\mu_\theta}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\|, \end{aligned} \tag{2.8}$$

where $\mu_\theta = \max\{\mu(\theta_{S,x}), \mu(\theta_{T,x})\}$. Hence, we have two cases as follow:
(i) If $0 \le \theta' < \theta_{S,x} \le \frac{\pi}{2}$ and $0 \le \theta'' < \theta_{T,x} \le \frac{\pi}{2}$ for all unit vectors $x \in \mathscr{H}$, then put $\theta = \min\{\theta', \theta''\}$. It follows from the monotonicity of μ, $\theta \le \theta'$ and $\theta \le \theta''$ that

$$\mu(\theta') \le \mu(\theta) \quad \text{and} \quad \mu(\theta'') \le \mu(\theta). \tag{2.9}$$

Moreover, by the monotonicity of μ, $\theta' < \theta_{T,x}$ and $\theta'' < \theta_{S,x}$, we have

$$\mu(\theta_{S,x}) \le \mu(\theta') \quad \text{and} \quad \mu(\theta_{T,x}) \le \mu(\theta''). \tag{2.10}$$

Therefore, applying (2.9) and (2.10), we obtain

$$\mu_\theta := \max\{\mu(\theta_{S,x}), \mu(\theta_{S,x})\} \le \max\{\mu(\theta'), \mu(\theta'')\} \le \mu(\theta). \tag{2.11}$$

Hence, using (2.8) and (2.11), we have

$$\begin{aligned} |\langle Xx,x\rangle| = |\langle (S+iT)x,x\rangle| &\le \frac{\mu_\theta}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\| \\ &\le \frac{\mu(\theta)}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\|, \end{aligned}$$

where $\theta = \min\{\theta', \theta''\}$. By taking the supremum in the above inequalities over $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$\omega(X) \le \frac{\mu(\theta)}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\|,$$

where $\theta = \min\{\theta', \theta''\}$.
(ii) If $\frac{\pi}{2} \le \theta_{S,x} < \theta' \le \pi$ and $\frac{\pi}{2} \le \theta_{T,x} < \theta'' \le \pi$ for all unit vectors $x \in \mathscr{H}$, then put $\theta = \max\{\theta', \theta''\}$. It follows from the monotonicity of μ, $\theta' \le \theta$ and $\theta'' \le \theta$ that

$$\mu(\theta') \le \mu(\theta) \quad \text{and} \quad \mu(\theta'') \le \mu(\theta). \tag{2.12}$$

Moreover, by the monotonicity of μ, $\theta_{S,x} < \theta'$ and $\theta_{T,x} < \theta''$, we get

$$\mu(\theta_{S,x}) \le \mu(\theta') \quad \text{and} \quad \mu(\theta_{T,x}) \le \mu(\theta''). \tag{2.13}$$

Thus, by using (2.12) and (2.13), we obtain

$$\mu_\theta := \max\{\mu(\theta_{S,x}), \mu(\theta_{T,x})\} \le \max\{\mu(\theta'), \mu(\theta'')\} \le \mu(\theta). \tag{2.14}$$

Applying (2.8) and (2.14), we have

$$\begin{aligned} |\langle Xx,x\rangle| = |\langle (S+iT)x,x\rangle| &\le \frac{\mu_\theta}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\| \\ &\le \frac{\mu(\theta)}{2}\left\| f^2(|S|)+f^2(|T|)+g^2(|S|)+g^2(|T|)\right\|, \end{aligned}$$

Now, by taking the supremum in the above inequalities over $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$\omega(X) \leq \frac{\mu(\theta)}{2} \left\| f^2(|S|) + f^2(|T|) + g^2(|S|) + g^2(|T|) \right\|,$$

where $\theta = \max\{\theta', \theta''\}$. As the required result. □

In the following theorem, we present a refinement and an improvement of the inequality (1.3).

Theorem 2.10 *Let $X, Y \in \mathscr{B}(\mathscr{H})$ have the polar decompositions $X = U|X|$, $Y = V|Y|$, $x \in \mathscr{H}$, $\theta_{X,x} = \angle_{f(|X|)x, g(|X|)U^*x}$, $\theta_{Y,x} = \angle_{f(|Y|)x, g(|Y|)V^*x}$, $0 \leq \alpha \leq 1$ and $r \geq 1$, and let $\mu(\psi) = \frac{1}{4}(2 + \cos\psi \cot\psi \log(\frac{1+\sin\psi}{1-\sin\psi}))$ and f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then*

(i) If $0 \leq \theta' < \theta_{X,x} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x} \leq \frac{\pi}{2}$ for all unit vectors $x \in \mathscr{H}$, then

$$\omega^r(X+Y) \leq 2^{r-2} \mu^r(\theta) \left\| f^2(|X|) + f^2(|Y|) + g^2(|X^*|) + g^2(|Y^*|) \right\|, \tag{2.15}$$

where $\theta = \min\{\theta', \theta''\}$;

(ii) If $\frac{\pi}{2} \leq \theta_{X,x} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x} < \theta'' \leq \pi$ for all unit vectors $x \in \mathscr{H}$, then

$$\omega^r(X+Y) \leq 2^{r-2} \mu^r(\theta) \left\| f^2(|X|) + f^2(|Y|) + g^2(|X^*|) + g^2(|Y^*|) \right\|, \tag{2.16}$$

where $\theta = \max\{\theta', \theta''\}$.

Proof: Assume $X, Y \in \mathscr{B}(\mathscr{H})$ have the polar decompositions $X = U|X|$, $Y = V|Y|$, $x, y \in \mathscr{H}$, and let $\theta_{X,x,y} = \angle_{f(|X|)x, g(|X|)U^*y}$ and $\theta_{Y,x,y} = \angle_{f(|Y|)x, g(|Y|)V^*y}$. Then, we have

$$\begin{aligned}
&|\langle (X+Y)x, y\rangle| \\
&\leq |\langle Xx, y\rangle| + |\langle Yx, y\rangle| \\
&\qquad \text{(by the triangle inequality)} \\
&\leq \mu(\theta_{X,x,y}) \sqrt{\langle f^2(|X|)x, x\rangle \langle g^2(|X^*|)y, y\rangle} + \mu(\theta_{Y,x,y}) \sqrt{\langle f^2(|Y|)x, x\rangle \langle g^2(|Y^*|)y, y\rangle} \\
&\qquad \text{(by the inequality (2.2))} \\
&\leq \mu(\theta_{X,x,y}) \left(\frac{(\langle f^2(|X|)x, x\rangle^r + \langle g^2(|X^*|)y, y\rangle^r)}{2} \right)^{\frac{1}{r}} \\
&\qquad + \mu(\theta_{Y,x,y}) \left(\frac{(\langle f^2(|Y|)x, x\rangle^r + \langle g^2(|Y^*|)y, y\rangle^r)}{2} \right)^{\frac{1}{r}} \text{(by Lemma 2.2)} \\
&\leq \mu(\theta_{X,x,y}) \left(\frac{(\langle f^{2r}(|X|)x, x\rangle + \langle g^{2r}(|X^*|)y, y\rangle)}{2} \right)^{\frac{1}{r}} \\
&\qquad + \mu(\theta_{Y,x,y}) \left(\frac{(\langle f^{2r}(|Y|)x, x\rangle + \langle g^{2r}(|Y^*|)y, y\rangle)}{2} \right)^{\frac{1}{r}} \text{(by Lemma 2.3)}
\end{aligned}$$

$$\begin{aligned}
&\leq \mu_\theta \times \left(\left(\frac{\langle f^{2r}(|X|)x, x\rangle + \langle g^{2r}(|X^*|)y, y\rangle}{2} \right)^{\frac{1}{r}} + \left(\frac{\langle f^{2r}(|Y|)x, x\rangle + \langle g^{2r}(|Y^*|)y, y\rangle}{2} \right)^{\frac{1}{r}} \right) \\
&\qquad \text{(by } \mu_\theta = \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\}) \\
&\leq \mu_\theta \times \left(2^{1-\frac{1}{r}} \left(\frac{\langle f^{2r}(|X|)x, x\rangle + \langle g^{2r}(|X^*|)y, y\rangle + \langle f^{2r}(|Y|)x, x\rangle + \langle g^{2r}(|Y^*|)y, y\rangle}{2} \right)^{\frac{1}{r}} \right) \\
&\qquad \text{(by the concavity of } f(t) = t^{\frac{1}{r}} \text{ for } r \geq 1),
\end{aligned}$$

where $\mu_\theta = \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\}$. Thus

$$\begin{aligned}|\langle (X+Y)x,x\rangle|^r &\le 2^{r-2}\mu_\theta^r \left(\langle (f^{2r}(|X|)+g^{2r}(|Y|)x,x\rangle + \langle (f^{2r}(|T^*|)+g^{2r}(|S^*|)x,x\rangle\right) \\ &= 2^{r-2}\mu_\theta^r\langle (f^{2r}(|X|)+g^{2r}(|Y|)+f^{2r}(|X^*|)+g^{2r}(|Y^*|))x,x\rangle, \end{aligned} \tag{2.17}$$

where $\mu_\theta = \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\}$. Now, we have two cases as follows:
(i) If $0 \le \theta' < \theta_{X,x} \le \frac{\pi}{2}$ and $0 \le \theta'' < \theta_{Y,x} \le \frac{\pi}{2}$ for all unit vectors $x \in \mathscr{H}$, then put $\theta = \min\{\theta', \theta''\}$. It follows from the monotonicity of μ, $\theta \le \theta'$ and $\theta \le \theta''$ that

$$\mu(\theta') \le \mu(\theta) \quad \text{and} \quad \mu(\theta'') \le \mu(\theta). \tag{2.18}$$

Moreover, by the monotonicity of μ, $\theta' < \theta_{X,x}$ and $\theta'' < \theta_{Y,x}$, we have

$$\mu(\theta_{X,x}) \le \mu(\theta') \quad \text{and} \quad \mu(\theta_{Y,x}) \le \mu(\theta''). \tag{2.19}$$

Applying (2.18) and (2.19), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\} \le \max\{\mu(\theta'), \mu(\theta'')\} \le \mu(\theta). \tag{2.20}$$

Hence, using (2.17) and (2.20), we have

$$\begin{aligned}|\langle (X+Y)x,x\rangle|^r &\le 2^{r-2}\mu_\theta^r\langle (|X|^{2\alpha r}+|Y|^{2\alpha r}+|X^*|^{2(1-\alpha)r}+|Y^*|^{2(1-\alpha)r})x,x\rangle \\ &\le 2^{r-2}\mu^r(\theta)\langle (|X|^{2\alpha r}+|Y|^{2\alpha r}+|X^*|^{2(1-\alpha)r}+|Y^*|^{2(1-\alpha)r})x,x\rangle.\end{aligned}$$

(ii) If $\frac{\pi}{2} \le \theta_{X,x} < \theta' \le \pi$ and $\frac{\pi}{2} \le \theta_{Y,x} < \theta'' \le \pi$ for all unit vectors $x \in \mathscr{H}$, then put $\theta = \max\{\theta', \theta''\}$. It follows from the monotonicity of μ, $\theta' \le \theta$ and $\theta'' \le \theta$ that

$$\mu(\theta') \le \mu(\theta) \quad \text{and} \quad \mu(\theta'') \le \mu(\theta). \tag{2.21}$$

Moreover, by the monotonicity of μ, $\theta_{T,x} < \theta'$ and $\theta_{S,x} < \theta''$, we have

$$\mu(\theta_{X,x}) \le \mu(\theta') \quad \text{and} \quad \mu(\theta_{Y,x}) \le \mu(\theta''). \tag{2.22}$$

Applying (2.21) and (2.22), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\} \le \max\{\mu(\theta'), \mu(\theta'')\} \le \mu(\theta). \tag{2.23}$$

Hence, using (2.17) and (2.23), we have

$$\begin{aligned}|\langle (X+Y)x,x\rangle|^r &\le 2^{r-2}\mu_\theta^r\langle (|X|^{2\alpha r}+|Y|^{2\alpha r}+|X^*|^{2(1-\alpha)r}+|Y^*|^{2(1-\alpha)r})x,x\rangle \\ &\le 2^{r-2}\mu^r(\theta)\langle (|X|^{2\alpha r}+|Y|^{2\alpha r}+|X^*|^{2(1-\alpha)r}+|Y^*|^{2(1-\alpha)r})x,x\rangle.\end{aligned}$$

where $\theta = \max\{\theta', \theta''\}$. Now, by taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequalities, we get desired results. □

By choosing $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ for $0 \le \alpha \le 1$, we confidently achieve the following result.

Corollary 2.11 *[21] Let $X, Y \in \mathscr{B}(\mathscr{H})$ have the polar decompositions $X = U|X|$, $Y = V|Y|$, $x \in \mathscr{H}$, $\theta_{X,x} = \angle_{|X|^\alpha x, |X|^{1-\alpha}U^*x}$, $\theta_{Y,x} = \angle_{|Y|^\alpha x, |Y|^{1-\alpha}V^*x}$, $0 \le \alpha \le 1$ and $r \ge 1$, and let $\mu(\psi) = \frac{1}{4}(2 + \cos\psi\cot\psi\log(\frac{1+\sin\psi}{1-\sin\psi}))$. Then*
(i) If $0 \le \theta' < \theta_{X,x} \le \frac{\pi}{2}$ and $0 \le \theta'' < \theta_{Y,x} \le \frac{\pi}{2}$ for all unit vectors $x \in \mathscr{H}$, then

$$\omega^r(X+Y) \le 2^{r-2}\mu^r(\theta)\left\| |X|^{2\alpha} + |Y|^{2\alpha} + |X^*|^{2-2\alpha} + |Y^*|^{2-2\alpha} \right\|,$$

where $\theta = \min\{\theta', \theta''\}$;
(ii) If $\frac{\pi}{2} \le \theta_{X,x} < \theta' \le \pi$ and $\frac{\pi}{2} \le \theta_{Y,x} < \theta'' \le \pi$ for all unit vectors $x \in \mathscr{H}$, then

$$\omega^r(X+Y) \le 2^{r-2}\mu^r(\theta)\left\| |X|^{2\alpha} + |Y|^{2\alpha} + |X^*|^{2-2\alpha} + |Y^*|^{2-2\alpha} \right\|,$$

where $\theta = \max\{\theta', \theta''\}$.

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