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Modified Versions of Fractional Inequalities via Double Integral Operators with Coordinated Convexity

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ABSTRACT: Integral inequalities are of great importance in both theoretical and applied analysis. There is no doubt that inequalities aim to develop various mathematical techniques. Therefore, in order to prove the accuracy and uniqueness of such mathematical techniques, it is necessary to find explicit inequalities. The behavior and properties of the convexity of the functions are of great importance in the field of inequalities. This paper aims to introduce a new class of coordinated convexity, pre-invexities, modify some well-known fractional inequalities, and its refinements by implementation of two-dimensional fractional integral operators. We present a novel form of Hermite-Hadamard (H-H) and trapezoidal-type fractional two-dimensional integral inequalities for coordinated convex and pre-invex functions, grounded in the h-Godunova-Levin framework. Moreover, we deduce some corollaries from the main results, which are the famous inequalities in the previously published articles.

Keywords: Generalized fractional integrals, inequalities, coordinated convexity.

Contents

1. Introduction

Fractional calculus plays an important role for the generalization of fractional operators, which resolved many issues of physical and theoretical problems in different area of mathematics, and have significant applications in applied mathematics and analysis of inequalities [1,3,4,5,6,35,36,41]. The fractional operators could be generalized by using the improved version of special functions as its kernels, which have been developed many fractional inequalities according to the advance research of analysis in aspects of different types convexities and pre-invexities. Many researchers worked for the extension and generalization of fractional operators by using extended beta function, fractional inequalities by means of convexities and discussed fruitful applications in the area of analysis [7,8,9].

Convex sets and convex functions are widely utilized across various scientific disciplines, such as geometry, information theory, control theory, operations research, optimization, and functional analysis. In modern approach of convexities like as η-convex function, (η_1, η_2)-convex function have been used to established the many fractional inequalities, and widely discussed in [10,11,12,13,14] due to immense applications in several area of mathematics. Many researchers are worked to extended new generation of fractional Hermite-Hadamard inequalities by utilized generalized fractional operators by means of general kernels, and discussed its several refinements [15,16,17,18,19,20].

The Hermite-Hadamard inequality provides both upper and lower bounds for the average (integral) value of a convex function over a given interval. Each side of the inequality reflects a condition that

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characterizes the convexity of the function. Additionally, it is well known that a function is convex if and only if it is twice differentiable and its second derivative is non-negative.

The Hermite-Hadamard type inequality [21] is defined for convex function $\alpha : [a, b] \to \mathbb{R}$ as follows

$$\alpha\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \alpha(x)dx \leq \frac{\alpha(u_1) + \alpha(u_2)}{2}.$$

Due to the modern approach of analysis, optimizations have increased the demand of convexity, pre-invexities and fractional operators equipped with the non-singular functions as their kernels in multi-dimensions, which deals to many solved problems of fractional mathematical models and has significant applications in the applied sciences and the theory of optimization by means of fractional inequalities for partial differentiable functions [22,23,24,25,26,27,28,29].

Wenfeng and Xiaowei [39] discussed the importance of Hermite-Hadamard type inequalities and convex functions in the domain of signal processing. By examining their mathematical foundations and practical applications, they have gained valuable insight into how these concepts significantly influence the design and development of signal processing algorithms. Larson's study [40], presents a refined multidimensional Hermite-Hadamard inequality, offering a robust mathematical framework to analyze the integral means of non-negative subharmonic functions. By establishing the sharpness of this inequality and advancing previous results, the research deepens our understanding of Hermite-Hadamard inequalities and their role in mathematical analysis. These findings contribute significantly to the theoretical foundation of integral inequalities, providing fresh insights into the behavior of subharmonic functions in multidimensional contexts.

2. Preliminaries

In this section, we present some fundamental definitions that are essential for understanding the main results. Moreover, we describe the two-dimensional different type of convexities and generalized Riemann-Liouville fractional operators in the contents of two-dimensional. Also, we discuss some remarks which are extracting our newly developed definitions.

Definition 2.1. *[2] A set S is called a convex set if, for all $x_1, x_2 \in S$ and for every $\theta \in [0, 1]$, the following condition is satisfied:*

$$\theta x_1 + (1 - \theta)x_2 \in S \tag{2.1}$$

Definition 2.2. *[30] Let $t \in [0, 1]$ and $\beta_1, \beta_2 \in \tau$, where $\tau \subset \mathbb{R}$ is a convex set. Then, a function $\alpha : \tau \to \mathbb{R}$ is said to be convex if it satisfies the following condition:*

$$\alpha(t\beta_1 + (1 - t)\beta_2) \leq t\alpha(\beta_1) + (1 - t)\alpha(\beta_2). \tag{2.2}$$

Definition 2.3. *[31] The invex set $\xi \subseteq \mathbb{R}$ with respect to real bi-function α, $\alpha : \xi \times \xi \to \mathbb{R}$ is defined , as follows*

$$\beta_2 + \lambda\alpha(\beta_1, \beta_2) \in \xi. \tag{2.3}$$

where $\beta_1, \beta_2 \in \xi, \lambda \in [0, 1]$

Definition 2.4. *[31] Let ξ be an invex set with respect to α. Then, a function $\Xi : \xi \to \mathbb{R}$ is called pre-invex if, for all $\beta_1, \beta_2 \in \xi$ and $\lambda \in [0, 1]$, it satisfies the following condition:*

$$\Xi(\beta_2 + \lambda\alpha(\beta_1, \beta_2) \leq \lambda\Xi(\beta_1) + (1 - \lambda)\Xi(\beta_2). \tag{2.4}$$

Definition 2.5. *[32] The Godunova-Levin function $\alpha : C \subseteq \mathbb{R} \to \mathbb{R}$, is defined for $\forall$ $\beta_1, \beta_2 \in C, t \in (0, 1)$, as follows*

$$\alpha(t\beta_1 + (1 - t)\beta_2 \leq \frac{\alpha(\beta_1)}{t} + \frac{\alpha(\beta_2)}{1 - t}. \tag{2.5}$$

Definition 2.6. *[33] Let $h : (0,1) \to \mathbb{R}$ be a function and $\alpha : C \to \mathbb{R}$ be a non-negative function. Then, the h-Godunova-Levin function is defined as follows:*

$$\alpha(t\beta_1 + (1-t)\beta_2) \le \frac{\alpha(\beta_1)}{h(t)} + \frac{\alpha(\beta_2)}{h(1-t)}. \tag{2.6}$$

where $\beta_1, \beta_2 \in C$ and $t \in (0,1)$.

Definition 2.7. *[33] The function $\Xi : \xi \to \mathbb{R}$ is said to be h-Godunova-Levin preinvex on the invex set ξ with respect to bi-function α, if for all $\beta_1, \beta_2 \in \xi$ and $t \in (0,1)$, it satisfies the following condition:*

$$\Xi(\beta_1 + t\alpha(\beta_2, \beta_1)) \le \frac{\Xi(\beta_1)}{h(1-t)} + \frac{\Xi(\beta_2)}{h(t)}. \tag{2.7}$$

Definition 2.8. *[34] The Gamma function is defined for $\Re(t)$ by the following expression:*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

Definition 2.9. *[34] The beta function is defined for $\Re(l) > 0,\ \Re(h) > 0$, as given below*

$$\beta(l,h) = \int_0^1 t^{l-1}(1-t)^{h-1} dt.$$

Definition 2.10. *[37]] The Riemann-Liouville fractional integral operators of order $\eta \ge 0$, denoted by $\mathrm{J}^{\eta}_{\beta 1^+}\alpha$ (left-sided) and $\mathrm{J}^{\eta}_{k3^-}\alpha$ (right-sided), for $\beta_1 \ge 0$, are defined as follows:*

$$\mathrm{J}^{\eta}_{\beta_1^+}\alpha(\beta_2) = \frac{1}{\Gamma(\eta)} \int_{\beta_1}^{\beta_2} (\beta_2 - t)^{\eta-1}\alpha(t) dt,$$

and

$$\mathrm{J}^{\eta}_{k_3^-}\alpha(\beta_2) = \frac{1}{\Gamma(\eta)} \int_{\beta_2}^{k_3} (t - \beta_2)^{\eta-1}\alpha(t) dt.$$

where $\beta_2 > \beta_1$ and $k_3 > \beta_2$ and $\alpha \in L^1[\beta_2, k_3]$.

Definition 2.11. *[22] Let $t \in [0,1]$, and $\forall\ (\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi$, then the two-dimensional convex function $F : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as follows*

$$F(t\beta_1 + (1-t)\beta_2, tk_3 + (1-t)k_4) \le tF(\beta_1, k_3) + (1-t)F(\beta_2, k_4). \tag{2.8}$$

Remark 2.1. If we replace $k_3 = 0$ and $k_4 = 0$ in the equation (2.8), then we get the definition of convex function in equation (2.2).

Definition 2.12. *[22] Let a real valued bi-function $\alpha : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$, then the invex set $\Xi \subseteq \mathbb{R}$ with respect to bi-function α is defined as follows:*

$$\beta_2 + \lambda\alpha(\beta_1, \beta_2), k_4 + \lambda\alpha(k_3, k_4) \in \Xi$$

where $(\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi, \lambda \in [0,1]$.

Definition 2.13. *Let $(\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi$ and $\lambda \in [0,1]$, then the two-dimensional pre-invex function $F : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as follows:*

$$F(\beta_2 + \lambda\alpha(\beta_1, \beta_2), k_4 + \lambda\alpha(k_3, k_4)) \le \lambda F(\beta_1, k_3) + (1-\lambda)F(\beta_2, k_4). \tag{2.9}$$

where Ξ is two-dimensional invex set with respect to bi-function α.

Remark 2.2. Putting the values of $k_3 = 0$ and $k_4 = 0$ in the equation (2.9), then we get definition of pre-invex function in equation (2.4).

Definition 2.14. *The Godunova-Levin two-dimensional function $\digamma : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$, is defined as follows:*

$$\digamma(t\beta_1 + (1-t)\beta_2, tk_3 + (1-t)k_4) \leq \frac{\digamma(\beta_1, k_3)}{t} + \frac{\digamma(\beta_2, k_4)}{1-t}. \tag{2.10}$$

where $(\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi, t \in (0,1)$.

Remark 2.3. If we replace $k_3 = 0$ and $k_4 = 0$ in the equation (2.10), then we get definition of Godunova-Levin function in equation (2.5).

Definition 2.15. *Let $h : (0,1) \to \mathbb{R}$ be a given function, and consider the function $\digamma : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$ be a non-negative, then the h-Godunova-Levin two- dimensional function $\forall$ $(\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi$ and $t \in (0,1)$ is defined as follows:*

$$\digamma(t\beta_1 + (1-t)\beta_2, tk_3 + (1-t)k_4) \leq \frac{\digamma(\beta_1, k_3)}{h(t)} + \frac{\digamma(\beta_2, k_4)}{h(1-t)}. \tag{2.11}$$

Remark 2.4. If we replace $k_3 = 0$ and $k_4 = 0$ in the equation (2.11), then we get definition of h-Godunova-Levin function in equation (2.6).

Definition 2.16. *The h-Godunova-Levin pre-invex two-dimensional function $\digamma : \Xi \times \Xi \subset \mathbb{R}^2 \to \mathbb{R}$ with respect to α is defined for $\forall$ $(\beta_1, \beta_2), (k_3, k_4) \in \Xi \times \Xi, t \in (0,1)$, as follows:*

$$\digamma(\beta_1 + t\alpha(\beta_2, \beta_1), k_3 + t\alpha(k_4, k_3)) \leq \frac{\digamma(\beta_1, k_3)}{h(1-t)} + \frac{\digamma(\beta_2, k_4)}{h(t)}. \tag{2.12}$$

Remark 2.5. If we replace $k_3 = 0$ and $k_4 = 0$ in the equation (2.12), then we get definition of h-Godunova-Levin preinvex function in equation (2.7).

Definition 2.17. *[38] Let $\Delta = \Xi_1 \times \Xi_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. The coordinated pre-invex function $\digamma : \Delta \to \mathbb{R}$ is on Δ, as follows:*

$$\begin{aligned}\digamma(\beta_1 + \theta\xi_1(\beta_2, \beta_1), k_3 + r\xi_2(k_4, k_3)) &= (1-\theta)(1-r)\digamma(\beta_1, k_3) + (1-\theta)r\digamma(\beta_1, k_4)\\ &\quad + \theta(1-r)\digamma(\beta_2, k_3) + \theta r\digamma(\beta_2, k_4),\end{aligned}$$

where (β_1, k_3), (β_1, k_4), (β_2, k_3), $(\beta_2, k_4) \in \Delta$ and $\theta, r \in [0,1]$.

Definition 2.18. *[38] Let $\Delta = \Xi_1 \times \Xi_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\Xi_1, \Xi_2 \in \mathbb{R}^n$ be two invex sets with respect to $\xi_1 : \Xi_1 \times \Xi_1 \to \mathbb{R}^n$ and $\xi_2 : \Xi_2 \times \Xi_2 \to \mathbb{R}^n$, then the continuous function $\digamma : \Delta \to \mathbb{R}$ is said to be Godunova-Levin type s-pre-invex on the co-ordinates with respect to ξ_1 on Ξ_1 and ξ_2 on Ξ_2 is defined for every $\beta_1, \beta_2 \in \Xi_1$ and $k_3, k_4 \in \Xi_2$, θ,$r \in (0,1)$ and $s \in [0,1]$, as follows*

$$\begin{aligned}\digamma(\beta_1 + \theta\xi_1(\beta_2, \beta_1), k_3 + r\xi_2(k_4, k_3)) &\leq \frac{\digamma(\beta_1, k_3)}{(1-\theta^s)(1-r)^s} + \frac{\digamma(\beta_1, k_4)}{(1-\theta)^s r^s}\\ &\quad + \frac{\digamma(\beta_2, k_3)}{\theta^s(1-r)^s} + \frac{\digamma(\beta_2, k_4)}{\theta^s r^s}.\end{aligned}$$

Remark 2.6. If we replace $s = 1$ in the definition [2.18], then we obtain the following inequality

$$\digamma(\beta_1 + \theta\xi_1(\beta_2, \beta_1), k_3 + r\xi_2(k_4, k_3)) \leq \frac{\digamma(\beta_1, k_3)}{(1-\theta)(1-r)} + \frac{\digamma(\beta_1, k_4)}{(1-\theta)r} + \frac{\digamma(\beta_2, k_3)}{\theta(1-r)} + \frac{\digamma(\beta_2, k_4)}{\theta r}.$$

Definition 2.19. *Let $\Delta = \Xi_1 \times \Xi_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\Xi_1, \Xi_2 \in \mathbb{R}^n$ be two invex sets with respect to $\xi_1 : \Xi_1 \times \Xi_1 \to \mathbb{R}^n$ and $\xi_2 : \Xi_2 \times \Xi_2 \to \mathbb{R}^n$, then the h-Godunova-Levin coordinated pre-invex function $\digamma : \Delta \to \mathbb{R}$ is defined as follows:*

$$\begin{aligned}\digamma(\beta_1 + \theta\xi_1(\beta_2, \beta_1), k_3 + r\xi_2(k_4, k_3)) &\leq \frac{\digamma(\beta_1, k_3)}{h[(1-\theta)(1-r)]} + \frac{\digamma(\beta_1, k_4)}{h[(1-\theta)r]}\\ &\quad + \frac{\digamma(\beta_2, k_3)}{h[\theta(1-r)]} + \frac{\digamma(\beta_2, k_4)}{h[\theta r]}.\end{aligned}$$

Definition 2.20. *The two-dimensional Riemann-Liouville fractional integral operators is defined, as follows:*

$$\mathrm{J}^{\eta}_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)=\frac{1}{\Gamma(\eta)}\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}(\beta_2-t)^{\eta-1}(k_4-s)^{\eta-1}\alpha(t,s)dtds, \tag{2.13}$$

and

$$\mathrm{J}^{\eta}_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)=\frac{1}{\Gamma(\eta)}\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}(t-\beta_1)^{\eta-1}(s-k_3)^{\eta-1}\alpha(t,s)dtds. \tag{2.14}$$

Remark 2.7.

1. If we replace $s=0$,$k_3=0$ and $k_4=1$ in the equation (2.13), we have left sided Riemann-Liouvill operator, which defined in [2.10].
2. if we put the parameters $s=0$,$k_3=1$ and $k_4=0$ in the equation (2.14), then we obtained right sided Riemann-Liouvill fractional operators defined in [2.10].

The aim of this paper is to explore a new class of convexity and pre-invexity within the framework of a two-dimensional system, focusing on two-dimensional h-Godunova-Levin convex functions and coordinated pre-invexity and to describe the two-dimensional Riemann-Liouville fractional operators (left-right sided). To investigate Hermite-Hadamard and Trapezoid-type integral inequalities by successfully implemented newly describe two-dimensional fractional integral operators. Our new inequalities have modified versions of classical inequalities, which is a great contribution to the literature.

3. Analyzing the behavior of Hermite-Hadamard type fractional integral inequalities for two-dimensional h-Godunova-Levin convex functions

Here, we focus on investigating Hermite-Hadamard type fractional integral inequalities for two-dimensional h-Godunova-Levin convex functions through the application of two-dimensional fractional integral operators. The outcomes are further elaborated via corollaries, offering detailed insights into the behavior of these inequalities under the stated convexity conditions.

Theorem 3.1. *Let $h:(0,1)\to\mathbb{R}$ be positive function, $h(\theta)\neq 0$, and $\alpha:[\beta_1,\beta_2]\times[k_3,k_4]\to\mathbb{R}$ be (coordinated) two-dimensional h-Godunova-Levin function for $0<\beta_1<\beta_2$, $0<k_3<k_4$, $\alpha\in L_1[\beta_1,\beta_2],L_1^*[k_3,k_4]$, then the following inequality holds*

$$\frac{h(\frac{1}{2})}{2}\alpha\left[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\right]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^{\eta}(k_4-k_3)^{\eta}}\left[\mathrm{J}^{\eta}_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^{\eta}_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\right]$$

$$\leq\eta\frac{\alpha\Big[\beta_1+\beta_2,k_3+k_4\Big]}{2}\int_0^1\int_0^1\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]\theta^{\eta-1}r^{\eta-1}d\theta dr.$$

Proof. Consider the definition of two-dimensional h-Godunova-Levin convex function

$$\alpha\big[\theta x+(1-\theta)z,\theta y+(1-\theta)w\big]\leq\frac{\alpha(x,y)}{h(\theta)}+\frac{\alpha(z,w)}{h(1-\theta)}. \tag{3.1}$$

Putting the values $x=\theta\beta_1+(1-\theta)\beta_2, y=rk_3+(1-r)k_4, z=(1-\theta)\beta_1+\theta\beta_2, w=(1-r)k_3+rk_4\delta=\frac{1}{2}$ in equation (3.1), we have

$$h(\frac{1}{2})\alpha\left[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\right]\leq\alpha\big[\theta\beta_1+(1-\theta)\beta_2,rk_3+(1-r)k_4\big]$$

$$+\alpha[(1-\theta)\beta_1+\theta\beta_2,(1-r)k_3+rk_4]. \tag{3.2}$$

After multiplication both sides of the equation (3.2) by $\theta^{\eta-1}r^{\eta-1}$ and then integrate with respect to θ, and r over the interval $[0,1]$, we obtain

$$\begin{aligned} &h(\frac{1}{2})\alpha\big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\big]\int_0^1\int_0^1\theta^{\eta-1}r^{\eta-1}d\theta dr\\ &\le\int_0^1\int_0^1\alpha\big[\theta\beta_1+(1-\theta)\beta_2,rk_3+(1-r)k_4\big]\theta^{\eta-1}r^{\eta-1}d\theta dr\\ &+\int_0^1\int_0^1\alpha\big[(1-\theta)\beta_1+\theta\beta_2,(1-r)k_3+rk_4\big]\theta^{\eta-1}r^{\eta-1}d\theta dr. \end{aligned}\tag{3.3}$$

By putting the values $\theta\beta_1+(1-\theta)\beta_2=t, rk_3+(1-r)k_4=s, (1-\theta)\beta_1+\theta\beta_2=v, (1-r)k_3+rk_4=w$ in the equation (3.3), we obtain

$$\begin{aligned} &h(\frac{1}{2})\alpha\left[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\right]\frac{1}{\eta^2}\\ &\le\int_{\beta_2}^{\beta_1}\int_{k_4}^{k_3}\alpha[t,s](\frac{\beta_2-t}{\beta_2-\beta_1})^{\eta-1}(\frac{k_4-s}{k_4-k_3})^{\eta-1}(-\frac{dt}{\beta_2-\beta_1})(-\frac{ds}{k_4-k_3})\\ &+\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}\alpha[v,w](\frac{v-\beta_1}{\beta_2-\beta_1})^{\eta-1}(\frac{w-k_3}{k_4-k_3})^{\eta-1}(\frac{dv}{\beta_2-\beta_1})(\frac{dw}{k_4-k_3})\\ &\le\frac{1}{(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}\alpha[t,s](\beta_2-t)^{\eta-1}(k_4-s)^{\eta-1}dtds\\ &+\frac{1}{(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}\alpha[v,w](v-\beta_1)^{\eta-1}(w-k_3)^{\eta-1}dvdw, \end{aligned}$$

and so,

$$\begin{aligned} \frac{h(\frac{1}{2})}{2}\alpha\left[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\right]\le{}&\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\\ &\times\left[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\right]. \end{aligned}\tag{3.4}$$

For the second part of inequality, we take the two- dimensional h-Godunova-Levin convex function

$$\alpha\big[\theta x+(1-\theta)z,\theta y+(1-\theta)w\big]\le\frac{\alpha(x,y)}{h(\theta)}+\frac{\alpha(z,w)}{h(1-\theta)},\tag{3.5}$$

and

$$\alpha\big[(1-\theta)x+\theta z,(1-\theta)y+\theta w\big]\le\frac{\alpha(x,y)}{h(1-\theta)}+\frac{\alpha(z,w)}{h(\theta)}.\tag{3.6}$$

Adding the equations (3.5) and (3.6), then putting the values $x=\theta\beta_1+(1-\theta)\beta_2, y=rk_3+(1-r)k_4, z=(1-\theta)\beta_1+\theta\beta_2, w=(1-r)k_3+rk_4$, we have

$$\begin{aligned} &\alpha\big[\theta\beta_1+(1-\theta)\beta_2,rk_3+(1-r)k_4\big]+\alpha\big[(1-\theta)\beta_1+\theta\beta_2,(1-r)k_3+rk_4\big]\\ &\le\alpha[\beta_1+\beta_2,k_3+k_4]\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]. \end{aligned}$$

Multiplying both sides by $\theta^{\eta-1}r^{\eta-1}$ and integrating with respect to θ, and r on interval $[0,1]$, we obtain

$$\int_0^1\int_0^1 \alpha\Big[\theta\beta_1+(1-\theta)\beta_2, rk_3+(1-r)k_4\Big]\theta^{\eta-1}r^{\eta-1}d\theta dr$$

$$+\int_0^1\int_0^1 \alpha\Big[(1-\theta)\beta_1+\theta\beta_2, (1-r)k_3+rk_4\Big]\theta^{\eta-1}r^{\eta-1}d\theta dr$$

$$\leq \alpha[\beta_1+\beta_2, k_3+k_4]\int_0^1\int_0^1\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]\theta^{\eta-1}r^{\eta-1}d\theta dr,$$

and

$$\frac{\gamma(\eta+1)}{(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\Big]$$

$$\leq \eta\alpha[\beta_1+\beta_2, k_3+k_4]\times\int_0^1\int_0^1\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]\theta^{\eta-1}r^{\eta-1}d\theta dr. \tag{3.7}$$

By combining equation (3.4) and (3.7), we reached the required inequality. □

Corollary 3.1. *Using parametric conditions of the remark 2.7 in the assumption of theorem 3.1, then we have inequality which proved in [20].*

Corollary 3.2. *By suitable value of $h(\theta)=\theta^s$ in theorem [3.1], we have Hermite-Hadamard fractional integral inequality for two-dimensional s-Godunova-Levin function holds*

$$\frac{(\frac{1}{2})^s}{2}\alpha\Big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\Big]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}$$

$$\times\big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\big]$$

$$\leq\eta\alpha[\beta_1+\beta_2,k_3+k_4]\int_0^1\int_0^1\left[\frac{1}{(\theta)^s}+\frac{1}{(1-\theta)^s}\right]\theta^{\eta-1}r^{\eta-1}d\theta dr.$$

Corollary 3.3. *Using parametric conditions of the remark 2.7 and $h(\theta)=\theta^s$ in the assumption of theorem 3.1, then we have the inequality which was proved in [20].*

Corollary 3.4. *By replacing the value of $h(\theta)=1$ in the theorem [3.1], we obtain the following Hermite-Hadamard type integral inequality for two-dimensional P-function holds*

$$\frac{1}{2}\alpha\Big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\Big]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\Big]$$

$$\leq\frac{1}{\eta}\alpha[\beta_1+\beta_2,k_3+k_4].$$

Corollary 3.5. *Using parametric conditions of the remark 2.7 and $h(\theta)=1$ in the assumption of theorem 3.1, then we have the inequality which was proved in [20].*

Corollary 3.6. *By replacing the value $h(\theta)=\dfrac{1}{\theta}$ in theorem [3.1], we have the following inequality*

$$\alpha\Big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\Big]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\Big]$$

$$\leq\frac{1}{\eta}\alpha\Big[\frac{\beta_1+\beta_2,k_3+k_4}{2}\Big].$$

Corollary 3.7. *Using parametric conditions of the remark 2.7 and $h(\theta)=\frac{1}{\theta}$ in the assumption of theorem 3.1, then we have the inequality which was proved in [20].*

Corollary 3.8. *Applying theorem [3.1] for $h(\theta)=\theta$, we obtain the following inequality.*

$$\frac{1}{4}\alpha\Big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\Big]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\Big]$$
$$\leq\eta^2\alpha\Big[\frac{\beta_1+\beta_2,k_3+k_4}{2}\Big]\int_0^1\int_0^1\frac{\theta^{\eta-2}}{1-\theta}r^{\eta-1}d\theta dr.$$

Corollary 3.9. *Using parametric conditions of the remark 2.7 and $h(\theta)=\theta$ in the assumption of theorem 3.1, then we have the inequality which was proved in [20].*

Corollary 3.10.
1. *If we put the suitable value of $h(\theta)=\dfrac{1}{\theta^s}$ in [3.1], we obtain the following inequality holds for s-convex function*

$$2^{s-1}\alpha\Big[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\Big]\leq\frac{\gamma(\eta+1)}{2(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^\eta_{(\beta_1,k_3)^+}\alpha(\beta_2,k_4)+\mathrm{J}^\eta_{(\beta_2,k_4)^-}\alpha(\beta_1,k_3)\Big]$$
$$\leq\eta^2\alpha\Big[\frac{\beta_1+\beta_2,k_3+k_4}{2}\Big]\int_0^1\int_0^1\Big[\theta^s+(1-\theta)^s\Big]\theta^{\eta-1}r^{\eta-1}d\theta dr.$$

2. *if we choosing $\eta=1$ in the theorem [3.1], we obtain*

$$2^{s-1}\alpha\left[\frac{\beta_1+\beta_2}{2},\frac{k_3+k_4}{2}\right]\leq\frac{1}{(\beta_2-\beta_1)(k_4-k_3)}\int_{\beta_1}^{\beta_2}\int_{k_3}^{k_4}\alpha(\theta_1,\theta_2)d\theta_1d\theta_2$$
$$\leq\alpha\left(\frac{\beta_1+\beta_2,k_3+k_4}{s+1}\right).$$

Corollary 3.11. *Using parametric conditions of the remark 2.7 and $h(\theta)=\frac{1}{\theta^s}$, $\eta=1$ in the assumption of theorem 3.1, then we have the inequality which was proved in [20].*

4. Modification of Trapezoid type inequalities for coordinated h-Godunova-Levin pre-invex function

In this section, we develop the lemma for coordinated h-Godunova-Levin pre-invex function, and discuss modified version of Trapezoid type integral inequalities by using the lemma. Moreover, we discuss some other inequalities in the form of corollaries.

Lemma 4.1. *Let $\alpha:[\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)]\to\mathbb{R}^2$ with $\beta_1,\beta_2\in\mathbb{R}$, $k_3,k_4\in\mathbb{R}$, $\alpha\in L_1[\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)]$ be a partial differentiable function where $\mathrm{J}=[\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)]$ be an open-invex set with respect to $\mathrm{J}_1:\mathrm{J}\times\mathrm{J}\to\mathbb{R}$, $\mathrm{J}_2:\mathrm{J}\times\mathrm{J}\to\mathbb{R}$ with $\mathrm{J}_1(\beta_2,\beta_1)>0$, $\mathrm{J}_2(k_4,k_3)>0$,* *then the following result hold*

$$\begin{aligned}
&\frac{\alpha\big(\beta_1, k_3) + \alpha(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)}{2} \\
&- \frac{\eta}{(\beta_2-\beta_1)^\eta} \int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)} (t-\beta_1)^{\eta-1}\alpha(t,k_4)dt \\
&- \frac{\eta}{(\beta_2-\beta_1)^\eta} \int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)} (\beta_2-t)^{\eta-1}\alpha(t,k_3)dt \\
&- \frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^\eta \mathrm{J}_2(k_4,k_3)^\eta}\Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)^+}\alpha\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1), k_3+\mathrm{J}_2(k_4,k_3)\big) \\
&+ \mathrm{J}^{\eta+1,\eta}_{\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)^-}\alpha(\beta_1,k_3)\Big] \\
&= \frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\int_0^1\int_0^1 \big[\theta^\eta r^\eta - (1-\theta)^\eta(1-r)^\eta\big] \\
&\times \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr.
\end{aligned}$$

Proof. Consider the integral

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 \big[\theta^\eta r^\eta - (1-\theta)^\eta(1-r)^\eta\big]\frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr \\
&= \int_0^1\int_0^1 \theta^\eta r^\eta \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr \\
&- \int_0^1\int_0^1 (1-\theta)^\eta(1-r)^\eta \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr. \qquad (4.1)
\end{aligned}$$

Put the value of first and second integrals are I_2 and I_3 respectively, in the equation (4.1), and then consider I_2, we have

$$\begin{aligned}
I_2 &= \int_0^1\int_0^1 \theta^\eta r^\eta \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr \\
&= \int_0^1 \theta^\eta\Big\{\int_0^1 r^\eta \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)dr\Big\}d\theta. \qquad (4.2)
\end{aligned}$$

Now, we put the values of integral with respect to r as the integral I_4 in equation (4.2), and then consider it as follows

$$I_4 = \int_0^1 r^\eta \frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1), k_3+r\mathrm{J}_2(k_4,k_3)\big)dr.$$

So,

$$\begin{aligned}
I_4 &= \Bigg[r^{\eta} \frac{\partial \alpha}{\partial \theta} \frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} \Bigg|_0^1 \\
&\quad - \eta \int_0^1 r^{\eta-1} \frac{\partial \alpha}{\partial \theta} \frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} dr \Bigg] \\
&= \Bigg[\frac{\partial \alpha}{\partial \theta} \frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} \\
&\quad - \eta \int_0^1 r^{\eta-1} \frac{\partial \alpha}{\partial \theta} \frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} dr \Bigg]. \qquad (4.3)
\end{aligned}$$

Using the equation (4.3) in equation (4.2), then the integral I_2 becomes

$$\begin{aligned}
I_2 = \frac{1}{\mathrm{J}_2(k_4,k_3)} \Bigg[& \int_0^1 \theta^{\eta} \frac{\partial \alpha}{\partial \theta} \big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big) d\theta \\
& - \eta \int_0^1 \int_0^1 \theta^{\eta} r^{\eta-1} \frac{\partial \alpha}{\partial \theta} \big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big) d\theta dr \Bigg]. \qquad (4.4)
\end{aligned}$$

Putting the values of the integrals I_5 and I_6 in the right side of the equation (4.4) respectively, then consider the integral I_5

$$I_5 = \frac{1}{\mathrm{J}_1(\beta_2,\beta_1)} \Bigg[\alpha\big(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big) - \eta \int_0^1 \theta^{\eta-1} \alpha\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_4\big) d\theta \Bigg]. \qquad (4.5)$$

Putting the value of second integral I_7 in the right side of the equation (4.5), and consider the simplification

$$\begin{aligned}
I_7 &= \eta \int_0^1 \theta^{\eta-1} \alpha\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_4\big) d\theta \\
&= \frac{\eta}{(\beta_2 - \beta_1)^{\eta}} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2,\beta_1)} (t - \beta_1)^{\eta-1} \alpha(t, k_4) dt. \qquad (4.6)
\end{aligned}$$

Now, consider I_6. So, we have

$$I_6 = \eta \int_0^1 \int_0^1 \theta^{\eta} r^{\eta-1} \frac{\partial \alpha}{\partial \theta} \big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big) d\theta dr. \qquad (4.7)$$

After simplification of the equation (4.7), we obtain

$$\text{L.H.S.} = \frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^{\eta} \mathrm{J}_2(k_4,k_3)^{\eta} \mathrm{J}_1(\beta_2,\beta_1)} \Bigg[\mathrm{J}^{\eta+1,\eta}_{\left(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\right)^-} \alpha(\beta_1, k_3) \Bigg]. \qquad (4.8)$$

Rearranging the values of the equations (4.5), (4.6) and (4.8), then the equation (4.2) becomes

$$\begin{aligned}
I_2 = \frac{1}{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)} \Bigg[& \Bigg\{ \alpha\big(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big) \\
& - \frac{\eta}{(\beta_2 - \beta_1)^{\eta}} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2,\beta_1)} (t - \beta_1)^{\eta-1} \alpha(t, k_4) dt \Bigg\} \\
& - \frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^{\eta} \mathrm{J}_2(k_4,k_3)^{\eta}} \Bigg\{ \mathrm{J}^{\eta+1,\eta}_{\left(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\right)^-} \alpha(\beta_1, k_3) \Bigg\} \Bigg]. \qquad (4.9)
\end{aligned}$$

Now, consider the integral I_3 from equation (4.1), we have

$$\begin{aligned} I_3 &= \int_0^1 \int_0^1 (1-\theta)^\eta (1-r)^\eta \frac{\partial^2 \alpha}{\partial\theta\partial r}\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big) d\theta dr \\ &= \int_0^1 (1-\theta)^\eta \Big\{ \int_0^1 (1-r)^\eta \frac{\partial^2 \alpha}{\partial\theta\partial r}\Big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\Big) dr \Big\} d\theta. \end{aligned} \tag{4.10}$$

Put the integral I_8 in the right hand side of the equation , we have

$$\begin{aligned} I_8 &= \int_0^1 (1-r)^\eta \frac{\partial^2 \alpha}{\partial\theta\partial r}\Big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\Big) dr \\ &= -\Big[\frac{\partial\alpha}{\partial\theta}\frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3\big)}{\mathrm{J}_2(k_4,k_3)}\Big] - \eta\int_0^1 (1-r)^{\eta-1}\frac{\partial\alpha}{\partial\theta}\frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} dr \\ &= -\int_0^1 (1-\theta)^\eta \frac{\partial\alpha}{\partial\theta}\frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3\big) d\theta}{\mathrm{J}_2(k_4,k_3)} \\ &- \eta\int_0^1\int_0^1 (1-\theta)^\eta (1-r)^\eta \frac{\partial\alpha}{\partial\theta}\frac{\big(\beta_1 + \theta \mathrm{J}_1(\beta_2,\beta_1), k_3 + r\mathrm{J}_2(k_4,k_3)\big)}{\mathrm{J}_2(k_4,k_3)} d\theta dr. \end{aligned} \tag{4.11}$$

Putting the values of the integrals I_9 and I_{10} in the right hand side of the equation (4.11), and then consider I_9 and after simplification, we have

$$I_9 = \frac{1}{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}\Big[-\big\{\alpha(\beta_1,k_3)\big\} + \eta\int_0^1 (1-\theta)^{\eta-1}\alpha\big(\beta_1 + \theta\mathrm{J}_1(\beta_2,\beta_1), k_3\big) d\theta\Big]. \tag{4.12}$$

Putting the value of the I_{11} i the right side of the integral, we have

$$\begin{aligned} I_{11} &= \eta\int_0^1 (1-\theta)^{\eta-1}\alpha\big(\beta_1 + \theta\mathrm{J}_1(\beta_2,\beta_1), k_3\big) d\theta \\ &= \frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)} (\beta_2 - t)^{\eta-1}\alpha(t,k_3) dt. \end{aligned} \tag{4.13}$$

Now, consider I_{10} and after simplification, we have

$$\text{L.H.S.} = \frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^\eta \mathrm{J}_2(k_4,k_3)^\eta \mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}\Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)+}\alpha\big(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big)\Big]. \tag{4.14}$$

By combining (4.12), (4.13), (4.14), (4.11) in equation (4.10), we have

$$\begin{aligned} I_3 &= \frac{1}{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}\Big[\Big\{-\big\{\alpha(\beta_1,k_3)\big\} + \frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)} (\beta_2 - t)^{\eta-1}\alpha(t,k_3) dt\Big\} \\ &- \frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^\eta \mathrm{J}_2(k_4,k_3)^\eta}\Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)^+}\alpha\big(\beta_1 + \mathrm{J}_1(\beta_2,\beta_1), k_3 + \mathrm{J}_2(k_4,k_3)\big)\Big]\Big]. \end{aligned} \tag{4.15}$$

By combining (4.9) and (4.15) in equation in (4.1), we get the required result. □

Theorem 4.1. *Let α be a function from $\mathrm{J} \subset \mathbb{R}^2$ into $(0,\infty)$, where $\mathrm{J} = [\beta_1 + \mathrm{J}1(\beta 2, \beta_1), , k_3 + \mathrm{J}_2(k_4, k_3)]$. Suppose α is partially differentiable on J, and that $|\frac{\partial^2 \alpha}{\partial\theta\partial r}|$ is a two-dimensional h-Godunova-Levin pre-invex function. Then the function satisfies the following integral inequality.*

$$\Bigg|\frac{\alpha\big(\beta_1, k_3) + \alpha(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)}{2}$$

$$- \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (t - \beta_1)^{\eta - 1} \alpha(t, k_4) dt$$

$$- \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (\beta_2 - t)^{\eta - 1} \alpha(t, k_3) dt$$

$$- \frac{\gamma(\eta + 1)}{\mathrm{J}_1(\beta_2, \beta_1)^\eta \mathrm{J}_2(k_4, k_3)^\eta} \Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1, k_3)+} \alpha\big(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)$$

$$+ \mathrm{J}^{\eta+1,\eta}_{\big(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)^-} \alpha(\beta_1, k_3)\Big]\Bigg|$$

$$\leq \frac{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)}{2} \left[\left|\frac{\partial^2 \alpha(\beta_1, k_3)}{\partial\theta\partial r}\right| + \left|\frac{\partial^2 \alpha(\beta_1, k_4)}{\partial\theta\partial r}\right| + \left|\frac{\partial^2 \alpha(\beta_2, k_3)}{\partial\theta\partial r}\right| + \left|\frac{\partial^2 \alpha(\beta_2, k_4)}{\partial\theta\partial r}\right|\right]$$

$$\times \int_0^1 \int_0^1 \left|\theta^\eta r^\eta - (1 - \theta)^\eta (1 - r)^\eta\right| \frac{1}{h(\theta r)} d\theta dr.$$

Proof. Considering the lemma given in [4.1], we have

$$\Bigg|\frac{\alpha\big(\beta_1, k_3) + \alpha(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)}{2} - \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (t - \beta_1)^{\eta - 1} \alpha(t, k_4) dt$$

$$- \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (\beta_2 - t)^{\eta - 1} \alpha(t, k_3) dt - \frac{\gamma(\eta + 1)}{\mathrm{J}_1(\beta_2, \beta_1)^\eta \mathrm{J}_2(k_4, k_3)^\eta}$$

$$\times \Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1, k_3)+} \alpha\big(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big) + \mathrm{J}^{\eta+1,\eta}_{\big(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)^-} \alpha(\beta_1, k_3)\Big]\Bigg|$$

$$= \Bigg|\frac{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left[\theta^\eta r^\eta - (1 - \theta)^\eta (1 - r)^\eta\right] \frac{\partial^2 \alpha}{\partial\theta\partial r}$$

$$\times \big(\beta_1 + \theta \mathrm{J}_1(\beta_2, \beta_1), k_3 + r\mathrm{J}_2(k_4, k_3)\big) d\theta dr\Bigg|$$

$$\leq \frac{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left|\theta^\eta r^\eta - (1 - \theta)^\eta (1 - r)^\eta\right| \Bigg|\frac{\partial^2 \alpha}{\partial\theta\partial r}$$

$$\times \big(\beta_1 + \theta \mathrm{J}_1(\beta_2, \beta_1), k_3 + r\mathrm{J}_2(k_4, k_3)\big)\Bigg| d\theta dr. \tag{4.16}$$

By utilizing the definition of the h-Godunova-Levin coordinated pre-invex function given in equation

(4.16), we obtain:

$$\begin{aligned}
\text{L.H.S.} \leq{}& \frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\int_0^1\int_0^1\Big|\theta^\eta r^\eta-(1-\theta)^\eta(1-r)^\eta\Big| \\
&\times\left|\frac{\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}}{h[(1-\theta)(1-r)]}+\frac{\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}}{h[(1-\theta)(r)]}+\frac{\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}}{h[(\theta)(1-r)]}+\frac{\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}}{h[(\theta)(r)]}\right|d\theta dr \\
\leq{}& \frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\int_0^1\int_0^1\Big|\theta^\eta r^\eta-(1-\theta)^\eta(1-r)^\eta\Big| \\
&\times\left[\frac{\left|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\right|}{h[(1-\theta)(1-r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\right|}{h[(1-\theta)(r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\right|}{h[(\theta)(1-r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\right|}{h[(\theta)(r)]}\right]d\theta dr \\
={}& \frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\left[\left|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\right|\right] \\
&\times\int_0^1\int_0^1\Big|\theta^\eta r^\eta-(1-\theta)^\eta(1-r)^\eta\Big|\frac{1}{h(\theta r)}d\theta dr,
\end{aligned}$$

which are the required result. □

Corollary 4.1. *If we replace $\eta=1$ in theorem [4.1], we have the following inequality*

$$\begin{aligned}
&\Bigg|\frac{\alpha\big(\beta_1,k_3)+\alpha(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)}{2}-\frac{1}{\mathrm{J}(\beta_2,\beta_1)}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}\alpha(t,k_4)dt \\
&-\frac{k_4-k_3}{\mathrm{J}_1(\beta_2,\beta_1)^2\mathrm{J}_2(k_4,k_3)^2}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(t-\beta_1)\alpha(t)dt \\
&-\frac{1}{\mathrm{J}(\beta_2,\beta_1)}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}\alpha(t,k_3)dt-\frac{k_4-k_3}{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(\beta_2-t)\alpha(t)dt\Bigg| \\
&\leq\frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\left[\left|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\right|+\left|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\right|\right] \\
&\times\int_0^1\int_0^1\big|\theta r-(1-\theta)(1-r)\big|\frac{1}{h(\theta r)}d\theta dr.
\end{aligned}$$

Corollary 4.2. *If we replace $\mathrm{J}_1(\beta_2,\beta_1)=\beta_2-\beta_1$ and $\mathrm{J}_2(k_4,k_3)=k_4-k_3$ in theorem [4.1] then we* *obtain the following inequality*

$$
\begin{aligned}
&\Bigg|\frac{\alpha\big(\beta_1,k_3\big)+\alpha(\beta_2,k_4\big)}{2}-\frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_2}(t-\beta_1)^{\eta-1}\alpha(t,k_4)dt\\
&-\frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_2}(\beta_2-t)^{\eta-1}\alpha(t,k_3)dt\\
&-\frac{\gamma(\eta+1)}{(\beta_2-\beta_1)^\eta(k_4-k_3)^\eta}\Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)^+}\alpha\big(\beta_2,k_4\big)+\mathrm{J}^{\eta+1,\eta}_{\big(\beta_2,k_4\big)^-}\alpha(\beta_1,k_3)\Big]\Bigg|\\
&\le\frac{(\beta_2-\beta_1)(k_4-k_3)}{2}\Bigg[\bigg|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\bigg|+\bigg|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\bigg|+\bigg|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\bigg|+\bigg|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\bigg|\Bigg]\\
&\times\int_0^1\int_0^1\Big|\theta^\eta r^\eta-(1-\theta)^\eta(1-r)^\eta\Big|\frac{1}{h(\theta r)}d\theta dr.
\end{aligned}
$$

Theorem 4.2. *Let* $\alpha:\mathrm{J}=[\beta_1+\mathrm{J}1(\beta2,\beta_1),,k_3+\mathrm{J}_2(k_4,k_3)]\to(0,\infty)$ *be a real-valued function that is partially differentiable on the domain* $\mathrm{J}\subset\mathbb{R}^2$*. Suppose that* $|\frac{\partial^2\alpha}{\partial\theta,\partial r}|^q$ *is a two-dimensional (coordinated) h-Godunova-Levin pre-invex function on* J*, where* $p>1$ *and* $q=\frac{p}{p-1}$*. Then, the following inequality is satisfied.*

$$
\begin{aligned}
&\Bigg|\frac{\alpha\big(\beta_1,k_3)+\alpha(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)}{2}-\frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(t-\beta_1)^{\eta-1}\alpha(t,k_4)dt\\
&-\frac{\eta}{(\beta_2-\beta_1)^\eta}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(\beta_2-t)^{\eta-1}\alpha(t,k_3)dt-\frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^\eta\mathrm{J}_2(k_4,k_3)^\eta}\\
&\times\Big[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)^+}\alpha\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)+\mathrm{J}^{\eta+1,\eta}_{\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)^-}\alpha(\beta_1,k_3)\Big]\Bigg|\\
&\le\frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\Big[\big|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\big|^q+\big|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\big|^q+\big|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\big|^q+\big|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\big|^q\Big]^{\frac{1}{q}}\\
&\times\Big(\int_0^1\int_0^1\big|\theta^\eta r^\eta-(1-\theta)^\eta(1-r)^\eta\big|^p d\theta dr\Big)^{\frac{1}{p}}\Big(\frac{1}{h(\theta r)}d\theta dr\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Consider the lemma [4.1], we have

$$\begin{aligned}
&\Bigg|\frac{\alpha\big(\beta_1,k_3\big)+\alpha\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)}{2}-\frac{\eta}{(\beta_2-\beta_1)^{\eta}}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(t-\beta_1)^{\eta-1}\alpha(t)dt\\
&-\frac{\eta}{(\beta_2-\beta_1)^{\eta}}\int_{\beta_1}^{\beta_1+\mathrm{J}(\beta_2,\beta_1)}(\beta_2-t)^{\eta-1}\alpha(t)dt-\frac{\gamma(\eta+1)}{\mathrm{J}_1(\beta_2,\beta_1)^{\eta}\mathrm{J}_2(k_4,k_3)^{\eta}}\\
&\times\left[\mathrm{J}^{\eta+1,\eta}_{(\beta_1,k_3)^+}\alpha\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)+\mathrm{J}^{\eta+1,\eta}_{\big(\beta_1+\mathrm{J}_1(\beta_2,\beta_1),k_3+\mathrm{J}_2(k_4,k_3)\big)^-}\alpha(\beta_1,k_3)\right]\Bigg|\\
&=\Bigg|\frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}\int_0^1\int_0^1\big[\theta^{\eta}r^{\eta}-(1-\theta)^{\eta}(1-r)^{\eta}\big]\\
&\times\frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1),k_3+r\mathrm{J}_2(k_4,k_3)\big)d\theta dr\Bigg|. \qquad (4.17)
\end{aligned}$$

After simplification the equation (4.17), we obtain

$$\begin{aligned}
\text{L.H.S.}\le\frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}&\int_0^1\int_0^1\Big|\theta^{\eta}r^{\eta}-(1-\theta)^{\eta}(1-r)^{\eta}\Big|\\
&\times\left|\frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1),k_3+r\mathrm{J}_2(k_4,k_3)\big)\right|d\theta dr.
\end{aligned}$$

Applying the *Holder's* inequality, we obtain

$$\begin{aligned}
\text{L.H.S.}\le\frac{\mathrm{J}_1(\beta_2,\beta_1)\mathrm{J}_2(k_4,k_3)}{2}&\left(\int_0^1\int_0^1\big|\theta^{\eta}r^{\eta}-(1-\theta)^{\eta}(1-r)^{\eta}\big|^p d\theta dr\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1),k_3+r\mathrm{J}_2(k_4,k_3)\big)\right|^q d\theta dr\right)^{\frac{1}{q}}. \qquad (4.18)
\end{aligned}$$

Consider the second integral of right hand side of the equation (4.18), and applying the definition of h-Goduvova-Levin coordinated pre-invex function, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2\alpha}{\partial\theta\partial r}\big(\beta_1+\theta\mathrm{J}_1(\beta_2,\beta_1),k_3+r\mathrm{J}_2(k_4,k_3)\big)\right|^q d\theta dr\\
&\le\int_0^1\int_0^1\left[\frac{\left|\frac{\partial^2\alpha(\beta_1,k_3)}{\partial\theta\partial r}\right|^q}{h[(1-\theta)(1-r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_1,k_4)}{\partial\theta\partial r}\right|^q}{h[(1-\theta)(r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_2,k_3)}{\partial\theta\partial r}\right|^q}{h[(\theta)(1-r)]}+\frac{\left|\frac{\partial^2\alpha(\beta_2,k_4)}{\partial\theta\partial r}\right|^q}{h[(\theta)(r)]}\right]d\theta dr\Bigg]\\
&\le\left[\left|\frac{\partial^2\alpha}{\partial\theta\partial r}(\beta_1,k_3)\right|^q+\left|\frac{\partial^2\alpha}{\partial\theta\partial r}(\beta_1,k_4)\right|^q+\left|\frac{\partial^2\alpha}{\partial\theta\partial r}(\beta_2,k_3)\right|^q+\left|\frac{\partial^2\alpha}{\partial\theta\partial r}(\beta_2,k_4)\right|^q\right]\\
&\times\int_0^1\int_0^1\frac{1}{h[(\theta)(r)]}d\theta dr. \qquad (4.19)
\end{aligned}$$

Using equation (4.19) in equation (4.18), we get the required result. □

Corollary 4.3. *If we replace $\eta = 1$ in theorem [4.2], then we have the following inequality*

$$\Bigg|\frac{\alpha\big(\beta_1, k_3) + \alpha(\beta_1 + \mathrm{J}_1(\beta_2, \beta_1), k_3 + \mathrm{J}_2(k_4, k_3)\big)}{2} - \frac{1}{\mathrm{J}(\beta_2, \beta_1)} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} \alpha(t, k_4)dt$$

$$- \frac{k_4 - k_3}{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (t - \beta_1)\alpha(t)dt - \frac{1}{\mathrm{J}(\beta_2, \beta_1)} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} \alpha(t, k_3)dt$$

$$- \frac{k_4 - k_3}{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)} \int_{\beta_1}^{\beta_1 + \mathrm{J}(\beta_2, \beta_1)} (\beta_2 - t)\alpha(t)dt\Bigg|$$

$$\leq \frac{\mathrm{J}_1(\beta_2, \beta_1)\mathrm{J}_2(k_4, k_3)}{2} \Bigg[\Big|\frac{\partial^2 \alpha(\beta_1, k_3)}{\partial\theta\partial r}\Big|^q + \Big|\frac{\partial^2 \alpha(\beta_1, k_4)}{\partial\theta\partial r}\Big|^q + \Big|\frac{\partial^2 \alpha(\beta_2, k_3)}{\partial\theta\partial r}\Big|^q + \Big|\frac{\partial^2 \alpha(\beta_2, k_4)}{\partial\theta\partial r}\Big|^q\Bigg]^{\frac{1}{q}}$$

$$\times \left(\int_0^1 \int_0^1 \big|\theta r - (1 - \theta)(1 - r)\big|^p d\theta dr\right)^{\frac{1}{p}} \left(\frac{1}{h(\theta r)} d\theta dr\right)^{\frac{1}{q}}.$$

Corollary 4.4. *In theorem [4.2], if we replace $\mathrm{J}_1(\beta_2, \beta_1) = \beta_2 - \beta_1$ and $\mathrm{J}_2(k_4, k_3) = k_4 - k_3$, then we obtain the following inequality*

$$\Bigg|\frac{\alpha\big(\beta_1, k_3) + \alpha(\beta_2, k_4\big)}{2} - \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_2} (t - \beta_1)^{\eta - 1} \alpha(t, k_4)dt$$

$$- \frac{\eta}{(\beta_2 - \beta_1)^\eta} \int_{\beta_1}^{\beta_2} (\beta_2 - t)^{\eta - 1} \alpha(t, k_3)dt$$

$$- \frac{\gamma(\eta + 1)}{(\beta_2 - \beta_1)^\eta (k_4 - k_3)^\eta} \left[\mathrm{J}^{\eta+1,\eta}_{(\beta_1, k_3)^+} \alpha\big(\beta_2, k_4\big) + \mathrm{J}^{\eta+1,\eta}_{\left(\beta_2, k_4\right)^-} \alpha(\beta_1, k_3)\right]\Bigg|$$

$$\leq \frac{(\beta_2 - \beta_1)(k_4 - k_3)}{2} \left[\left|\frac{\partial^2 \alpha(\beta_1, k_3)}{\partial\theta\partial r}\right|^q + \left|\frac{\partial^2 \alpha(\beta_1, k_4)}{\partial\theta\partial r}\right|^q + \left|\frac{\partial^2 \alpha(\beta_2, k_3)}{\partial\theta\partial r}\right|^q + \left|\frac{\partial^2 \alpha(\beta_2, k_4)}{\partial\theta\partial r}\right|^q\right]^{\frac{1}{q}}$$

$$\times \left(\int_0^1 \int_0^1 \big|\theta^\eta r^\eta - (1 - \theta)^\eta (1 - r)^\eta\big|^p d\theta dr\right)^{\frac{1}{p}} \left(\frac{1}{h(\theta r)} d\theta dr\right)^{\frac{1}{q}}.$$

5. Conclusion

In this research paper, we discuss the idea for exploring the modification of well-known inequalities for the two-dimensional h-Godunova-Levin convex function and coordinated h-Godunova-Levin pre-invex function. To investigate a generalized version of Hermite-Hadamard and trapezoid-type inequalities by successfully implementation of newly described fractional operators with two-dimensional h-Godunova-Levin convex function and coordinated h-Godunova-Levin pre-invex function, we extract some other inequalities for partial differentiable functions. Due to this, two-dimensional inequalities could be possible by the implementation of double integral operators, and can be extended to higher order of convexity, which will be a great contribution in the field of analysis and resolved many issues in the theory of optimization.

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