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On Ramanujan-Type Congruences for Partition Triples and Quadruples with 7-Cores

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ABSTRACT: Let $B_t(k)$ and $C_t(k)$ denote the number of partitons of 3 and 4-tuples of k with 7-core partition respectively. In this article, we establish some Ramanujan-type congruences modulo 3 for $B_t(k)$ and modulo 2 for $C_t(k)$.

Keywords: Congruence, partition, t-Core.

Contents

1. Introduction

A partition of k is where k broken down into sum of positive integers in a non-increasing order, written as $k = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be expressed as λ.

Represented graphically using a Ferrers-Young diagram, a collection of k boxes arranged in adjusted left rows, where the i-th row contains exactly λ_i boxes. Each dot has a coordinate label (i, j), where i and j refer to rows and columns just like entries in a matrix, respectively.

If you look at the columns instead of the rows, the number of boxes in column j is denoted λ'_j. The sequence $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is called the conjugate partition of λ.

Now, attached to each box at position (i, j) is something called its hook number, written as $H(i, j)$. This counts the box itself, plus all the boxes directly below it in the same column and all the boxes directly to its right in the same row. A handy formula for this is given by $H(i, j) = \lambda_i + \lambda'_j - i - j + 1$.

Finally, following the convention in [1], we say a partition is a t-core if none of its hook numbers is divisible by t. In other words, no hook length is a product of t.

Example. Consider the partition $\lambda = (4, 2, 1)$ of $k = 7$. Its Ferrers-Young diagram is shown below

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Looking at Ferrers-Young diagram, we can compute the hook number for each box. For the box in the top-left corner at position $(1, 1)$, its hook includes itself, the three boxes to its right in the first row, and the two boxes below it in the first column, giving a total of $H(1, 1) = 6$.

If we continue this process for every box $(1, 2)$ has a hook number of 4, $(1, 3)$ has a hook number of 2, $(1, 4)$ has a hook number of 1, $(2, 1)$ has a hook number of 3, $(2, 2)$ has a hook number of 1, $(3, 1)$ has a hook number of 1.

The complete list of hook numbers is therefore $6, 4, 2, 1, 3, 1, 1$. We can see that none of these is a multiple of 5, which means the partition λ is a 5-core.

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In fact, since the largest hook number is 6, this partition is automatically a t-core for any t larger than 6, such as $t \geq 7$.

We denote by $a_t(n)$ the count of all t-core partitions of n. The following generating function identity for $a_t(n)$ is well-established

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1.1}$$

where we follow the conventional q-series notation with $|q| < 1$ [1]:

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^n).$$

The study of arithmetic properties of t-core partitions has attracted considerable attention from researchers in recent years (see for instance, [3,6,7,8]).
Now, let us extend our view from single partitions to collections of them. A *partition k-tuple* is an ordered list of k partitions, written as $(\lambda^1, \lambda^2, \ldots, \lambda^k)$, whose individual sizes sum up to a positive integer k. In other words, if you count up all the boxes from all the Ferrers diagrams in the tuple, the total is k.

To make this concrete, consider three example partitions

$$\lambda_1 = (3,2),$$
$$\lambda_2 = (2,1,1),$$
$$\lambda_3 = (2).$$

The combination (λ_1, λ_2) forms a partition *pair* (a 2-tuple) of $k = 9$, because the total number of boxes is $3+2+2+1+1 = 9$. If we include the third partition, $(\lambda_1, \lambda_2, \lambda_3)$ becomes a partition *triple* (a 3-tuple) of $k = 11$, since we add the final 2 boxes $3+2+2+1+1+2 = 11$.

We can now add a powerful constraint. Suppose we require that *every* partition λ^i in the k-tuple is a t-core, means no hook length in the diagram can be evenly divided by t. A collection that meets this condition is called a *k-tuple of t-cores*. This gives us a structured family where each component individually possesses the special core property.
Throughout this work, $A_7(k)$ represents the 7 core partition of k and $B_t(k)$ represents the number of partition triples of k where all three partitions are t-cores, while $C_t(k)$ denotes the corresponding count for partition quadruples. As direct consequences of (1.1), the generating functions for these sequences are

$$\sum_{k=0}^{\infty} A_7(k)q^k = \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}, \tag{1.2}$$

$$\sum_{k=0}^{\infty} B_t(k)q^k = \frac{(q^t;q^t)_\infty^{3t}}{(q;q)_\infty^3}, \tag{1.3}$$

and

$$\sum_{k=0}^{\infty} C_t(k)q^k = \frac{(q^t;q^t)_\infty^{4t}}{(q;q)_\infty^4}. \tag{1.4}$$

In 1996, Granville and Ono [4] made an elegant discovery using modular forms. They found a surprisingly simple formula for $p_3(n)$

$$p_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1),$$

where $d_{r,3}(n)$ counts how many positive divisors of n leave a remainder of r when divided by 3, and $p_3(n)$ counts the number of 3-core partition.

The function $d_{r,3}(m)$ gives us an elegant way to count positive divisors of m that leave a remainder of r when divided by 3. To find $p_3(n)$, we simply work with the number $3n+1$, then subtract how many of its divisors are congruent to $2 \pmod 3$ from those that are congruent to $1 \pmod 3$.

There is another important function worth discussing $X_t(n)$. This counts bipartitions of n where both partitions happen to be t-cores. In other words, we are counting ordered pairs (λ, μ) of t-core partitions whose combined size adds up to exactly n.
What really caught the attention of mathematicians are the congruence properties hiding in $X_t(n)$. Just as Ramanujan uncovered famous patterns in the ordinary partition function, researchers have found that $X_t(n)$ behaves in surprisingly regular ways modulo certain integers. These findings bridge classical partition theory with deeper arithmetic phenomena, keeping Ramanujan's spirit alive. Lin's work [6] revealed several congruences for $X_3(n)$, which Xia and Yao [12] later expanded upon.
In 2015, Wang [10] discovered many infinite families of arithmetic identities and congruences involving partition triples with 3-cores. Building on this work, Ranganatha [3] proved something quite beautiful about $X_5(n)$—the number of 5-core bipartitions of n that echoes Ramanujan's classical results.

Ranganatha proved that for any positive integer α and $n \geq 0$,

$$X_5\left(5^\alpha n + 5^\alpha - 2\right) \equiv 0 \pmod{5^\alpha}.$$

The idea here is both simple and striking pick any number n, plug it into $5^\alpha(n+1) - 2$, and count the 5-core bipartitions of that result, you will always get a multiple of 5^α. This divisibility holds no matter which α you choose, giving us infinitely many congruences. In this way, Ranganatha's theorem places $X_5(n)$ in the same spirit as the partition function Ramanujan studied, though we are now working in the richer world of 5-core bipartitions.

Majid and Fathima [7] pushed this even further, uncovering deep arithmetic structure in partition tuples. They proved a Ramanujan-type congruence modulo 5^α for $Y_5(n)$, which counts triples of partitions summing to n where each partition in the triple is a 5-core. Discovering such a congruence reveals an unexpected regularity in how $Y_5(n)$ behaves patterns reminiscent of what Ramanujan found for ordinary partitions.

Meanwhile, Mahadeva Naika and Shivaprasada Nayaka [8] made important discoveries about $Z_t(n)$, where

$$\sum_{k=0}^{\infty} Z_t(n) q^n = \frac{(q^t;q^t)_\infty^{4t}}{(q;q)_\infty^4},$$

which counts quadruples t-core partitions of n. Looking at cases where $t = 3, 5, 7, 25$, they showed that these counts often vanish modulo certain small primes along predictable arithmetic progressions. Their findings include:

$$Z_3(16n+14) \equiv 0 \pmod{8},$$
$$Z_5(5n+4) \equiv 0 \pmod{5},$$

and

$$Z_7(7n+6) \equiv 0 \pmod{7}.$$

These results showcase a hidden arithmetic regularity within the seemingly unstructured world of integer partitions. Building upon these developments, the present paper establishes new congruence relations specifically, congruences modulo 3 for $B_t(k)$ and congruences modulo 2 for $C_t(k)$.

2. Preliminaries

Here, we gather a few key identities that serve in establishing our main results.

Lemma 2.1 *[8, Eq. (2.2)] We have for any prime p and positive integer n,*

$$(q;q)_\infty^{p^n} \equiv (q^p;q^p)_\infty^{p^{n-1}} \pmod{p^n}.$$

For convinience we define $f_n = f(-q^n) = (q^n;q^n)_\infty$ for $n \geq 1$.

Lemma 2.2 *[5, Eq. (15)] We have*

$$\frac{1}{f_1^3} = a^2(q^3)\frac{f_9^3}{f_3^{10}} + 3qa(q^3)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}, \tag{2.1}$$

where Borwein's cubic theta function is given by

$$a_n = a(q^n) := \sum_{j,k=-\infty}^{\infty} q^{n.(j^2+jk+k^2)}.$$

Lemma 2.3 *[11, Eq. (2.6)] We have*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

Lemma 2.4 *In [2, P. 460, Entry 3(i)], it is shown that*

$$a^2(q) = 1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}. \tag{2.3}$$

Lemma 2.5 *[9, Eq. (2.6)] We have*

$$a_1 = a_3 + 6q\frac{f_9^3}{f_3}. \tag{2.4}$$

3. Main Results

In this section, we build on the groundwork laid earlier to prove our main results.

Theorem 3.1 *For all $k \geq 0$, we have*

$$B_7(3k+1) \equiv 0 \pmod 3, \tag{3.1}$$
$$B_7(3k+2) \equiv 0 \pmod 3, \tag{3.2}$$
$$B_7(3k) \equiv A_7(k) \pmod 3, \tag{3.3}$$
$$B_7(9k+3) \equiv 0 \pmod 3, \tag{3.4}$$

and

$$B_7(9k+6) \equiv 0 \pmod 3. \tag{3.5}$$

Proof: Setting $t = 7$ in equation (1.3), we have

$$\sum_{k=0}^{\infty} B_7(k)q^k = \frac{f_7^{21}}{f_1^3}.$$

Substituting (2.1) in the above equation, we get

$$\sum_{k=0}^{\infty} B_7(k)q^k = a_3^2\frac{f_7^{21}f_9^3}{f_3^{10}} + 3q\ a_3\frac{f_7^{21}f_9^6}{f_3^{11}} + 9q^2\frac{f_7^{21}f_9^9}{f_3^{12}}. \tag{3.6}$$

Collecting the terms of the form q^{3k+n} (for $n = 1, 2$) from both sides of (3.6), this gives us (3.1) and (3.2). Now, if we instead collect the q^{3k} terms from both sides of (3.6) and replace $q^3 \to q$, we find

$$\sum_{k=0}^{\infty} B_7(3k)q^k \equiv a_1^2 \frac{f_7^7 f_3^3}{f_1^{10}} \pmod 3,$$
$$\equiv a_1^2 \frac{f_7^7}{f_1} \pmod 3. \tag{3.7}$$

Using (2.3) and (1.2) in (3.7), we have

$$\sum_{k=0}^{\infty} B_7(3k)q^k \equiv \sum_{k=0}^{\infty} A_7(k)q^k \pmod 3.$$

Equating the coefficient of q^k, we get (3.3).
Substituting the value of (2.4) in (3.7), we get

$$\sum_{k=0}^{\infty} B_7(3k)q^k \equiv \left(a_3 + 6q\frac{f_9^3}{f_3}\right)^2 \frac{f_7^7}{f_1} \pmod 3.$$

$$\sum_{k=0}^{\infty} B_7(3k)q^k \equiv \left(a_3^2 + 12q\ a_3\frac{f_9^3}{f_3} + 36\ q^2\frac{f_9^6}{f_3^2}\right)^2 \frac{f_7^7}{f_1} \pmod 3. \tag{3.8}$$

collecting the terms of the form q^{3k+n} for $n = 1, 2$ from both sides of (3.8), we get (3.4) and (3.5).

□

Theorem 3.2 *For all $k \geq 0$, we have*

$$C_7(2k+1) \equiv 0 \pmod 2, \tag{3.9}$$
$$C_7(4k+2) \equiv 0 \pmod 2, \tag{3.10}$$

and

$$C_7(4k) \equiv A_7(k) \pmod 2. \tag{3.11}$$

Proof: Setting $t = 7$ in equation (1.4), we have

$$\sum_{k=0}^{\infty} C_7(k)q^k = \frac{f_7^{28}}{f_1^4}. \tag{3.12}$$

Substituting (2.2) in (3.12), we get

$$\sum_{k=0}^{\infty} C_7(k)q^k = \frac{f_7^{28} f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_7^{28} f_4^2 f_8^4}{f_2^{10}}. \tag{3.13}$$

Collecting the terms of the form q^{2k+1} from both sides of (3.13) leads to (3.9).
Similarly, extracting the q^{2k} terms from (3.13) and replacing q^2 by q, we obtain

$$\sum_{k=0}^{\infty} C_7(2k)q^k \equiv \frac{f_7^{14} f_2^{14}}{f_1^{14} f_4^4} \pmod 2. \tag{3.14}$$

Substituting (2.2) in (3.14), we get

$$\sum_{k=0}^{\infty} C_7(2k)q^k \equiv \frac{f_7^{14} f_2^{14}}{f_1^{10} f_4^4}\left[\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right] \pmod{2}. \tag{3.15}$$

Collecting all terms of the form q^{2k+n} where $n = 0, 1$ from both sides of the (3.15), we get, (3.10) and (3.11).

□

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