Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.79415

On Mersenne-Narayana and Mersenne-Narayana-Lucas Sequences

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ABSTRACT: The objective is to generate Mersenne-Narayana and Mersenne-Narayana-Lucas Sequences. Third-order recurrence relations corresponding to Mersenne-Narayana and Mersenne-Narayana-Lucas Sequences are introduced. For the aforementioned sequences, recurrence relations, generating functions, and Binet formulas are then found. These sequences have been verified through some well-known identities. Some identities have been provided using the matrix approach. This paper presents the introduction of two novel sequences, specifically the Mersenne-Narayana and Mersenne-Narayana-Lucas Sequences, accompanied by their respective recurrence relations, Binet formulas and some connected identities.

Key Words: Mersenne-Narayana, sequences, matrix approach.

Contents

1. Introduction

The Fibonacci sequence is the most popular positive integer series. This series, with different starting circumstances, can be used to create numerous fundamental arrangements, including the Lucas, Pell, Padovan and Jacobsthal sequences [1]-[4]. A number of the type $M_n = 2^n - 1$, where n is an integer, was first developed in 1644 by the French mathematician Marin Mersenne. The Mersenne sequences have been investigated as part of numerous studies. The definition of Mersenne-Lucas sequences is $ML_n = 2^n + 1,\ n \geq 2$, with $ML_0 = 2,\ ML_1 = 3$. In [5]-[7] the Mersenne-Lucas sequences, including its generating functions and Binet formulas were discussed.
The usual Mersenne sequence is defined as $M_0 = 0,\ M_1 = 1$ and $M_{n+1} = 3M_n - 2M_{n-1},\ n \geq 1$ and with the initial parameters $M_0 = 2,\ M_1 = 3$, the Mersenne-Lucas sequence satisfies the same recurrence relation.
The Narayana sequence is a third-order one and is defined as $N_0 = 0,\ N_1 = N_2 = 1$ and $N_{n+1} = N_n + N_{n-2}$ for $n \geq 2$
The Binet's formula for the Narayana sequence is given by $N_n = \frac{A\Omega^n + B\mu^n + \epsilon^n}{\Delta}$
where $\Omega = \frac{1+a+b}{3},\ \mu = \frac{1-\frac{1}{2}(a+b)-i\frac{\sqrt{3}}{2}(a-b)}{3},\ \epsilon = \frac{1-\frac{1}{2}(a+b)+i\frac{\sqrt{3}}{2}(a-b)}{3}, a = \sqrt[3]{\frac{29+3\sqrt{93}}{2}}$,
$b = \sqrt[3]{\frac{29-3\sqrt{93}}{2}}, \Delta = (\Omega - \mu)(\Omega - \epsilon)(\mu - \epsilon), A = (\mu - \epsilon)(1 - \mu - \epsilon)$.

The Jacobsthal-Narayana and Jacobstal-Narayana-Lucas sequences were studied in [8], which inspired us to embark on this article. One may refer to [9], [10] to know more on this context.

2. Main Definitions and Results

Definition 2.1 *We define the Mersenne-Narayana sequence MN_n by the recurrence relation $MN_n = 3MN_{n-1} - 2MN_{n-3},\ n \geq 3$ with the initial values $MN_0 = 0,\ MN_1 = MN_2 = 1$.*
The first few values of Mersenne-Narayana sequence are 0, 1, 1, 3, 7, 19, 51, 139, . . .

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2020 *Mathematics Subject Classification:* 11B37, 11B39, 11B83.
Submitted October 08, 2025. Published December 20, 2025

Definition 2.2 *The Mersenne-Narayana-Lucas sequence* MNL_n *is defined by the recurrence relation* $MNL_n = 3MNL_{n-1} - 2MNL_{n-3},\ n \geq 3$ *with the initial values* $MNL_0 = 2,\ MNL_1 = MNL_2 = 3.$ *The first few values of Mersenne-Narayana sequence are 2, 3, 3, 5, 9, 21, 53, 141 ...*

According to our definitions, there is a third order linear homogeneous difference equation, with constant coefficients, in the form

$$x_n = 3x_{n-1} - 2x_{n-3}. \tag{2.1}$$

It can then explore a solution to the above equation as $x_n = \varphi^n$, where φ is an unknown constant. On the substitution of this linear solution into our difference equation, we obtain

$$\varphi^3 = 3\varphi^2 - 2. \tag{2.2}$$

From the cubic formula for the roots, three independent solutions are obtained as follows:

$$\psi = 1 - \lambda_1 - i$$

$$\zeta = 1 + \left(\frac{1 + i\sqrt{3}}{2}\right)\lambda_1 + \left(\frac{\sqrt{3} + i}{2}\right)$$

$$\tau = 1 + \left(\frac{1 - i\sqrt{3}}{2}\right)\lambda_1 + \left(\frac{-\sqrt{3} + i}{2}\right)$$

where $i = \sqrt{-1},\ \lambda_1 = \sqrt[3]{i}$. Furthermore, a linear combination of the solutions in the above equations satisfies equation (2.2). Hence, $MN_n = C_1\psi^n + C_2\zeta^n + C_3\tau^n$, with the initial terms given by

$$C_1 + C_2 + C_3 = 0$$

$$\psi C_1 + \zeta C_2 + \tau C_3 = 1$$

$$\psi^2 C_1 + \zeta^2 C_2 + \tau^2 C_3 = 1$$

and obtain the solution

$$C_1 = \frac{\psi - 2}{(\psi - \zeta)(\psi - \tau)}$$

$$C_2 = \frac{\zeta - 2}{(\zeta - \psi)(\zeta - \tau)}$$

$$C_3 = \frac{\tau - 2}{(\tau - \psi)(\tau - \zeta)}$$

Besides, repeating the same technique for the Mersenne-Narayana-Lucas sequence gives the solution

$$C_1 = \frac{3(\psi - 2) + 2\zeta\tau}{(\psi - \zeta)(\psi - \tau)}$$

$$C_2 = \frac{3(\zeta - 2) + 2\psi\gamma}{(\zeta - \psi)(\zeta - \tau)}$$

$C_3 = \frac{3(\tau-2)+2\psi\zeta}{(\tau-\psi)(\tau-\zeta)}$ respectively.

Theorem 2.1 (Binet's formulas) *Let* n *be any integer. Binet's formula for the Mersenne-Narayana sequence and the Mersenne-Narayana-Lucas sequence are*

$$MN_n = \frac{\psi - 2}{(\psi - \zeta)(\psi - \tau)}\psi^n + \frac{\zeta - 2}{(\zeta - \psi)(\zeta - \tau)}\zeta^n + \frac{\tau - 2}{(\tau - \psi)(\tau - \beta)}\gamma^n$$

and $MNL_n = \frac{3(\psi-2)+2\zeta\tau}{(\psi-\zeta)(\psi-\tau)}\psi^n + \frac{3(\zeta-2)+2\psi\tau}{(\zeta-\psi)(\zeta-\tau)}\zeta^n + \frac{3(\tau-2)+2\psi\zeta}{(\tau-\psi)(\tau-\zeta)}\tau^n$ *respectively.*

Remark 2.1 For ψ, ζ and τ some interesting properties between the roots of the cubic equation $x^3-3x^2+2=0$ are given by

1. $\psi+\zeta+\tau=3$
2. $\psi\zeta\tau=-2$
3. $\psi\zeta+\zeta\tau+\psi\tau=0$
4. $K_1+K_2+K_3=0$
5. $K_1+K_2=\frac{2-\tau}{(\gamma-\psi)(\tau-\zeta)}$
6. $K_1+K_3=\frac{2-\zeta}{(\zeta-\psi)(\zeta-\tau)}$
7. $K_2+K_3=\frac{2-\psi}{(\psi-\zeta)(\psi-\tau)}$

where $K_1=\frac{\psi-2}{(\psi-\zeta)(\psi-\tau)}$, $K_2=\frac{\zeta-2}{(\zeta-\psi)(\zeta-\tau)}$, $K_3=\frac{\tau-2}{(\tau-\psi)(\tau-\zeta)}$.

Theorem 2.2 *The generating functions for the Mersenne-Narayana sequence and the Mersenne-Narayana-Lucas sequence are*
$\sum_{r=0}^{\infty} MN_r x^r=\frac{x(1-2x)}{1-3x+2x^3}$ *and* $\sum_{r=0}^{\infty} MNL_r x^r=\frac{2-3x-6x^2}{1-3x+2x^3}$ *respectively.*

Proof:
Define $g(x)=\sum_{r=0}^{\infty} MN_r x^r$. Then, summing the statements $g(x), -3xg(x)$, $2x^3 g(x)$, with some mathematical manipulations, the proof can be completed.

Theorem 2.3 *Let $r>0$ be an integer and k be an arbitrary integer. Then,*

$$MN_{r+k}+MN_{r-k}=\frac{\left(\psi^{2k}+1\right)(\psi-2)\psi^{r-k}}{(\psi-\zeta)(\psi-\tau)}+\frac{\left(\zeta^{2k}+1\right)(\zeta-2)\zeta^{r-k}}{(\zeta-\psi)(\zeta-\tau)}+\frac{\left(\tau^{2k}+1\right)(\tau-2)\tau^{r-k}}{(\tau-\psi)(\tau-\zeta)}$$

$$MN_{r+k}-MN_{r-k}=\frac{\left(\psi^{2k}-1\right)(\psi-2)\psi^{r-k}}{(\psi-\zeta)(\psi-\tau)}+\frac{\left(\zeta^{2k}-1\right)(\zeta-2)\zeta^{r-k}}{(\zeta-\psi)(\zeta-\tau)}+\frac{\left(\tau^{2k}-1\right)(\tau-2)\tau^{r-k}}{(\tau-\psi)(\tau-\zeta)}$$

$$MNL_{r+k}+MNL_{r-k}=\frac{(3(\psi-2)+2\zeta\tau)\left(\psi^{2k}+1\right)\psi^{r-k}}{(\psi-\zeta)(\psi-\tau)}+\frac{(3(\zeta-2)+2\psi\tau)\left(\zeta^{2k}+1\right)\zeta^{r-k}}{(\zeta-\psi)(\zeta-\tau)}+\frac{(3(\tau-2)+2\psi\zeta)\left(\tau^{2k}+1\right)\tau^{r-k}}{(\tau-\psi)(\tau-\zeta)}$$

$$MNL_{r+k}-MNL_{r-k}=\frac{(3(\psi-2)+2\zeta\tau)\left(\psi^{2k}-1\right)\psi^{r-k}}{(\psi-\zeta)(\psi-\tau)}+\frac{(3(\zeta-2)+2\psi\tau)\left(\zeta^{2k}-1\right)\zeta^{r-k}}{(\zeta-\psi)(\zeta-\tau)}+\frac{(3(\tau-2)+2\psi\zeta)\left(\tau^{2k}-1\right)\tau^{r-k}}{(\tau-\psi)(\tau-\zeta)}$$

In particular, for $k=1$, we get the following cases:

$$MN_{r+1}+MN_{r-1}=\frac{\left(\psi^{2}+1\right)(\psi-2)\psi^{r-1}}{(\psi-\zeta)(\psi-\tau)}+\frac{\left(\zeta^{2}+1\right)(\zeta-2)\beta^{r-1}}{(\zeta-\psi)(\zeta-\tau)}+\frac{\left(\tau^{2}+1\right)(\tau-2)\tau^{r-1}}{(\tau-\psi)(\tau-\zeta)}$$

$$MN_{r+1}-MN_{r-1}=\frac{\left(\psi^{2}-1\right)(\psi-2)\psi^{r-1}}{(\psi-\zeta)(\psi-\tau)}+\frac{\left(\zeta^{2}-1\right)(\zeta-2)\zeta^{r-1}}{(\zeta-\psi)(\zeta-\tau)}+\frac{\left(\tau^{2}-1\right)(\tau-2)\tau^{r-1}}{(\tau-\psi)(\tau-\zeta)}$$

$$MNL_{r+1}+MNL_{r-1}=\frac{(3(\psi-2)+2\zeta\tau)(\psi^2+1)\psi^{r-1}}{(\psi-\zeta)(\psi-\tau)}+\frac{(3(\zeta-2)+2\psi\tau)(\zeta^2+1)\zeta^{r-1}}{(\zeta-\psi)(\zeta-\tau)}+\frac{(3(\tau-2)+2\psi\zeta)(\tau^2+1)\tau^{r-1}}{(\tau-\psi)(\tau-\zeta)}$$

$$MNL_{r+1}-MNL_{r-1}=\frac{(3(\psi-2)+2\zeta\tau)(\psi^2-1)\psi^{r-1}}{(\psi-\zeta)(\psi-\tau)}+\frac{(3(\zeta-2)+2\psi\tau)(\zeta^2-1)\zeta^{r-1}}{(\beta-\psi)(\zeta-\tau)}+\frac{(3(\tau-2)+2\psi\zeta)(\tau^2-1)\tau^{r-1}}{(\tau-\psi)(\tau-\zeta)}.$$

Theorem 2.4 (Vajda identity) *Let n, $r$and s be positive integers. Then, we have*

$$MN_{n+r}MN_{n+s}-MN_nMN_{n+r+s}=\frac{(\psi-2)(\tau-2)}{(\psi-\zeta)(\zeta-\tau)(\psi-\tau)^2}(\psi^r-\tau^r)(\tau^s-\psi^s)(\psi\tau)^n$$
$$+\frac{(\psi-2)(\zeta-2)}{(\psi-\tau)(\zeta-\tau)(\psi-\zeta)^2}(\psi^r-\zeta^r)(\psi^s-\zeta^s)(\psi\zeta)^n$$
$$+\frac{(\zeta-2)(\tau-2)}{(\psi-\zeta)(\psi-\tau)(\zeta-\tau)^2}(\zeta^r-\tau^r)(\zeta^s-\tau^s)(\zeta\tau)^n$$

For $r=-s$, the Catalan's identity is obtained:

$$MN_{n-s}MN_{n+s}-MN_n^2=\frac{(\psi-2)(\tau-2)}{(\psi-\zeta)(\zeta-\tau)(\psi-\tau)^2}(\tau^s-\psi^s)^2(\psi\tau)^{n-s}$$
$$-\frac{(\psi-2)(\zeta-2)}{(\psi-\tau)(\zeta-\tau)(\psi-\zeta)^2}(\psi^s-\zeta^s)^2(\psi\zeta)^{n-s}$$
$$-\frac{(\zeta-2)(\tau-2)}{(\psi-\zeta)(\psi-\tau)(\zeta-\tau)^2}(\zeta^s-\tau^s)^2(\zeta\tau)^{n-s}.$$

For s$=-r=1$, the Cassini's identity is obtained:

$$MN_{n-1}MN_{n+1}-MN_n^2=\frac{(\psi-2)(\tau-2)}{(\psi-\zeta)(\zeta-\tau)}(\psi\tau)^{n-1}-\frac{(\psi-2)(\zeta-2)}{(\psi-\tau)(\zeta-\tau)}(\psi\zeta)^{n-1}$$
$$-\frac{(\zeta-2)(\tau-2)}{(\psi-\zeta)(\psi-\tau)}(\zeta\tau)^{n-1}.$$

For s$=m-n$ and $r=1$, the d'Ocagne's identity is obtained:

$$MN_{n+1}MN_m-MN_nMN_{m+1}=\frac{1}{(\psi-\zeta)(\zeta-\tau)(\psi-\tau)}[(\psi-2)(\zeta-2)(\psi^m\zeta^n-\psi^n\zeta^m)$$
$$+(\zeta-2)(\tau-2)(\zeta^m\tau^n-\zeta^n\tau^m)+(\psi-2)(\tau-2)(\tau^m\psi^n-\tau^n\psi^m)].$$

Definition 2.3 (Matrix approach) *Define the matrix* $\varphi=\begin{bmatrix}0&1&0\\0&0&-1\\2&0&3\end{bmatrix}$.
This is a spiral matrix that satisfies the characteristic equation $\varphi^3-3\varphi^2+2I=0$ where I is the identity matrix. This result can be seen from the well-known Cayley Hamilton theorem.

Theorem 2.5 *For any integer n, we have*

$$|\varphi|=-2$$
$$|\varphi^n|=(-1)^n2^n=(-2)^n.$$

Theorem 2.6 *For the matrix φ, we have the matrix polynomial identity*

$$\varphi^{n+5}=3\varphi^{n+4}-\varphi^{n+3}+\varphi^{n+2}-2\varphi^n$$

Proof: From the equation $\varphi^3-3\varphi^2+2I=0$ we can write

$$I=\frac{1}{2}\varphi^2(3I-\varphi)=\frac{1}{2}\varphi^2\left(3\varphi^2-\varphi^3+I-\varphi\right)$$

$$I=\frac{1}{2}\left(\varphi^2-\varphi^3+3\varphi^4-\varphi^5\right)$$

Multiplying both sides of the above equality by φ^n, the proof is completed.

Theorem 2.7 *Let $r \geq 0$ be an integer. Then, we have*

$$MN_{r+5} = 3MN_{r+4} - MN_{r+3} + MN_{r+2} - 2MN_r$$

and $MNL_{r+5} = 3MNL_{r+4} - MNL_{r+3} + MNL_{r+2} - 2MNL_r$

Proof: We prove by induction on r.
Since, $MN_6 = 3MN_5 - MN_4 + MN_3 - 2MN_1$, the result is true for $r = 1$.
Now based on the assumption that
$MN_{t+5} = 3MN_{t+4} - MN_{t+3} + MN_{t+2} - 2MN_t$ is satisfied for all $t < r$.
Then we write $MN_{n+5} = 3MN_{n+4} - 2MN_{n+2}$

$$= 3\left(3MN_{r+3} - MN_{r+2} + MN_{r+1} - 2MN_{r-1}\right) - 2\left(3MN_{r+1} - MN_r + MN_{r-1} - 2MN_{r-3}\right)$$

$$= 3\left(3MN_{r+3} - 2MN_{r+1}\right) - \left(3MN_{r+2} - 2MN_r\right) + 3\left(3MN_{r+1} - 2MN_{r-1}\right) - 2\left(3MN_{r-1} - 2MN_{r-3}\right)$$

$= 3MN_{r+4} - MN_{r+3} + MN_{r+2} - 2MN_r$ which is the desired result.

Declaration of the Authors:
There are no conflicts of interest.
The manuscript does not contain any animal study.
There are no human studies in the manuscript.
The project received ethical clearance from Madurai Kamaraj University's local ethical committee.
Funding statement: The authors affirm that they did not receive any grants, funding, or other forms of assistance in order to prepare this paper.
Data Availability: Upon a reasonable request, the corresponding author will make the datasets created and/or examined during the current investigation available.

References

1. Akkuş, Hakan & Üregen, Rabia & Özkan, Engin. A New Approach to k-Jacobsthal Lucas Sequences. Sakarya University Journal of Science. 2021;25:969-973. https://www.researchgate.net/publication/352846326_A_New_Approac h_to_k-Jacobsthal_Lucas_Sequences
2. C. Kizilates. A new generalization of Fibonacci hybrid and Lucas hybrid numbers. Chaos, Solitons and Fractals. 2020;130:109449. https://doi.org/10.1016/j.chaos.2019.109449
3. Dresden, G., & Wang, Y. A Universal Convolution Identity for Fibonacci-type and Lucas-type Sequences. Math. Mag. 2022;95:1-7. https://dresden.academic.wlu.edu/files/2021/08/Universal.pdf
4. Mine Uysal and Engin Ozkan. Padovan Hybrid Quaternions and Some properties. Journal of Science and Arts. 2022;22(1):121-132. https://www.josa.ro/index.html?https%3A//www.josa.ro/josa.html
5. B. Malini Devi and S. Devibala. On the Split Mersenne and Mersenne-Lucas Hybrid Quaternions. Baghdad Science Journal. 2023; 10(1):311-317. https://doi.org/10.21123/bsj.2023.8429
6. Awad, Yahia, Hindi, Ramiz, and Chehade Haissam. A Comparative Study between a Novel Deterministic Test for Mersenne Primes and the Well-Known Primality Tests. Baghdad Science Journal. 2023; 20(5):202-2055. https://doi.org/10.21123/bsj.2023.7791
7. Altassan A and Alan M. Mersenne Numbers in Generalized Lucas Sequences. Proceedings of the Bulgarian Academy of Sciences. 2024;77(1):3–10. https://doi.org/10.7546/CRABS.2024.01.01
8. S. H. Jafari Petroudi, A. Dasdemir and M. Pirouz. On Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas Sequences. Konuralp Journal of Mathematics. 2024;12(1):55-61. https://dergipark.org.tr/en/pub/konuralpjournalmath/issue/84314/12 15445
9. T. Koshy. Fibonacci and Lucas Numbers with Applications. Canada: John Wiley & sons, inc. 2001.
10. T. Koshy. Pell and Pell-Lucas Numbers with Applications. New York: Springer 2014.

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