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A Parametric Kind of Fubini-Fibonacci Polynomials and their Generalizations

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ABSTRACT: In this paper, we introduce bivariate kind of three-variable Fubini-Fibonacci polynomials and their associated numbers within the approach of Golden F- Calculus. Utilizing generating functions, we derive several fundamental properties, including summation theorems, recurrence relations, symmetry properties, and F-derivative identities. We further establish connections with, Bernoulli-Fibonacci, Euler-Fibonacci, Genocchi-Fibonacci Stirling-Fibonacci numbers of the second kind and present mu,ltiple summation formulas and convolution-type identities. The proposed approach enriches the theory of Fibonacci-based special polynomials and opens new avenues for applications in combinatorics, number theory, approximation theory, and matrix analysis.

Key Words: Golden calculus, Fubini polynomials, Fubini-Fibonacci polynomials, generating functions, Stirling-Fibonacci numbers of the second kind.

Contents

1. Introduction

It must be emphasized that special numbers, special polynomials, generating functions, and trigonometric functions are not only essential in applied mathematics but also play a crucial role in various branches of mathematics, including number theory, matrix theory, and mathematical analysis (see [5,7,8,9,10,24]). Lately, a number of scholars have advanced and obtained generating functions for several new families of special polynomials, including those with one and two parametric types, such as Bernoulli, Euler, Genocchi, Fubini, and others. We have presented a formal investigation into the relationships between trigonometric functions and special polynomials, utilizing generating functions. By implementing the partial derivative operator to these generating functions, we have derived a number of formulae involving finite combinatorial sums and the aforementioned polynomials. These formulae provide new insights into the behavior of these functions and numbers. (see [1,2,4,6,22,3,23].

As a special type of Appell polynomials the Fubini polynomials are defined as the sum

$$\mathbf{F}_w(\xi) = \sum_{k=0}^{n} S_2(w,k)k!\xi^k \tag{1.1}$$

where $S_2(w,k)$ are the Stirling numbers of second kind [8]. These polynomials can be generated by [20]

$$\frac{1}{1-\xi(e^d-1)} = \sum_{w=0}^{\infty} \mathbf{F}_w(\xi)\frac{d^w}{w!}$$

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Substituting $\xi=1$ in Equation (1.1), we obtain the Fubini numbers. They have

$$\frac{1}{1-(e^{d}-1)}=\sum_{w=0}^{\infty}\mathbf{F}_{w}\frac{d^{w}}{w!}$$

as the exponential generating function. For more details, please see [21,22,23].

We present certain definitions and properties associated with Golden Calculus (also known as F-calculus). The Fibonacci sequence is characterized by the following recurrence relation:

$$F_{w}=F_{w-1}+F_{w-2},\quad w\geq 2$$

where $F_0=0$, $F_1=1$. Fibonacci numbers can be expressed explicitly as

$$F_{w}=\frac{\alpha^{w}-\beta^{w}}{\alpha-\beta},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ (Golden ratio) and $\beta=\frac{1-\sqrt{5}}{2}$.

The use of the golden ratio in various branches of science and mathematics is well-documented. Additionally, this mystical number also makes appearances in the fields of architecture and art. Recently, the properties of F-calculus have been exhaustively defined and studied by Pashaev and Nalci [11] for the first time. Please refer to the following papers for further reading: Krot [13], Özvatan [14], Pashaev [12], and Kus et al. [15]. These sources provide additional information and insights on the topic at hand.

The F-factorial, a product arising from the Fibonacci numbers, was formally introduced as follows:

$$F_1F_2F_3\ldots F_w=F_w!,$$

where $F_0!=1$. The following is the formal expression of the binomial theorem for the F-analogues, also referred to as the Golden binomial theorem:

$$(a+b)_{F}^{w}=\sum_{s=0}^{w}(-1)^{\binom{s}{2}}\binom{w}{s}_{F}a^{w-s}b^{s},\tag{1.2}$$

in terms of the Golden binomial coefficients, called as Fibonomials

$$\binom{w}{s}_{F}=\frac{F_w!}{F_{w-s}!F_s!}$$

with n and k being nonnegative integers, $w\geq s$. The Golden derivative defined as follows:

$$\frac{\partial_F}{\partial_F\xi}\left(g(\xi)\right)=\frac{g\left(\alpha\xi\right)-g\left(-\frac{\xi}{\alpha}\right)}{\left(\alpha-\left(-\frac{1}{\alpha}\right)\right)\xi}=\frac{g\left(\alpha\xi\right)-g\left(\beta\xi\right)}{\left(\alpha-\beta\right)\xi}.\tag{1.3}$$

respectively. The first and second types of Golden exponential functions are denoted as

$$e_{F}^{\xi}=\sum_{s=0}^{\infty}\frac{(\xi)_{F}^{s}}{F_s!},$$

and

$$E_{F}^{\xi}=\sum_{s=0}^{\infty}(-1)^{\binom{s}{2}}\frac{(\xi)_{F}^{s}}{F_s!}.$$

Briefly, we use this notation throughout the article

$$e_{F}^{\xi}=\sum_{s=0}^{\infty}\frac{\xi^{s}}{F_s!},\tag{1.4}$$

and

$$E_F^{\xi} = \sum_{s=0}^{\infty} (-1)^{\binom{s}{2}} \frac{\xi^s}{F_s!}. \tag{1.5}$$

Using the Equations (1.4), and (1.5), the following equation can be given

$$e_F^{\xi} E_F^{\eta} = e_F^{\xi+\eta}. \tag{1.6}$$

The Fibonacci cosine and sine functions, also known as the Golden trigonometric functions, are denoted by the power series

$$\cos_F(\xi) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s}}{F_{2s}!}, \tag{1.7}$$

and

$$\sin_F(\xi) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s+1}}{F_{2s+1}!}. \tag{1.8}$$

For arbitrary number ϕ, Golden derivatives of $e_F^{\phi\xi}$, $E_F^{\phi\xi}$, $\cos_F(\phi\xi)$, and $\sin_F(\phi\xi)$ functions are

$$\frac{\partial_F}{\partial_F \xi}\left(e_F^{\phi\xi}\right) = \phi e_F^{k\xi}, \tag{1.9}$$

$$\frac{\partial_F}{\partial_F \xi}\left(E_F^{\phi\xi}\right) = \phi E_F^{-\phi\xi}, \tag{1.10}$$

$$\frac{\partial_F}{\partial_F \xi}\left(\cos_F(\phi\xi)\right) = -\phi \sin_F(\phi\xi), \tag{1.11}$$

and

$$\frac{\partial_F}{\partial_F \xi}\left(\sin_F(\phi\xi)\right) = \phi \cos_F(\phi\xi). \tag{1.12}$$

By virtue of (1), Pashaev and Özvatan [16] formalized the concept of Bernoulli-Fibonacci polynomials. Subsequently, Kuş et al. [15] defined the Euler-Fibonacci numbers and polynomials, and provided identities and matrix representations for both Bernoulli-Fibonacci polynomials and Euler-Fibonacci polynomials. Very recently, Tuglu and Ercan [17,18] generalized the concept of Bernoulli-Fibonacci polynomials and Euler-Fibonacci polynomials of order α as follows:

$$\left(\frac{d}{\lambda e_F^d - 1}\right)^{\alpha} e_F^{\xi d} = \sum_{w=0}^{\infty} \mathcal{B}_{w,F}^{\alpha}(\xi;\lambda) \frac{d^w}{F_w!}, \tag{1.13}$$

and

$$\left(\frac{2}{\lambda e_F^d + 1}\right)^{\alpha} e_F^{\xi d} = \sum_{w=0}^{\infty} \mathcal{E}_{w,F}^{\alpha}(\xi;\lambda) \frac{d^w}{F_w!}. \tag{1.14}$$

In [19] Kızılateş and Öztürk defined the define two parametric types of the Apostol Bernoulli-Fibonacci polynomials, the Apostol Euler-Fibonacci polynomials, and the Apostol Genocchi-Fibonacci polynomials of order α, as follows:

$$\left(\frac{d}{\lambda e_F^d - 1}\right)^{\alpha} e_F^{pd} \cos_F(qd) = \sum_{w=0}^{\infty} \mathcal{B}_{w,F}^{(c,\alpha)}(p,q;\lambda) \frac{d^w}{F_w!}, \tag{1.15}$$

$$\left(\frac{d}{\lambda e_F^d - 1}\right)^{\alpha} e_F^{pd} \sin_F(qd) = \sum_{w=0}^{\infty} \mathcal{B}_{w,F}^{(s,\alpha)}(p,q;\lambda) \frac{d^w}{F_w!}, \tag{1.16}$$

$$\left(\frac{2}{\lambda e_F^d + 1}\right)^{\alpha} e_F^{pd} \cos_F(qd) = \sum_{w=0}^{\infty} \mathcal{E}_{w,F}^{(c,\alpha)}(p,q;\lambda) \frac{d^w}{F_w!}, \tag{1.17}$$

$$\left(\frac{2}{\lambda e_F^d+1}\right)^{\alpha} e_F^{pd}\sin_F(qd)=\sum_{w=0}^{\infty}\mathcal{E}_{w,F}^{(s,\alpha)}(p,q;\lambda)\frac{d^w}{F_w!}, \tag{1.18}$$

$$\left(\frac{2d}{\lambda e_F^d+1}\right)^{\alpha} e_F^{pd}\cos_F(qd)=\sum_{w=0}^{\infty}\mathcal{G}_{w,F}^{(c,\alpha)}(p,q;\lambda)\frac{d^w}{F_w!}, \tag{1.19}$$

$$\left(\frac{2d}{\lambda e_F^d+1}\right)^{\alpha} e_F^{pd}\sin_F(qd)=\sum_{w=0}^{\infty}\mathcal{G}_{w,F}^{(s,\alpha)}(p,q;\lambda)\frac{d^w}{F_w!}. \tag{1.20}$$

In this paper, we define the bivariate kind of Fubini-Fibonacci polynomials and investigate their basic properties and other associated results using Golden Calculus. We also establish relationships between the bivariate type of Fubini-like polynomials and other polynomials. We introduce the generalized Fubini-Fibonacci polynomials numbers and establish some properties of these newly defined sequences utilizing generating functions and their functional equations. Furthermore, we introduce the concept of generalized Fubini-Fibonacci numbers and establish a relationship between these numbers and the parametric Fubini-Fibonacci polynomials. Finally, we derive relation expressions for parametric for Fubini-Fibonacci polynomials.

2. A parametric kind of Fubini-Fibonacci Numbers and Polynomials

This section extends the three-variable Fubini-Fibonacci polynomials by integrating parametric trigonometric elements through the application of the golden cosine and sine functions. The fibonomial convolution of two sequences, u_w and v_w, was formally defined by Krot [13] as follows:

$$U_F(d)=\sum_{w=0}^{\infty}u_w\frac{d^w}{F_w!} \text{ and } V_F(d)=\sum_{w=0}^{\infty}v_w\frac{d^w}{F_w!},$$

then their F-convolution is defined as

$$l_w=u_w *_F v_w=\sum_{l=0}^{w}\binom{w}{k}_F u_l\ v_{w-l}.$$

So, the generating function takes the form

$$L_F(d)=U_F(d)V_F(d)=\sum_{w=0}^{\infty}l_w\frac{d^w}{F_w!}.$$

Let $p,q\in\mathbb{R}$. The Taylor series of the functions $e_F^{pd}\cos_F(qd)$ and $e_F^{pd}\sin_F(qd)$ can be express as follows:

$$e_F^{pd}\cos_F(qd)=\sum_{w=0}^{\infty}\mathcal{C}_{w,F}(p,q)\frac{d^w}{F_w!}, \tag{2.1}$$

and

$$e_F^{pd}\sin_F(qd)=\sum_{w=0}^{\infty}\mathcal{S}_{w,F}(p,q)\frac{d^w}{F_w!}, \tag{2.2}$$

where

$$\mathcal{C}_{F,w}(p,q)=\sum_{k=0}^{\left\lfloor\frac{w}{2}\right\rfloor}(-1)^k\binom{w}{2k}_F p^{w-2k}q^{2k}, \tag{2.3}$$

$$\mathcal{S}_{F,w}(p,q)=\sum_{k=0}^{\left\lfloor\frac{w-1}{2}\right\rfloor}(-1)^k\binom{w}{2k+1}_F p^{w-2k-1}q^{2k+1}. \tag{2.4}$$

Based on the aforementioned definitions of $\mathcal{C}_{n,F}(p,q)$ and $\mathcal{S}_{n,F}(p,q)$, as well as the number of a $\mathbb{F}_{F,w}$, we can specify two parameters for Fubini-Fibonacci polynomials as follows:

$$\frac{1}{1-\gamma(e_F^d-1)}e_F^{\xi d}\cos_F \eta d=\sum_{w=0}^{\infty}\mathbb{F}_{F,w}^{(c)}(\xi,\eta;\gamma)\frac{d^w}{F_w!} \tag{2.5}$$

and

$$\frac{1}{1-\gamma(e_F^d-1)}e_F^{\xi d}\sin_F \eta d=\sum_{w=0}^{\infty}\mathbb{F}_{F,w}^{(s)}(\xi,\eta;\gamma)\frac{d^w}{F_w!}, \tag{2.6}$$

respectively.

Remark 2.1 *For $\xi=0$ in (2.5) and (2.6), we get new type of F-cosine Fubini polynomials $\mathbb{F}_{w,F}^{(c)}(\eta;\gamma)$ and F-sine Fubini polynomials $\mathbb{F}_{w,F}^{(s)}(\eta;\gamma)$ as*

$$\frac{1}{1-\gamma(e_F^d-1)}\cos_F \eta d=\sum_{w=0}^{\infty}\mathbb{F}_{F,w}^{(c)}(\eta;\gamma)\frac{d^w}{F_w!}, \tag{2.7}$$

and

$$\frac{1}{1-\gamma(e_F^d-1)}\sin_F \eta d=\sum_{w=0}^{\infty}\mathbb{F}_{F,w}^{(s)}(\eta;\gamma)\frac{d^w}{F_w!}. \tag{2.8}$$

respectively.

Based on the aforementioned definitions, we have arrived at the following principal results.

Theorem 2.1 *The following identities hold true:*

$$\mathbb{F}_{w,F}^{(c)}(\eta;\gamma)=\sum_{v=0}^{[\frac{w}{2}]}\left(\begin{array}{c} w+v \\ 2v \end{array}\right)_F(-1)^v\eta^{2v}\mathbb{F}_{w-2v,F}(\gamma), \tag{2.9}$$

and

$$\mathbb{F}_{w,F}^{(s)}(\eta;\gamma)=\sum_{v=0}^{[\frac{w-1}{2}]}\left(\begin{array}{c} w+v \\ 2v+1 \end{array}\right)_F(-1)^v\eta^{2v+1}\mathbb{F}_{w-2v-1,F}(\gamma). \tag{2.10}$$

Proof: By (2.7) and (2.8), we can derive the following equations

$$\begin{aligned}\sum_{w=0}^{\infty}\mathbb{F}_{w,F}^{(c)}(\eta;\gamma)\frac{d^w}{[w]_q!}&=\frac{1}{1-\gamma(e_F^d-1)}cos_F(\eta d)\\ &=\sum_{w=0}^{\infty}\mathbb{F}_{w,F}(\gamma)\frac{t^n}{[n]_q!}\sum_{v=0}^{\infty}(-1)^v q^{(2v-1)v}\eta^{2v}\frac{d^v}{F_{2v}!}\\ &=\sum_{w=0}^{\infty}\left(\sum_{v=0}^{[\frac{w}{2}]}\left(\begin{array}{c} w+v \\ 2v \end{array}\right)_F(-1)^v\eta^{2v}\mathbb{F}_{w-2v,F}(\gamma)\right)\frac{d^w}{F_w!},\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sum_{w=0}^{\infty}\mathbb{F}_{w,q}^{(S)}(\eta;\gamma)\frac{d^w}{[w]_q!}&=\frac{1}{1-\gamma(e_F^d-1)}sin_F(\eta d)\\ &=\sum_{w=0}^{\infty}\left(\sum_{v=0}^{[\frac{w-1}{2}]}\left(\begin{array}{c} w \\ 2v+1 \end{array}\right)_F(-1)^v\eta^{2v+1}\mathbb{F}_{w-2v-1,F}(\gamma)\right)\frac{d^w}{F_w!}.\end{aligned} \tag{2.12}$$

Therefore, by (2.11) and (2.12), we get (2.9) and (2.10). □

Theorem 2.2 *Let $w \geq 0$. Then*

$$\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)=\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}_{k,F}(\gamma)\mathbb{C}_{w-k,F}(\xi,\eta), \tag{2.13}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma)=\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}_{k,F}(\gamma)\mathbb{S}_{w-k,F}(\xi,\eta). \tag{2.14}$$

Proof: Consider

$$\left(\sum_{w=0}^{\infty} a_w \frac{d^w}{w!}\right)\left(\sum_{k=0}^{\infty} b_k \frac{d^k}{k!}\right)=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w} a_{w-k}b_k\right)\frac{d^w}{w!}.$$

Now

$$\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}=\frac{1}{1-\gamma(e^d_F-1)}e^{\xi d}_F cos_F(\eta d)$$
$$=\left(\sum_{k=0}^{\infty}\mathbb{F}_{k,F}(\gamma)\frac{d^k}{[k]_F!}\right)\left(\sum_{w=0}^{\infty}\mathbb{C}_{w,q}(\xi,\eta)\frac{d^w}{F_w!}\right)$$
$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}_{k,F}(\gamma)\mathbb{C}_{w-k,q}(\xi,\eta)\right)\frac{d^w}{F_w!},$$

which proves (2.13). The proof of (2.14) is similar. □

Theorem 2.3 *Let $w \geq 0$. Then*

$$\mathbb{F}^{(c)}_{w,F}(\xi+r,\eta;\gamma)=\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}^{(c)}_{k,F}(\xi,\eta;\gamma)r^{w-k}, \tag{2.15}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi+r,\eta;\gamma)=\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}^{(s)}_{k,q}(\xi,\eta;\gamma)r^{w-k}. \tag{2.16}$$

Proof: By changing ξ with $\xi+r$ in (2.5) , we have

$$\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi+r,\eta;\gamma)\frac{d^w}{[w]_F!}=\frac{1}{1-\gamma(e^d_F-1)}e^{\xi d}_F \cos_F(\eta d)e^{rd}_F$$
$$=\left(\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}\right)\left(\sum_{k=0}^{\infty}r^k\frac{d^k}{F_k!}\right)$$
$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\left(\begin{array}{c} w \\ k \end{array}\right)_F \mathbb{F}^{(c)}_{k,F}(\xi,\eta;\gamma)r^{w-k}\right)\frac{d^w}{F_w!},$$

which complete the proof (2.15). The result (2.16) can be similarly proved.

□

Theorem 2.4 *Let $w \geq 1$. Then*

$$\frac{\partial}{\partial\xi}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)=F_w\mathbb{F}^{(c)}_{w-1,F}(\xi,\eta;\gamma), \tag{2.17}$$

$$\frac{\partial}{\partial \eta}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma) = -F_w\mathbb{F}^{(s)}_{w-1,F}(\xi,\eta;\gamma), \tag{2.18}$$

and

$$\frac{\partial}{\partial \xi}\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma) = F_w\mathbb{F}^{(s)}_{w-1,F}(\xi,\eta;\gamma), \tag{2.19}$$

$$\frac{\partial}{\partial \eta}\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma) = F_w\mathbb{F}^{(c)}_{w-1,F}(\xi,\eta;\gamma). \tag{2.20}$$

Proof: Equation (2.5) yields

$$\sum_{w=1}^{\infty}\frac{\partial}{\partial \xi}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!} = \frac{1}{1-\gamma(e_F^d-1)}\frac{\partial}{\partial \xi}e_F^{\xi d}\cos_F(\eta d) = \sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^{w+1}}{F_w!}$$
$$= \sum_{w=1}^{\infty}\mathbb{F}^{(c)}_{w-1,F}(\xi,\eta;\gamma)\frac{d^w}{[F_w!} = \sum_{w=1}^{\infty}F_w\mathbb{F}^{(c)}_{w-1,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!},$$

proving (2.17). Other (2.18), (2.19) and (2.20) can be similarly derived. □

Theorem 2.5 *Let $w \in \mathbb{N}^*$, the following formula holds true. Then*

$$\mathbb{F}^{(c)}_{w,F}(2\xi,\eta;\gamma) = \sum_{k=0}^{w}\binom{w}{k}_F\mathbb{F}^{(c)}_{k,F}(\xi,\eta;\gamma)\xi^{w-k}, \tag{2.21}$$

and

$$\mathbb{F}^{(s)}_{w,F}(2\xi,\eta;\gamma) = \sum_{k=0}^{w}\binom{w}{k}_F\mathbb{F}^{(s)}_{k,F}(\xi,\eta;\gamma)\xi^{w-k}. \tag{2.22}$$

Proof: By using equations (2.5) and (2.6), we can easily proof of equations (2.21) and (2.22). We omit the proof. □

Theorem 2.6 *For $w \geq 0$, the following formula holds true:*

$$\mathbb{C}_{w,F}(\xi,\eta) = \mathbb{F}^{(x)}_{w,F}(\xi,\eta;\gamma) - \gamma\mathbb{F}^{(c)}_{w,F}(\xi+1,\eta;\gamma) + \gamma\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma), \tag{2.23}$$

and

$$\mathbb{S}_{w,F}(\xi,\eta) = \mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma) - \gamma\mathbb{F}^{(s)}_{w,F}(\xi+1,\eta;\gamma) + \gamma\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma). \tag{2.24}$$

Proof: By (2.5) and write

$$e_F^{\xi d}\cos_F(\eta d) = \frac{1-\gamma(e_F^d-1)}{1-\gamma(e_F^d-1)}e_F^{\xi d}\cos_F(\eta d) = \frac{e_F^{\xi d}\cos_F(\eta d)}{1-\gamma(e_F^d-1)} - \frac{z(e_q(t)-1)}{1-\gamma(e_F^d-1)}e_F^{\xi d}\cos_F(\eta d).$$

Then using the equations (2.1) and (2.5), we have

$$\sum_{w=0}^{\infty}\mathbb{C}_{w,F}(\xi,\eta)\frac{d^w}{F_w!} = \sum_{w=0}^{\infty}\left[\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma) - \gamma\mathbb{F}^{(c)}_{w,F}(\xi+1,\eta;\gamma) + \gamma\mathbb{F}^{(C)}_{w,F}(\xi,\eta;\gamma)\right]\frac{d^w}{F_w!}.$$

Finally, comparing the coefficients of $\frac{d^w}{F_w!}$, we get (2.23). The proof of (2.24) is similar. □

Theorem 2.7 *For $w \geq 0$, we have*

$$\gamma\mathbb{F}^{(c)}_{w,F}(\xi+1,\eta;\gamma) = (1+\gamma)\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma) - \mathbb{C}_{w,F}(\xi,\eta), \tag{2.25}$$

and

$$\gamma\mathbb{F}^{(s)}_{w,F}(\xi+1,\eta;\gamma) = (1+\gamma)\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma) - \mathbb{S}_{w,F}(\xi,\eta). \tag{2.26}$$

Proof: From (2.5), we have

$$\sum_{w=0}^{\infty}\left[\mathbb{F}_{w,F}^{(c)}(\xi+1,\eta;\gamma)-\mathbb{F}_{w,F}^{(c)}(\xi,\eta;\gamma)\right]\frac{d^w}{F_w!}=\frac{e_F^{\xi d}\cos_F(\eta d)}{1-\gamma(e_F^d-1)}(e_q(t)-1)=\frac{1}{\gamma}\left[\frac{e_F^{\xi d}\cos_F(\eta d)}{1-\gamma(e_F^d-1)}-e_F^{\xi d}\cos_F(\eta d)\right]$$

$$=\frac{1}{\gamma}\sum_{w=0}^{\infty}\left[\mathbb{F}_{w,F}^{(c)}(\xi,\eta;\gamma)-\mathbb{C}_{w,F}(\xi,\eta)\right]\frac{d^w}{F_w!}.$$

Comparing the coefficients of $\frac{d^w}{F_w!}$ on both sides, we obtain (2.25). The proof of (2.26) is similar. □

Theorem 2.8 *Let w be integer. Then*

$$\sum_{k=0}^{w}\left(\begin{array}{c}w\\k\end{array}\right)_F\mathbb{F}_{w-k,F}^{(c)}(\xi_1,\eta_1;\gamma_1)\mathbb{F}_{k,F}^{(c)}(\xi_2,\eta_2;\gamma_2)=\frac{\gamma_2\mathbb{F}_{w,F}^{(c)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_2)-\gamma_1\mathbb{F}_{w,F}^{(c)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_1)}{\gamma_2-\gamma_1},\tag{2.27}$$

and

$$\sum_{k=0}^{w}\left(\begin{array}{c}w\\k\end{array}\right)_F\mathbb{F}_{w-k,F}^{(s)}(\xi_1,\eta_1;\gamma_1)\mathbb{F}_{k,F}^{(s)}(\xi_2,\eta_2;\gamma_2)=\frac{\gamma_2\mathbb{F}_{w,F}^{(s)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_2)-\gamma_1\mathbb{F}_{w,F}^{(s)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_1)}{\gamma_2-\gamma_1}.\tag{2.28}$$

Proof: The products of (2.5) can be written as

$$\sum_{w=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{F}_{w,q}^{(c)}(\xi_1,\eta_1;\gamma_1)\mathbb{F}_{k,q}^{(c)}(\xi_2,\eta_2;\gamma_2)\frac{d^w}{F_w!}\frac{d^k}{F_w!}=\frac{e_F^{\xi_1 d}\cos_F(\eta_1 d)}{1-\gamma_1(e_F^d-1)}\frac{e_F(\xi_2 d)\cos_F(\eta_2 d)}{1-\gamma_2(e_F^d-1)}$$

$$\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\left(\begin{array}{c}w\\k\end{array}\right)_F\mathbb{F}_{w-k,F}^{(c)}(\xi_1,\eta_1;\gamma_1)\mathbb{F}_{k,F}^{(c)}(\xi_2,\eta_2;\gamma_2)\right)\frac{d^w}{F_w!}$$

$$=\frac{\gamma_2}{\gamma_2-\gamma_1}\frac{e_F^{(\xi_1+\xi_2)d}\cos_F(\eta_1 d)\cos_F(\eta_2 d)}{1-\gamma_1(e_F^d-1)}-\frac{\gamma_1}{\gamma_2-\gamma_1}\frac{e_F^{(\xi_1+\xi_2)d}\cos_F(\eta_1 d)\cos_F(\eta_2 d)}{1-\gamma_2(e_F^d-1)}$$

$$=\sum_{w=0}^{\infty}\left(\frac{\gamma_2\mathbb{F}_{w,q}^{(c)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_2)-\gamma_1\mathbb{F}_{w,q}^{(c)}(\xi_1+\xi_2,\eta_1+\eta_2;\gamma_1)}{\gamma_2-\gamma_1}\right)\frac{d^w}{F_w!}.$$

By equating the coefficients of $\frac{d^w}{F_w!}$ on both sides, we get (2.27). The proof of (2.28) is similar. □

Theorem 2.9 *Let w be integer. Then*

$$(1+\gamma)\mathbb{F}_{w,F}^{(c)}(\xi,\eta;\gamma)=\gamma\sum_{k=0}^{w}\left(\begin{array}{c}w\\k\end{array}\right)_F\mathbb{F}_{w-k,F}^{(c)}(\xi,\eta;\gamma)+\mathbb{C}_{w,F}\tag{2.29}$$

and

$$(1+\gamma)\mathbb{F}_{w,F}^{(s)}(\xi,\eta;\gamma)=\gamma\sum_{k=0}^{w}\left(\begin{array}{c}w\\k\end{array}\right)_F\mathbb{F}_{w-k,F}^{(s)}(\xi,\eta;\gamma)+\mathbb{S}_{w,F}\tag{2.30}$$

Proof: Consider the following identity

$$\frac{1+\gamma}{(1-\gamma(e_F^d-1))\gamma e_F^d}=\frac{1}{1-\gamma(e_F^d-1)}+\frac{1}{ze_q(t)}.$$

Evaluating the following fraction using above identity, we find

$$\frac{(1+\gamma)e_F^{\xi d}\cos_F(\eta d)}{(1-\gamma(e_F^d-1))\gamma e_F^d}=\frac{e_F^{\xi d}\cos_F(\eta d)}{1-\gamma(e_F^d-1)}+\frac{e_F^{\xi d}\cos_F(\eta d)}{\gamma e_F^d}$$

$$(1+\gamma)\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}=\gamma\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}\sum_{k=0}^{\infty}\frac{d^k}{F_k!}+\sum_{w=0}^{\infty}\mathbb{C}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of d in both sides of the resultant equation, assertion (2.29) follows. The proof of (2.30) is similar. □

3. Relationship between F-Bernoulli, F-Euler and F-Genocchi polynomials and F-Stirling numbers of the second kind

In this section, we prove some relationships for two bivariate kind of F-cosine Fubini polynomials and F-sine Fubini polynomials related to F-Bernoulli polynomials, F-Euler polynomials and F-Genocchi polynomials and F-Stirling numbers of the second kind. We start a following theorem.

Theorem 3.1 *Each of the following relationships holds true:*

$$\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)=\sum_{p=0}^{w+1}\binom{w+1}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{B}_{p-k,F}(\xi)-\mathbb{B}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(c)}_{w+1-p,F}(\eta;\gamma)}{F_{w+1}} \tag{3.1}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma)=\sum_{p=0}^{w+1}\binom{w+1}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{B}_{p-k,F}(\xi)-\mathbb{B}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(s)}_{w+1-p,F}(\eta;\gamma)}{F_{w+1}} \tag{3.2}$$

Proof: By using (1.15) and (2.5), we have

$$\begin{aligned}\left(\frac{1}{1-\gamma(e_F^d-1)}\right)e_F^{\xi d}\cos_F(\eta d)&=\left(\frac{1}{1-\gamma(e_F^d-1)}\right)\frac{d}{e_F^d-1}\frac{e_F^d-1}{d}e_F^{\xi d}\cos_F(\eta d)\\
&=\frac{1}{d}\sum_{w=0}^{\infty}\left(\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{B}_{p-k,F}(\xi)\right)\frac{d^p}{F_p!}\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\eta;\gamma)\frac{d^w}{F_w!}\\
&\quad-\frac{1}{d}\sum_{p=0}^{\infty}\mathbb{B}_{p,F}(\xi)\frac{d^p}{F_p!}\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\eta;\gamma)\frac{d^w}{F_w!}\\
&=\frac{1}{d}\sum_{w=0}^{\infty}\left[\sum_{p=0}^{w}\binom{w}{p}_F\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{B}_{p-k,F}(\xi)\right]\mathbb{F}^{(c)}_{w-p,F}(\eta;\gamma)\frac{d^w}{F_w!}\\
&\quad-\frac{1}{d}\sum_{w=0}^{\infty}\left[\sum_{p=0}^{w}\binom{w}{p}_F\mathbb{B}_{p,F}(\xi)\right]\mathbb{F}^{(c)}_{w-p,F}(\eta;\gamma)\frac{d^w}{F_w!}.\end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{d^w}{F_w!}$, we arrive at the required result (3.1). The proof of (3.2) is similar. □

Theorem 3.2 *Each of the following relationships holds true:*

$$\mathbb{F}^{(c)}_{w,q}(\xi,\eta;\gamma)=\sum_{p=0}^{w}\binom{w}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{E}_{p-k,F}(\xi)+\mathbb{E}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(c)}_{w-p,F}(\eta;\gamma)}{2}, \tag{3.3}$$

and

$$\mathbb{F}^{s}_{w,F}(\xi,\eta;\gamma)=\sum_{p=0}^{w}\binom{w}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{E}_{p-k,F}(\xi)+\mathbb{E}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(s)}_{w-p,F}(\eta;\gamma)}{2}, \tag{3.4}$$

Proof: By using definitions (1.16) and (2.5), we have

$$\left(\frac{1}{1-\gamma(e_F^d-1)}\right)e_F^{\xi d}\cos_F(\eta d)=\left(\frac{1}{1-\gamma(e_F^d-1)}\right)\frac{2}{e_F^d+1}\frac{e_F^d+1}{2}e_F^{\xi d}\cos_F(\eta d)$$
$$=\frac{1}{2}\left[\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F\mathbb{E}_{w-k,F}(\xi)\right)\frac{d^w}{F_w!}+\sum_{w=0}^{\infty}\mathbb{E}_{w,F}(\xi)\frac{d^w}{F_w!}\right]\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\eta;\gamma)\frac{d^w}{F_w!}$$
$$=\frac{1}{2}\sum_{w=0}^{\infty}\left[\sum_{p=0}^{w}\binom{w}{p}_F\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{E}_{p-k,F}(\xi)+\sum_{p=0}^{w}\binom{w}{p}_F\mathbb{E}_{p,F}(\xi)\right]\mathbb{F}^{(c)}_{w-p,F}(\eta;\gamma)\frac{d^w}{F_w!}.$$

Comparing the coefficients of $\frac{d^w}{F_w!}$, we arrive at the desired result (3.3). The proof of (3.4) is similar. □

Theorem 3.3 *Each of the following relationships holds true:*

$$\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)=\sum_{p=0}^{w}\binom{w+1}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{G}_{p-k,F}(\xi)+\mathbb{G}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(c)}_{w+1-p,F}(\eta;\gamma)}{2F_{w+1}},\tag{3.5}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma)=\sum_{p=0}^{w}\binom{w+1}{p}_F\left[\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{G}_{p-k,F}(\xi)+\mathbb{G}_{p,F}(\xi)\right]\frac{\mathbb{F}^{(s)}_{w+1-p,F}(\eta;\gamma)}{2F_{w+1}},\tag{3.6}$$

Proof: By (1.17) and (2.5), we have

$$\left(\frac{1}{1-\gamma(e_F^d-1)}\right)e_F^{\xi d}\cos_F(\eta d)=\left(\frac{1}{1-\gamma(e_F^d-1)}\right)\frac{2d}{e_F^d+1}\frac{e_F^d+1}{2d}e_F^{\xi d}\cos_F(\eta d)$$
$$=\frac{1}{2d}\left[\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F\mathbb{G}_{w-k,F}(\xi)\right)\frac{d^w}{F_w!}+\sum_{w=0}^{\infty}\mathbb{G}_{w,F}(\xi)\frac{d^w}{F_w!}\right]\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\eta;\gamma)\frac{d^w}{F_w!}$$
$$=\frac{1}{2}\sum_{w=0}^{\infty}\left[\sum_{p=0}^{w}\binom{w}{p}_F\sum_{k=0}^{p}\binom{p}{k}_F\mathbb{G}_{p-k,F}(x)+\sum_{p=0}^{w}\binom{w}{p}_F\mathbb{G}_{p,F}(\xi)\right]\mathbb{F}^{(c)}_{w+1-p,F}(\eta;\gamma)\frac{d^w}{F_{w+1}!}.$$

Comparing the coefficients of $\frac{d^w}{F_w!}$, then we have the asserted result (3.5). The proof of (3.6) is similar. □

The subsequent definition delineates the r-Stirling–Fibonacci numbers of the second kind.

$$\sum_{k=m}^{\infty}S^F_{2,r}(k+r,m+r)\frac{d^k}{F_k!}=e_F^{rd}\frac{(e_F^d-1)^m}{F_m!}.\tag{3.7}$$

For $r=0$, we have Stirling-Fibonacci numbers of the second kind [2]

$$\sum_{k=m}^{\infty}S^F_2(k,m)\frac{t^k}{F_k!}=\frac{(e_F^d-1)^m}{F_m!}.\tag{3.8}$$

Theorem 3.4 *Let w be integer. Then*

$$\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)=\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{C}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l,k),\tag{3.9}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi,\eta;\gamma)=\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{S}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l,k).\tag{3.10}$$

Proof: Using (2.5) and (3.8), we have

$$\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi,\eta;\gamma)\frac{d^w}{F_w!}=\frac{1}{1-\gamma(e_F^d-1)}e_F^{\xi d}\cos_F(\eta d)=e_F^{\xi d}\cos_F(\eta d)\sum_{k=0}^{\infty}\gamma^k(e_F^d-1)^k$$

$$=e_F^{\xi d}\cos_F(\eta d)\sum_{k=0}^{\infty}\gamma^k\sum_{l=k}^{\infty}k!S_{2,F}(l,k)\frac{d^l}{F_l!}=\sum_{w=0}^{\infty}\mathbb{C}_{F,q}(\xi,\eta)\frac{d^w}{F_n!}\sum_{l=0}^{\infty}\gamma^k\sum_{k=0}^{l}k!S_{2,F}(l,k)\frac{d^l}{F_l!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\left(\begin{array}{c}w\\l\end{array}\right)_F\mathbb{C}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l,k)\right)\frac{d^w}{F_w!}.$$

Comparing the coefficients of $\frac{d^w}{F_w!}$ in both sides, we get (3.9). The proof of (3.10) is similar. □

Theorem 3.5 *Let w integer. Then*

$$\mathbb{F}^{(c)}_{w,F}(\xi+r,\eta;\gamma)=\sum_{l=0}^{w}\left(\begin{array}{c}w\\l\end{array}\right)_F\mathbb{C}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l+r,k+r),\tag{3.11}$$

and

$$\mathbb{F}^{(s)}_{w,F}(\xi+r,\eta;\gamma)=\sum_{l=0}^{w}\left(\begin{array}{c}w\\l\end{array}\right)_F\mathbb{S}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l+r,k+r).\tag{3.12}$$

Proof: Using (2.5) and (3.7), we have

$$\sum_{w=0}^{\infty}\mathbb{F}^{(c)}_{w,F}(\xi+r,\eta;\gamma)\frac{d^w}{F_w!}=\frac{1}{1-\gamma(e_F^d-1)}e_F^{(\xi+r)d}\cos_F(\eta d)=e_F^{(\xi+r)d}\cos_F(\eta d)\sum_{k=0}^{\infty}\gamma^k(e_F^d-1)^k$$

$$=e_F^{\xi d}\cos_F(\eta d)\sum_{k=0}^{\infty}\gamma^k\sum_{l=k}^{\infty}k!S_{2,F}(l+r,k+r)\frac{d^l}{F_l!}=\sum_{w=0}^{\infty}\mathbb{C}_{F,q}(\xi,\eta)\frac{d^w}{F_n!}\sum_{l=0}^{\infty}\gamma^k\sum_{k=0}^{l}k!S_{2,F}(l+r,k+r)\frac{d^l}{F_l!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\left(\begin{array}{c}w\\l\end{array}\right)_F\mathbb{C}_{w-l,F}(\xi,\eta)\sum_{k=0}^{l}\gamma^k k!S_{2,F}(l+r,k+r)\right)\frac{d^w}{F_w!}.$$

Comparing the coefficients of $\frac{d^w}{F_w!}$ in both sides, we get (3.11). The proof of (3.12) is similar.

□

4. Conclusion

In the present paper, we have examined the F-analogues parametric types of the Fubini-Fibonacci polynomials. By utilizing their generating functions, we have derived several fundamental properties of these parametric kind of Fubini-Fibonacci polynomials. Furthermore, we have introduced the generalized Fubini-Fibonacci numbers and derived some properties of these newly established numbers. We have also presented some results related to these numbers and polynomials. Finally, we have provided some relation expressions of parametric kind Fubini-Fibonacci polynomials. Our work suggests that the results presented here may inspire further research in the field of other polynomial types using the Golden Calculus.

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