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Qualitative Results for Iterative Fractional Differential Equations with Mixed Boundary Conditions

Ramdas Narute, V. V. Kharat and Subhash Kendre

ABSTRACT: This paper is about existence and uniqueness results of iterative fractional differential equations with mixed fractional derivative equipped with integral boundary conditions. We establish an important equivalence between our problem and a homogeneous integral equation. Using Banach fixed point theorem and Krasnoselskii fixed point theorem, existence and uniqueness of solutions for our problem are proved. Further, Various Ulam-Hyers stability results are studied along with examples.

Keywords: Fractional differential equations, mixed fractional derivative, integral boundary conditions, iterative.

Contents

1. Introduction

Fractional calculus is a branch of Mathematics that deals with the derivative and integral of fractional orders. It has wide range of applications in various fields like Biology, Bioengineering, Control theory, Hydrology, Elasticity, Thermodynamics and many more! Readers can refer the books ([2], [4], [5], [14], [19]). Different fractional derivatives have been introduced by several mathematicians till date of which 'Caputo fractional derivative' and 'Riemann-Liouville fractional derivative' are popular ones. In [16], Khalil introduced a new kind of fractional derivative known as 'conformable fractional derivative.

Researchers have explored different fields of fractional calculus such as integrodifferential equations, implicit fractional differential equations, fractional differential equations with integral boundary conditions. Further, many researchers also focused on stability analysis of solutions to the fractional differential equations by studying 'Ulam-Hyers stability' which was introduced in the mid of 19th century. It has become on of the important and popular subject in mathematical analysis.

Fractional differential equations involving left as well as right fractional derivatives has also been a study of interest for a long time. A lot of recent work deals with the study of mixed fractional derivatives with integral boundary conditions.

Iterative fractional differential equations have grabbed the attention of many mathematicians. In [11], Ibrahim R.W. studied iterative fractional differential equations of the type

$$D^{\gamma}u(s) = g(s, u(s), u(u(s))).$$

A study of existence of an infective disease processes is done in [8]. In [9], Ibrahim et. al. studied initial value problem in iterative fractional differential equations.

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In [18], Ntyous et. al. established existence and uniqueness results for solutions of single and multi-valued boundary value problems which involves Riemann-Liouville and Caputo fractional derivatives. Lakoud et.al. studied the existence of solutions for a mixed fractional boundary value problem in which left Riemann-Liouville and right Caputo fractional derivatives were involved in [17]. Also in [12,13] V. V. Kharat et. al, authors studied existence of fractional differential equations.

In this article, we investigate existence and uniqueness of solutions for the iterative fractional differential equation with mixed fractional derivatives namely right conformable fractional derivative and left Caputo fractional derivative with boundary conditions

$$D^{\beta}_{1^-}({}^{C}D^{\alpha}_{0+}x)(t) = f\big(t, x(x(t))\big) \tag{1.1a}$$

$$x(0) = \gamma \int_0^1 x(t)dt \tag{1.1b}$$

$${}^{C}D^{\alpha}_{0+}x(1) = 0, \tag{1.1c}$$

where $\alpha, \beta \in (0, 1], \gamma \in (0, 1), D^{\beta}_{1^-}$ denotes the right conformable fractional derivative, ${}^{C}D^{\alpha}_{0+}$ denotes the left Caputo fractional derivative, x is the unknown function and $f : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous function.

2. Preliminaries

In this section, we see some important definitions, lemmas and theorems on fractional calculus.

Definition 2.1 ([1], [16]) The left conformable fractional derivative starting from a of a function $x : [a, +\infty) \to \mathbb{R}$ of order $0 < \beta \le 1$ is defined by

$$D^{\beta}_{a^+}x(t) = \lim_{\epsilon\to 0} \frac{x\big(t + \epsilon(t - a)^{(1-\beta)}\big) - x(t)}{\epsilon}, \forall\ t > 0. \tag{2.1}$$

If $D^{\beta}_{a^+}x(t)$ exists on (a, b) then $D^{\beta}_{a^+}x(a) = \lim_{t\to a^+} x(t)$.

The right conformable fractional derivative of order $0 < \beta \le 1$ terminating at b of x is defined by

$$D^{\beta}_{b^-}x(t) = -\lim_{\epsilon\to 0} \frac{x\big(t + \epsilon(b - t)^{(1-\beta)}\big) - x(t)}{\epsilon}, \forall\ t > 0. \tag{2.2}$$

If $D^{\beta}_{b^-}x(t)$ exists on (a, b) then $D^{\beta}_{b^-}x(b) = \lim_{t\to b^-} x(t)$.

Definition 2.2 ([1], [16]) The left and right conformable fractional integral of a function x of order $0 < \beta \le 1$ is defined respectively by

$$I^{\beta}_{a^+}x(t) = \int_a^t (s - a)^{\beta-1}x(s)ds \tag{2.3a}$$

$$I^{\beta}_{b^-}x(t) = \int_t^b (b - s)^{\beta-1}x(s)ds \tag{2.3b}$$

Definition 2.3 ([14]) The left Caputo fractional derivative of order $0 < \alpha \le 1$ of an absolutely continuous function $x : [a, b] \to \mathbb{R}$ is given by

$${}^{C}D^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - s)^{-\alpha}x'(s)ds \tag{2.4}$$

The right Caputo fractional derivative of order $0 < \alpha \le 1$ terminating at b of x is defined by

$${}^{C}D^{\alpha}_{b-}x(t) = -\frac{1}{\Gamma(1 - \alpha)} \int_t^b (s - t)^{-\alpha}x'(s)ds \tag{2.5}$$

Definition 2.4 ([14]) The left and right Riemann-Liouville fractional integral of order $0 < \alpha \leq 1$ of a function x are defined respectively by

$$J^{\alpha}_{a^+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds \tag{2.6a}$$

$$J^{\alpha}_{b^-}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}x(s)ds \tag{2.6b}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.1 *([1], [14], [16])*

1. *If x is a continuous function on $(a, b]$ then*

$$D^{(\beta)}_{b^-}\left(I^{\beta}_{b^-}x(t)\right) = D^{(\beta)}_{a^+}\left(I^{\beta}_{a^+}x(t)\right) =^C D^{\alpha}_{b^-}\left(J^{\alpha}_{b_-}x(t)\right) =^C D^{\alpha}_{a^+}\left(J^{\alpha}_{a_+}x(t)\right) = x(t) \tag{2.7}$$

2. *If $D^{\beta}_{a^+}x, D^{\beta}_{b^-}x,^C D^{\alpha}_{a^+}x,^C D^{\alpha}_{b^-}x$ are continuous on (a, b) then*

$$I^{\beta}_{a^+}\left(D^{\beta}_{a^+}x(t)\right) = J^{\alpha}_{a^+}\left({}^C D^{\alpha}_{a^+}x(t)\right) = x(t) - x(a), \tag{2.8a}$$

$$I^{\beta}_{b^-}\left(D^{\beta}_{b^-}x(t)\right) = J^{\alpha}_{b^-}\left({}^C D^{\alpha}_{b^-}x(t)\right) = x(t) - x(b) \tag{2.8b}$$

3. *If x is differentiable on (a, b) then*

$$D^{\beta}_{a^+}x(t) = (t-a)^{1-\beta}x'(t) \tag{2.9a}$$

$$D^{\beta}_{b^-}x(t) = -(b-t)^{1-\beta}x'(t) \tag{2.9b}$$

Theorem 2.1 *([2], [5]) (Contraction Mapping Principle) Let $(E, ||\cdot||)$ be a Banach space, P is a non-empty closed subset. If $T : P \to P$ is a strictly contraction, i.e. if there exists $L \in (0, 1)$ such that $||Tx - Ty|| \leq L||x - y||, \forall x, y \in P$. Then T has a fixed point.*

Theorem 2.2 *([15]) (Kransnoselskii Fixed Point Theorem) Let M be a closed convex and non-empty subset of a Banach space X. Let T_1, T_2 be the operators such that*

1. *$T_1x + T_2y \in M, \ \forall \ x, y \in M$.*
2. *T_1 is compact and continuous.*
3. *T_2 is a contraction mapping.*

Then there exists $z \in M$ such that $z = T_1(z) + T_2(z)$.

3. Main Results

To proceed, we define the following.
Let us denote $E = C([0, 1], [0, 1])$,

$C_\alpha = \left\{x \in E : |x(t_1) - x(t_2)| \leq K|t_1 - t_2|\right\}$
$C_{\alpha,r} = \{x \in C_\alpha : ||x|| \leq r\}$ and

$$\Delta = \frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta} + \frac{1}{\alpha+1}\right] \tag{3.1}$$

Hypotheses:

$(H1)$ There exists a real number $L > 0$ such that

$$||f(t,x) - f(t,y)|| \leq L||x-y||, \forall t \in [0,1], x, y \in [0,1]. \tag{3.2}$$

$(H2)$ There exists a continuous function $\mu : [0,1] \to [0,1]$ such that

$$||f(t,x)|| \leq \mu(t), \forall (t,x) \in [0,1] \times [0,1]. \tag{3.3}$$

Lemma 3.1 *([6]) Let $\alpha, \beta \in (0,1]$ and $\gamma \in (0,1), f : [0,1] \times [0,1] \to [0,1]$ be continuous function. Then $x \in E$ is a solution of (1.1a)-(1.1c) iff $x \in E$ satisfies*

$$x(t) = \int_0^1 G(t,s) f\big(s, x(x(s))\big) ds \tag{3.4}$$

where

$$G(t,s) = \begin{cases} \left[\dfrac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + t^\alpha - (t-s)^\alpha\right] \dfrac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} & if \quad 0 \leq s \leq t \leq 1 \\ \left[\dfrac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + t^\alpha\right] \dfrac{(1-s)^{\beta-1}}{\Gamma(\alpha+1)} & if \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{3.5}$$

Theorem 3.1 *Let $\alpha, \beta \in (0,1]$ and $\gamma \in (0,1), f : [0,1] \times [0,1] \to [0,1]$ be continuous function. Then $x \in E$ is a solution of (1.1a)-(1.1c) iff $x \in E$ satisfies*

$$\begin{aligned} x(t) =& \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\big(s, x(x(s))\big) ds \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^\alpha (1-s)^{\beta-1} f\big(s, x(x(s))\big) ds \\ &- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^\alpha (1-s)^{\beta-1} f\big(s, x(x(s))\big) ds \end{aligned} \tag{3.6}$$

Proof Using the Lemma 3.1, rewriting the solution, we get the result.

Theorem 3.2 *Assume that $f \in C([0,1],[0,1])$ is continuous and satisfies (H1) with $L < \dfrac{1}{\Delta(\bar{K}+1)}$. Then the BVP (1.1a)-(1.1c) has a unique solution.*

Proof Let $M = \sup_{t\in[0,1]} |f(t,0)|$, choose $r \geq \dfrac{M\Delta}{1-L\Delta}$. Define $T : C_{\alpha,r} \to C_{\alpha,r}$ given by

$$\begin{aligned} (Tx)(t) =& \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, x(x(s))\right) ds \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^\alpha (1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \\ &- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^\alpha (1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \end{aligned} \tag{3.7}$$

Consider

$$
\begin{aligned}
||(Tx)(t)|| \leq & \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big] || \big[f\big(s, x(x(s))\big) - f(s,0)\big] + f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} || \big[f\big(s, x(x(s))\big) - f(s,0)\big] + f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} || \big[f\big(s, x(x(s))\big) - f(s,0)\big] + f(s,0)|| ds \\
\leq & \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big] ||x(s)|| ds \\
& + \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big] ||f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} ||f\big(s, x(x(s))\big) - f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} ||f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} ||f\big(s, x(x(s))\big) - f(s,0)|| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} ||f(s,0)|| ds \\
\leq & \frac{Lr+M}{\Gamma(\alpha+1)} \left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta} + \frac{1}{\alpha+1} \right] \\
\leq & (Lr+M)\Delta \\
\leq & r,
\end{aligned}
$$

where Δ is given by (3.1) and $r \geq \dfrac{M\Delta}{1-L\Delta}$. This proves that $(Tx) \in C_{\alpha,r}$. Consider

$$
\begin{aligned}
||Tx - Ty|| \leq & \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \Big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\Big] |f\big(s, x(x(s))\big) - f\big(s, y(y(s))\big)| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} |f\big(s, x(x(s))\big) - f\big(s, y(y(s))\big)| ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} |f\big(s, x(x(s))\big) - f\big(s, y(y(s))\big)| ds \\
\leq & \frac{L\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \Big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\Big] |x(x(s)) - y(y(s))| ds \\
& + \frac{L}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} |x(x(s)) - y(y(s))| ds \\
& + \frac{L}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} |x(x(s)) - y(y(s))| ds
\end{aligned}
$$

by using (H1)

$$\begin{aligned}||Tx-Ty|| \le & \frac{L\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \Big[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\Big]|x(x(s))-x(y(s))+x(y(s))-y(y(s))|ds\\ &+\frac{L}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1}|x(x(s))-x(y(s))+x(y(s))-y(y(s))|ds\\ &+\frac{L}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}(1-s)^{\beta-1}|x(x(s))-x(y(s))+x(y(s))-y(y(s))|ds\\ &\le L\left(\frac{k}{\Gamma(\alpha+1)}+1\right)\Delta||x-y||\\ &\le L(\bar{K}+1)\Delta||x-y||,\end{aligned}$$

where $\bar{K}=\max\frac{K}{\Gamma(\alpha+1)}$. As $L<\frac{1}{\Delta(\bar{K}+1)}$, we can see that T is a contraction. Thus, by Banach fixed point theorem, the operator defined by (3.7) has unique solution.

Theorem 3.3 *Let $f\in C([0,1]\times[0,1],[0,1])$ be a jointly continuous function mapping bounded sets of $[0,1]\times[0,1]$ to $[0,1]$ and the assumptions (H1),(H2) hold with*

$$P=\frac{L(\bar{K}+1)}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)}+\frac{1}{\beta}\right]<1 \tag{3.8}$$

then, the BVP (1.1a)-(1.1c) has at least one solution on $[0,1]$.

Proof Define $B_{\bar{r}}=\{x\in C_{\alpha,r}:||x||\le\bar{r}\}$ and $\sup_{t\in[0,1]}|\mu(t)|=||\mu||$ and fix

$$\bar{r}\ge\frac{||\mu||}{\Gamma(\alpha+1)}\Big[\frac{1+(1-\gamma)(\alpha+1)+\beta(1-\gamma)}{(1-\gamma)\beta(\alpha+1)}\Big] \tag{3.9}$$

Define operators T_1, T_2 on $B_{\bar{r}}$ by

$$(T_1x)(t)=-\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}(1-s)^{\beta-1}f\left(s,x(x(s))\right)ds \tag{3.10}$$

$$\begin{aligned}(T_2x)(t)=&\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\big]f\left(s,x(x(s))\right)ds\\ &+\frac{1}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1}f\left(s,x(x(s))\right)ds\end{aligned} \tag{3.11}$$

Claim 1: $T(B_{\bar{r}})\subset B_{\bar{r}}$. For $x,y\in B_{\bar{r}}$, consider

$$\begin{aligned}||T_1x+T_2y||\le &\Big|-\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}(1-s)^{\beta-1}f\left(s,x(x(s))\right)ds\\ &+\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\big]f\left(s,y(y(s))\right)ds\\ &+\frac{1}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1}f\left(s,y(y(s))\right)ds\Big|\\ &\le||\mu||\Big[\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}ds\\ &+\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\big]ds+\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\beta-1}ds\Big]\\ &\le||\mu||\Big\{\frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)}+\frac{1}{\beta}+\frac{1}{\alpha+1}\right]\Big\}\le\bar{r}.\end{aligned}$$

Hence, $T_1x + T_2y \in B_{\bar{r}}$.
Claim 2: The hypothesis (H_1) and (3.8) that T_2 is a contraction mapping. For

$$\begin{aligned}
\Big|(T_2x)(t) - (T_2y)(t)\Big| &\leq \Big|\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, x(x(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \\
&- \Bigg[\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds\Bigg]\Big| \\
&\leq \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right]\Big|f\left(s, x(x(s))\right) - f\left(s, y(y(s))\right)\Big| ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\int_0^1 t^{\alpha}(1-s)^{\beta-1}\Big|f\left(s, x(x(s))\right) - f\left(s, y(y(s))\right)\Big| ds \\
||T_2(x) - T_2(y)|| &\leq \frac{L(\bar{K}+1)}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta}\right]||x-y|| \\
&\leq P||x-y||
\end{aligned}$$

where P is given by (3.8).

Also, f is continuous so is T_1. Further, T_1 is uniformly bounded on $B_{\bar{r}}$ since

$$||T_1x|| \leq ||\mu||\frac{1}{\Gamma(\alpha+1)} \tag{3.12}$$

Claim 3: T_1 is compact. For $t_1, t_2 \in [0,1]$ with $t_2 > t_1$ and $x \in B_{\bar{r}}$. Consider

$$\begin{aligned}
||(T_1x)(t_2) - (T_1x)(t_1)|| &= \Bigg|\Bigg| -\frac{1}{\Gamma(\alpha+1)}\int_0^{t_2}(t_2-s)^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \\
&- \left(-\frac{1}{\Gamma(\alpha+1)}\int_0^{t_1}(t_1-s)^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds\right)\Bigg|\Bigg| \\
&\leq \frac{||\mu||}{\Gamma(\alpha+1)}\Bigg|\int_0^{t_2}(t_2-s)^{\alpha} ds - \int_0^{t_1}(t_1-s)^{\alpha} ds\Bigg| \\
&\leq \frac{||\mu||}{\Gamma(\alpha+1)}\Bigg|\int_0^{t_1}\Big((t_2-s)^{\alpha} - (t_1-s)^{\alpha}\Big) ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha} ds\Bigg| \\
&\leq \frac{||\mu||}{\Gamma(\alpha+1)}\left|2(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_1^{\alpha}\right|
\end{aligned}$$

which is independent of x. Hence, T_1 is equicontinuous. As f maps bounded subsets into relatively compact subsets,i.e. $T_1(B)(t)$ is relatively compact subset of E for every t for every bounded subset B of C_{α}. Hence, T_1 is relatively compact on $B_{\bar{r}}$. Hence, all assumptions of the Kransnoseleskii theorem are satisfied, hence, the BVP (1.1a)-(1.1c) has at least one solution.

4. Stability Results

4.1. Definitions

Let $\epsilon > 0$ be a positive real number, $f : [0,1] \times [0,1] \to [0,1]$ and $\varphi : [0,1] \to R_+$be a continuous function. For the BVP (1.1a)-(1.1c), we concentrate on the following inequalities.

$$\left|D_{1^-}^{\beta}({}^C D_{0^+}^{\alpha} y)(t) - f(t, y(y(t)))\right| \leq \epsilon, t \in [0,1] \tag{4.1}$$

$$\left|D_{1^-}^{\beta}({}^C D_{0^+}^{\alpha} y)(t) - f(t, y(y(t)))\right| \leq \varphi(t), t \in [0,1] \tag{4.2}$$

$$\left|D_{1^-}^{\beta}({}^C D_{0^+}^{\alpha} y)(t) - f(t, y(y(t)))\right| < \epsilon\varphi(t), t \in [0,1] \tag{4.3}$$

Definition 4.1 ([20], [22]) The BVP (1.1a)-(1.1c) is Ulam-Hyers stable if there exist constants $\lambda > 0$ such that for each $\epsilon > 0$ and for each solution $y \in E$ of the inequality (4.1), there exists a solution $x \in E$ of the BVP (1.1a)-(1.1c) such that

$$|y(t) - x(t)| \leq \lambda\epsilon, \ t \in [0, 1].$$

Definition 4.2 ([20], [22]) The BVP (1.1a)-(1.1c) is generalized Ulam-Hyers stable if there exists $\theta \in C(\mathbb{R}_+, \mathbb{R}_+), \theta(0) = 0$ such that for each $\epsilon > 0$ and for each solution $y \in E$ of the inequality (4.1), there exists a solution $x \in E$ of the BVP (1.1a)-(1.1c) such that

$$|y(t) - x(t)| \leq \theta(\epsilon), t \in [0, 1].$$

Definition 4.3 ([20], [22]) The BVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c > 0$ for each $\epsilon > 0$ and for each solution $y \in E$ of the inequality (4.2), there exists a solution $x \in E$ of the BVP (1.1a)-(1.1c) such that

$$|y(t) - x(t)| \leq c\epsilon\varphi(t), t \in [0, 1].$$

Definition 4.4 ([20], [22]) The BVP (1.1a)-(1.1c) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c > 0$ such that for each solution $y \in E$ of the inequality (4.2), there exists a solution $x \in E$ of the BVP (1.1a)-(1.1c) such that

$$|y(t) - x(t)| \leq c\varphi(t), t \in [0, 1].$$

Remark([22])

1. Definition (4.1) $\Rightarrow$ Definition (4.2).
2. Definition (4.3) $\Rightarrow$ Definition (4.4).

Remark([20], [22])

1. A function $y \in E$ is a solution of inequality (4.1) if and only if there exists a function $w \in C([0, 1], \mathbb{R}_+)$ such that
   (a) $|w(t)| \leq \epsilon, t \in [0, 1]$.
   (b) $D^{\beta}_{1-}(^{C}D^{\alpha}_{0+}y)(t) = f(t, y(y(t))) + w(t), t \in [0, 1]$.
2. Also a function $y \in E$ is a solution of the inequality (4.2) iff there exists $h \in C([0, 1], \mathbb{R}_+)$ such that
   (a) $|h(t)| \leq \epsilon\varphi(t), t \in [0, 1]$,
   (b) $D^{\beta}_{1-}(^{C}D^{\alpha}_{0+}y)(t) = f(t, y(y(t))) + h(t), t \in [0, 1]$.
3. For, (4.3), there exists a function $\Phi \in C([0, 1], \mathbb{R}_+)$ such that
   (a) $|\Phi(t)| \leq \epsilon\varphi(t), t \in [0, 1]$,
   (b) $D^{\beta}_{1-}(^{C}D^{\alpha}_{0+}y)(t) = f(t, y(y(t))) + \Phi(t), t \in [0, 1]$.

4.2. Ulam-Hyers Stability

In this part, we present our next result about Ulam-Hyers stability for the given BVP (1.1a)-(1.1c).

Theorem 4.1 *Suppose $f : [0, 1] \times [0, 1] \to [0, 1]$ is continuous and satisfying (H_1) and $L < \dfrac{1}{\Delta(\bar{K} + 1)}$. Then*

1. *The BVP (1.1a)-(1.1c) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*
2. *If $\varphi : [0, 1] \to [0, 1]$ is differentiable and increasing function such that $\varphi(0) \neq 0$, then the BVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable.*

Proof (i) Let $y \in E$ be any solution of the inequality (4.1), then by Remark 4.1, we have $D^{\beta}_{1^-}({}^C D^{\alpha}_{0^+} y)(t) = f(t, y(y(t))) + w(t)$ Then

$$\begin{aligned}
y(t) =& \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&+ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] w(s) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} w(s) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} w(s) ds
\end{aligned}$$

$$\begin{aligned}
&\Bigg| y(t) - \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \Bigg\} \Bigg| \\
&= \Bigg| \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] w(s) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} w(s) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} w(s) ds \Bigg| \\
&\leq \Delta \epsilon
\end{aligned} \tag{4.4}$$

Let $x \in E$ be a solution of the BVP (1.1a)-(1.1c). Then we get

$$\begin{aligned}
|y(t)-x(t)| = \Bigg| y(t) - \Bigg\{ & \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, x(x(s))\right) ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \\
& - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \Bigg\} \Bigg| \\
& \le \Bigg| y(t) - \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
& - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \Bigg\} \Bigg| \\
& + \Bigg| \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
& - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
& - \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f\left(s, x(x(s))\right) ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \\
& - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, x(x(s))\right) ds \Bigg\} \Bigg| \\
& \le \Delta\epsilon + L(\bar{K}+1)\Delta ||y-x||
\end{aligned}$$

From equation (3.1) and (4.3),

$$||y-x|| \le \Delta\epsilon + L(\bar{K}+1)\Delta|y-x|,$$

which implies that $||y-x|| \le \dfrac{\Delta}{[1-L\Delta(\bar{K}+1)]}\epsilon$ where $\lambda = \dfrac{\Delta}{[1-L\Delta(\bar{K}+1)]}\epsilon > 0$. Therefore, the problem (1.1a)-(1.1c) is Ulam-Hyers stable. Further, let us set $\theta(\epsilon) = \lambda\epsilon$, then the BVP (1.1a)-(1.1c) is generalized Ulam-Hyers-stable.
(ii) Let $y \in E$ be the solution of the inequality (4.2), then by Remark 4.1, we have

$$D_{1-}^{\beta}({}^{C}D_{0+}^{\alpha}y)(t) = f(t, y(y(t))) + \Phi(t), t \in [0,1]$$

Using Theorem 3.1, we get

$$\begin{aligned}
y(t) =&\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&+ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] \Phi(s) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} \Phi(s) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} \Phi(s) ds
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\Bigg| y(t) - \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \Bigg\} \Bigg| \\
&= \Bigg| \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] \Phi(s) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} \Phi(s) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} \Phi(s) ds \Bigg| \\
&\leq \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] \left|\Phi(s)\right| ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} \left|\Phi(s)\right| ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} \left|\Phi(s)\right| ds \\
&\leq \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right] \varphi(s) ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} \varphi(s) ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} \varphi(s) ds
\end{aligned} \tag{4.6}$$

As $\varphi : [0,1] \to [0,1]$ is increasing, so In $0 \le s \le t$ and hence

$$s \le t \Rightarrow \varphi(s) \le \varphi(t)$$

$$\int_0^t \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]\varphi(s)\big]ds \le \varphi(t)\int_0^t \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta} ds \tag{4.7}$$

$$\int_0^t (1-s)^{\beta-1}\varphi(s)ds \le \varphi(1)\int_0^t (1-s)^{\beta-1}ds \tag{4.8}$$

and

$$s \le 1 \Rightarrow \varphi(s) \le \varphi(1)$$

$$\int_t^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]\varphi(s)ds \le \varphi(1)\int_0^t \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta} ds \tag{4.9}$$

$$\int_t^1 (1-s)^{\beta-1}\varphi(s)ds \le \varphi(1)\int_t^1 (1-s)^{\beta-1}ds \tag{4.10}$$

Consider

$$\begin{aligned}
&\Bigg|y(t) - \Bigg\{\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big] f\left(s, y(y(s))\right) ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\int_0^1 t^\alpha (1-s)^{\beta-1} f\left(s, y(y(s))\right) ds \\
&- \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha (1-s)^{\beta-1} f\left(s, y(y(s))\right) ds\Bigg\}\Bigg| \\
&\le \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]\big|\Phi(s)\big|ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\int_0^1 t^\alpha (1-s)^{\beta-1}\big|\Phi(s)\big|ds \\
&- \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha (1-s)^{\beta-1}\big|\Phi(s)\big|ds \\
&\le \epsilon\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\int_0^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]\varphi(s)ds \\
&+ \epsilon\frac{1}{\Gamma(\alpha+1)}\int_0^1 t^\alpha (1-s)^{\beta-1}\varphi(s)ds \\
&- \epsilon\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha (1-s)^{\beta-1}\varphi(s)ds \\
&\le \epsilon\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\varphi(t)\int_0^t \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]ds \\
&+ \epsilon\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\varphi(1)\int_t^1 \big[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\big]ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\epsilon\varphi(t)\int_0^t (1-s)^{\beta-1}ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\epsilon\varphi(1)\int_t^1 (1-s)^{\beta-1}ds \\
&+ \frac{1}{\Gamma(\alpha+1)}\epsilon\varphi(t)\int_0^t (t-s)^\alpha ds \\
&\le \epsilon\varphi(t)\frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta} + \frac{1}{\alpha+1}\right] + \epsilon\varphi(1)\left[\Delta - \frac{1}{\Gamma(\alpha+2)}\right]
\end{aligned}$$

Define a function $\nu : [0,1] \to [0,1]$ by

$$\nu(t) = \Delta\varphi(t) + \varphi(1)\left[\Delta - \frac{1}{\Gamma(\alpha+2)}\right] - \varphi(t)\left[\Delta + \frac{\varphi(1)}{\varphi(0)}\Delta - \frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right] \tag{4.11}$$

Then, $\nu'(t) = -\phi'(t)\left[\frac{\varphi(1)}{\varphi(0)}\Delta - \frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]$ Since, φ is increasing and differentiable, we have $\nu'(t) \le 0$. Also, $\nu(0) = 0$. Then for any solution $x \in E$ of a BVP (1.1a)-(1.1c), then

$$\begin{aligned}
|y(t) - x(t)| \le \Bigg| y(t) - \Bigg\{ &\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f(s, x(x(s)))\, ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f(s, x(x(s)))\, ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f(s, x(x(s)))\, ds \Bigg\} \Bigg| \\
&\le \Bigg| y(t) - \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f(s, y(y(s)))\, ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f(s, y(y(s)))\, ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f(s, y(y(s)))\, ds \Bigg\} \Bigg| \\
&+ \Bigg| \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f(s, y(y(s)))\, ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f(s, y(y(s)))\, ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f(s, y(y(s)))\, ds \\
&- \Bigg\{ \frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)} \int_0^1 \left[(1-s)^{\beta-1} - (1-s)^{\alpha+\beta}\right] f(s, x(x(s)))\, ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 t^{\alpha}(1-s)^{\beta-1} f(s, x(x(s)))\, ds \\
&- \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}(1-s)^{\beta-1} f(s, x(x(s)))\, ds \Bigg\} \Bigg|
\end{aligned}$$

$$\begin{aligned}
|y(t)-x(t)| &\le \epsilon\frac{\varphi(1)}{\varphi(0)}\Delta\varphi(t)+\frac{L\gamma}{(1-\gamma)(\Gamma(\alpha+2))}\int_0^1\left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right]|y(y(s))-y(x(s))|ds\\
&+\frac{L\gamma}{(1-\gamma)(\Gamma(\alpha+2))}\int_0^1\left[(1-s)^{\beta-1}-(1-s)^{\alpha+\beta}\right]|y(x(s))-x(x(s))|ds\\
&+\frac{L}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\beta-1}|y(y(s))-y(x(s))|ds\\
&+\frac{L}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\beta-1}|y(x(s))-x(x(s))|ds\\
&+\frac{L}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}|y(y(s))-y(x(s))|ds\\
&+\frac{L}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}|y(x(s))-x(x(s))|ds\\
&\le \epsilon\varphi(t)\left[\Delta+\frac{\varphi(1)}{\varphi(0)}\Delta-\frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]+L(\bar{K}+1)\Delta||y-x||
\end{aligned}$$

Hence, we get

$$||y-x|| \le \epsilon\varphi(t)\left[\Delta+\frac{\varphi(1)}{\varphi(0)}\Delta-\frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]+L(\bar{K}+1)\Delta||y-x||$$

$$||y-x|| \le \frac{\left[\Delta+\frac{\varphi(1)}{\varphi(0)}\Delta-\frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]}{\left[1-L(\bar{K}+1)\Delta\right]}\varphi(t)$$

or we can write

$$||y-x|| \le c\epsilon\varphi(t)$$

where $c=\frac{\left[\Delta+\frac{\varphi(1)}{\varphi(0)}\Delta-\frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]}{\left[1-L(\bar{K}+1)\Delta\right]}>0$. Hence,the BVP (1.1a)-(1.1c) is Ulam-Hyers-Rassias stable and consequently generalized Ulam-Hyers-Rassias stable.

5. Examples

Example 5.1

$$\begin{cases}
D^{\frac{1}{2}}_{1^-}\left({}^{C}D^{\frac{1}{2}}_{0^+}x\right)(t)=\frac{1}{7}\Big(t+x(x(t))\Big), 0<t<1\\
x(0)=\frac{1}{2}\int_0^1 x(t)dt\\
{}^{C}D^{\alpha}_{0^+}x(1)=0
\end{cases} \tag{5.1}$$

Solution:
Comparing the given example with (1.1a) we get,
$f(t,x(x(t)))=\frac{1}{7}\Big(t+x(x(t))\Big), \alpha=\beta=\gamma=\frac{1}{2}$.
Let us take $\bar{x}:=x(x(t)), \bar{y}:=y(y(t))$

$$\begin{aligned}
|f(t,\bar{x})-f(t,\bar{y})| &= \left|\left(\frac{1}{7}(t+\bar{x})-\frac{1}{7}(t+\bar{y})\right)\right|\\
&\le \frac{1}{7}|\bar{x}-\bar{y}|
\end{aligned}$$

So, the Lipschitz constant $L = \frac{1}{7}$. Also, by using the values of $\alpha,\ \beta,\ \gamma$ in Δ, we get

$$\begin{aligned}\Delta &= \frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta} + \frac{1}{\alpha+1}\right] \\ &\approxeq 4.137399\end{aligned}$$

$\bar{K} = \frac{L}{\Gamma(\alpha+1)} = 0.16119and(\bar{K}+1)\Delta L = 0.16119 \times 4.137399 \times \frac{1}{7} = 0.09528 < 1$. Hence the BVP (5.1) has a unique solution. Let $y \in E$ be the solution of the inequality

$$\left|D^{\frac{1}{2}}_{1^-}\left({}^C D^{\frac{1}{2}}_{0^+}y\right)(t) - \frac{1}{7}\Big(t + y(y(t))\Big)\right| \leq \epsilon, t \in [0,1]$$

$\lambda = \frac{\Delta}{[1 - L\Delta(\bar{K}+1)]} = \frac{4.137399}{1 - 0.09528} > 0$ Hence, the BVP is Ulam-Hyers stable. Also, let $y \in E$ be the solution of

$$\left|D^{\frac{1}{2}}_{1^-}\left({}^C D^{\frac{1}{2}}_{0^+}y\right)(t) - \frac{1}{7}\Big(t + y(y(t))\Big)\right| \leq \epsilon\varphi(t), t \in [0,1]$$

where $\varphi(t) = \frac{t+1}{t+2}$ which is increasing in $[0,1]$. Then

$c = \frac{\left[\Delta + \frac{\varphi(1)}{\varphi(0)}\Delta - \frac{\varphi(1)}{\varphi(0)}\frac{1}{\Gamma(\alpha+2)}\right]}{[1 - L(\bar{K}+1)\Delta]} > 0$. Hence, the BVP (5.1) is Ulam-Hyers-Rassias stable.

Example 5.2 Now we consider another equation

$$\begin{cases} D^{\frac{1}{4}}_{1^-}\left({}^C D^{\frac{1}{2}}_{0^+}x\right)(t) = \dfrac{\Big(t + cos(x(x(t)))\Big)}{t+9}, 0 < t < 1 \\ x(0) = \dfrac{3}{4}\displaystyle\int_0^1 x(t)dt \\ {}^C D^{\alpha}_{0^+}x(1) = 0 \end{cases} \tag{5.2}$$

Solution:

Here, $f(t, x(x(t))) = \frac{\Big(t + cos(x(x(t)))\Big)}{t+9}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \gamma = \frac{3}{4}$. Let us take $\bar{x} := x(x(t)), \bar{y} := y(y(t))$

$$\begin{aligned}|f(t,\bar{x}) - f(t,\bar{y})| &= \left|\frac{\Big(t + \cos\bar{x}\Big)}{t+9} - \frac{\Big(t + cos\bar{y}\Big)}{t+9}\right| \\ &\leq \frac{1}{9}|\bar{x} - \bar{y}|\end{aligned}$$

So, the Lipschitz constant $L = \frac{1}{9}$. Also, by using the values of $\alpha,\ \beta,\ \gamma$ in Δ, we get

$$\begin{aligned}\Delta &= \frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)\beta(\alpha+\beta+1)} + \frac{1}{\beta} + \frac{1}{\alpha+1}\right] \\ &\approxeq 6.39416\end{aligned}$$

$\bar{K} = \frac{L}{\Gamma(\alpha+1)} = 0.281698$ and $(\bar{K}+1)\Delta L = 1.281698 \times 6.39416 \times \frac{1}{9} \approxeq 0.9101598 < 1$. Hence the BVP (5.2) has a unique solution. Let $y \in E$ be the solution of the inequality

$$\left|D^{\frac{1}{2}}_{1^-}\left({}^C D^{\frac{1}{2}}_{0^+}y\right)(t) - \frac{1}{9}\Big(t + cos\big(y(y(t))\big)\Big)\right| \leq \epsilon, t \in [0,1]$$

Then as $\lambda = \dfrac{\Delta}{[1 - L\Delta(\bar{K}+1)]} = \dfrac{6.39416}{1 - 0.9101598} > 0$,the BVP is Ulam-Hyers stable. Also, let $y \in E$ be the solution of

$$\left| D_{1^-}^{\frac{1}{2}} \left({}^C D_{0^+}^{\frac{1}{2}} y \right)(t) - \frac{1}{9}\Big(t + cos\big(y(y(t))\big)\Big) \right| \leq \epsilon\varphi(t), t \in [0,1]$$

where $\varphi(t) = \dfrac{t^2+1}{2}$ which is increasing in $[0,1]$. Then

$c = \dfrac{\left[\Delta + \dfrac{\varphi(1)}{\varphi(0)}\Delta - \dfrac{\varphi(1)}{\varphi(0)}\dfrac{1}{\Gamma(\alpha+2)}\right]}{\big[1 - L(\bar{K}+1)\Delta\big]} = \dfrac{6.39416}{(0.5)[1 - 0.9101598]} > 0$. Hence, the BVP (5.2) is Ulam-Hyers-Rassias stable.

References

1. T. Abdeljawad: On conformable fractional calculus. J. Comput. Appl. Math. 279(2015), 57-66.
2. R. Agarwal, M. Meehan, D. O'Regan: Fixed Point Theory and Applications. Cambridge University Press, Cambridge, 2001.
3. B. Ahmad, S. K. Ntouyas, A. Alsaedi: New Existence Results for Nonlinear Fractional Differential Equations with Three-Point Integral Boudnary Conditions, Advances in Difference Equations, 2011, doi:10.1155/2011/107384.
4. K. Diemling: Nonlinear Fractional Analysis. Springer, Berlin/Heidenberg, 1985.
5. K. Diethelm: The Analysis of Fractional Differential Equations. Springer, 2010.
6. S. Dijab, N. Brahim: A new class of mixed fractional differential equations with integral boundary conditions, Moroccan Journal of Pure and Applied Analysis 7.2 (2021): 227-247.
7. R. W. Ibrahim: Existence of deviating fractional differential equation. Cubo (Temuco) 14(3)(2012), 129–142.
8. R. W. Ibrahim, M. Darus: Infective disease processes based on fractional differential equation. In: Proceedings of the 3rd International Conference on Mathematical Sciences, vol. 1602. No. 1. AIP Publishing(2014).
9. R. W. Ibrahim, A. Kilicman, F. H. Damag: Existence and uniqueness for a class of iterative fractional differential equations. Adv. Differ. Equ. 2015(1), 1–13 (2015). doi:10.1186/s13662-015-0421-y.
10. R. W. Ibrahim, A. Kilicman, F. Damag: Existence and uniqueness for a class of iterative fractional differential equations. Advances in Difference Equations (2015) 2015:78.
11. R. W. Ibrahim: Existence of Iterative Cauchy Fractional Differential Equations, Journal of Mathematics, 2013.
12. V. V. Kharat, T. B. Jagtap: Existence of iterative fractional differential equation with non local condition. The Journal of Indian Mathematical Society, 83(2016), 97-106.
13. V. V. Kharat, Shivaji Tate, A. R. Reshimkar: Some existence results on implicit fractional differential equations. Filomat, 35(12) (2021), 4257-4265.
14. A. Kilbas, H. Shrivatsava, J. Trujillo: Theory and Applications of Fractional Differential Equations. Elsevier: Amsterdam, The Netherlands, 2006.
15. M. A. Krasnoselskii: Two remarks on the method of successive approximations, Uspekhi Matematicheskikh Nauk, vol. 10, pp. 123–127, 1955.
16. R. Khalil, M. A. Horani, A. Yousuf, Sababheh M.: A new definition of fractional derivative. J. Comput. Appl. Math. 264(2014), 65-70.
17. A. Lakoud, R. Khaldi, A. Kilicman: Existence of solutions for a mixed fractional boundary value problem. Adv. Difference Equ. 2017, 1 (2017), 164.
18. S. Ntouyas, A. Alsaedi, B. Ahmad: Existence theorems for mixed Riemann-Liouville and Caputo fractional differential equations and inclusions with nonlocal fractional integro-differential boundary conditions. Fract. Fract. 3, 2(2019), 21.
19. I. Podlubny; Fractional differential equations, *Academic Press, San Diego*, 1999.
20. I. A. Rus: Ulam stability of ordinary differential equations. Stud. Univ. Babes-Bolyai, Math, LIV, 4 (2009), 125-133.
21. S. Samko, A. Kilbas, O. Marichev: Fractional integral and Derivatives(Theory and Applications). Gordonnd Breach, Switzerland, 1993.
22. J. Wang, L. Lv, Y. Zhou: Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electron. J. Qual. Theory Differ. Equ. 63(2011), 1-10.

Ramdas Narute,
Department of Mathematics,
Savitribai Phule Pune University, Pune-411007, India (M. S.)
E-mail address: ramthemagic@gmail.com

and

V. V. Kharat,
Department of Mathematics,
School of Computational Sciences,
JSPM University Pune-412207, India (M.S.).
E-mail address: vvkvinod9@gmail.com

and

Subhash Kendre,
Department of Mathematics,
Savitribai Phule Pune University, Pune-411007, India (M. S.)
E-mail address: drsubhash2010@gmail.com