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Applications of the Neutrosophic Poisson Distribution to Biunivalent Functions Involving Chebyshev Polynomials and q-Derivative Operator

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ABSTRACT: In this paper, we use the q-derivative operator to present a new subclass of analytic and bi-univalent functions defined in the open unit disk associated with generalised neutrosophic Poisson distribution series and Chebyshev polynomials. We obtain the bounds of the initial two coefficients and the Fekete Szego problem for this class.

Keywords: Analytic functions, univalent and Bi-univalent functions, neutrosophic Poisson distribution, Chebyshev polynomials, Fekete-Szegö inequality, q-derivative operator.

Contents

1. Introduction

1cm We denote by $\mathcal{A}$ the collection of functions, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

which satisfies the normalized condition $f(0) = f'(0) - 1 = 0$. We also denote by $\mathcal{S}$ the sub-collection of the set $\mathcal{A}$ consisting of functions which are univalent (one-one) in $\mathbb{D}$. The Koebe one-quarter theorem [10] asserts that the image of $\mathbb{D}$ under each univalent function f in $\mathcal{S}$ contains a disk of radius $1/4$. According to this, every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad \{z \in \mathbb{D}\}$$

and

$$f(f^{-1}(w)) = w, \qquad \left\{|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right\},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Let Σ stand for the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). For more basic results one may refer to [16,2,8,12] and the results by Srivastava et al. [25,26].

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{D}$. We say that the function f is subordinate to g if there exists a Schwarz function w, which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{D})$, such that $f(z) = g(w(z))$. This subordination relation is denoted by $f(z) \prec g(z)$ for $z \in \mathbb{D}$.

It is well known that, if the function g is univalent in $\mathbb{D}$, then (see [18])

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

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For $q \in (0,1)$, the Jackson q-derivative of a function $f \in \mathcal{A}$ is given by (see [14,15])

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0. \end{cases} \tag{1.3}$$

From (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

where $\quad [n]_q = \dfrac{1-q^n}{1-q}$. If $q \to 1^-$, then $[n]_q \to n$.

In this paper, we used Chebyshev polynomials and it play a considerable role in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and research articles related to specific orthogonal polynomials of Chebyshev family contain essentially results of Chebyshev polynomials of first and second kinds $T_n(t)$ and $U_n(t)$ and their numerous uses in different applications, see [9,11,17,20]. The Chebyshev polynomials of the first and second kinds are orthogonal for $t \in [-1,1]$ and defined as follows:

Definition 1.1 The Chebyshev polynomials of the first kind are defined by the following recurrence relation:

$$T_0(t) = 1, \quad T_1(t) = t,$$
$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

The first few of the Chebyshev polynomials of the first kind are

$$T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1, ... \tag{1.5}$$

The generating function for the Chebyshev polynomials of the first kind, $T_n(t)$, is given by:

$$F(z,t) = \sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad \text{for} \quad z \in \mathbb{D}.$$

Definition 1.2 The Chebyshev polynomials of the second kind are defined by the following recurrence relation:

$$U_0(t) = 1, \quad U_1(t) = 2t,$$
$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t).$$

The first few of the Chebyshev polynomials of the second kind are

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, ... \tag{1.6}$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$, is given by:

$$H(z,t) = \sum_{n=0}^{\infty} U_n(t) z^n = \frac{1}{1-2tz+z^2} \quad \text{for} \quad z \in \mathbb{D}.$$

The Chebyshev polynomials of the first and second kinds are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

A variable x is said to have Poisson distribution if it takes the values $0, 1, 2, 3, ...$ with probabilities

$$e^{-m}, \quad m\frac{e^{-m}}{1!}, \quad m^2\frac{e^{-m}}{2!}, \quad m^3\frac{e^{-m}}{3!}, \ldots.$$

respectively, where m is called the parameter. Thus

$$P(x=r)=\frac{m^r e^{-m}}{r!}, \qquad r=0,1,2,3,\ldots.$$

Recently, Porwal [21] introduced a power series whose coefficients are probabilities of Poisson distribution

$$K(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n \quad \text{for} \quad m>0, \quad z\in\mathbb{D}. \tag{1.7}$$

We note that, by ratio test, the radius of convergence of the above series is infinity.

The concept of neutrosophic theory was introduced by Smarandache in 1995. This new field of philosophy is a generalisation of both intrinsic and fuzzy logic. Fuzzy logic is a new branch of philosophy that provides a new foundation for dealing with issues that have indeterminate data (see [23,1,24,13] for neutrosophic numbers) and the references therein. The application of neutrosophic crisp sets theory with the classical probability distributions particularly Poisson distribution, Exponential distribution and uniform distribution, which opens way for handling problems involving the classical distributions and at the same time contain data not specified accurately. Neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of x, but its parameter is imprecise. For example, m can be set with two or more elements. A variable x is said to be a neutrosophic Poisson distribution if it takes values $0,1,2,\ldots$ the probability $e^{-m_N}, \frac{m_N e^{-m_N}}{1!}, \frac{m_N^2 e^{-m_N}}{2!},\ldots$ respectively and m_N is called distribution parameters which are equal to the expected values and the variance. Hence,

$$NP(x=k)=e^{-m_N}\frac{(m_N)^k}{k!}, \quad k=0,1,2,\ldots$$

That is

$$NE(x)=NV(x)=m_N$$

and $N=d+I$ (where d is determinate part and I is an indeterminate part) is a neutrosophic statistical number (see [23]). Now we modify (1.7) as follows

$$K(m_N,z)=z+\sum_{n=2}^{\infty}\frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}z^n. \tag{1.8}$$

We define a linear operator by

$$I_{m_N}f(z)\equiv K(m_N,z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}a_n z^n \quad \text{for} \quad z\in\mathbb{D}, \tag{1.9}$$

where $*$ denote the convolution (or Hadamard product) of two series.

This linear operator possesses an elegant structure combining exponential generating functions and combinatorial coefficients, with the factor e^{-m_N} suggesting a connection to the Poisson distribution. This form enables the construction of new sub classes of analytic and bi-univalent functions, facilitating the study of coefficient estimates, growth, and distortion properties. The operator's formulation allows for meaningful subordination conditions and links to probabilistic and geometric function theory, thereby enriching the analytical framework and offering deep insight into the behavior of complex functions.

Motivated by earlier studies on bi-univalent functions [12,22,27] and present investigation of bi-univalent functions associated with various polynomials and neutrosophic poisson distribution as well as by many recent works on the Fekete-Szegö functional and other coefficient estimates (see [3,19]). In this work we present and investigate a new subclass $\mathfrak{N}_\Sigma(\alpha,\beta,m_N,t,q)$ of the function class Σ involving q-derivative operator related with Neutrosophic-poisson distribution and Chebyshev polynomials.

Definition 1.3 For $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $t \in (0,1]$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{N}_\Sigma(\alpha, \beta, m_N, t)$ if it satisfies the subordinations

$$(1-\alpha)\left[\beta\frac{(z(I_{m_N}f(z))')'}{(I_{m_N}f(z))'}+(1-\beta)\frac{z(I_{m_N}f(z))'}{I_{m_N}f(z)}\right]+\alpha\left[\frac{(1-\beta)z(I_{m_N}f(z))'+\beta z(z(I_{m_N}f(z)')')'}{(1-\beta)I_{m_N}f(z)+\beta z(I_{m_N}f(z))'}\right] \prec \frac{1}{1-2tz+z^2} =: H(z,t)$$

and

$$(1-\alpha)\left[\beta\frac{(w(I_{m_N}g(w))')'}{(I_{m_N}g(w))'}+(1-\beta)\frac{w(I_{m_N}g(w))'}{I_{m_N}g(w)}\right]+\alpha\left[\frac{(1-\beta)w(I_{m_N}g(w))'+\beta w(w(I_{m_N}g(w))')'}{(1-\beta)I_{m_N}g(w)+\beta w(I_{m_N}g(w))'}\right] \prec \frac{1}{1-2tw+w^2} =: H(w,t)$$

where the function $g = f^{-1}$ is given by (1.2).

The q-analogue to the function class $\mathfrak{N}_\Sigma(\alpha,\ \beta, m_N, t)$ is provided in the following manner:

Definition 1.4 For $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $t \in (0,1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{N}_\Sigma(\alpha, \beta, m_N, t, q)$ if it satisfies the subordinations

$$(1-\alpha)\left[\beta\frac{D_q(zD_q(I_{m_N}f(z)))}{D_q(I_{m_N}f(z))}+(1-\beta)\frac{zD_q(I_{m_N}f(z))}{I_{m_N}f(z)}\right]+\alpha\left[\frac{(1-\beta)zD_q(I_{m_N}f(z))+\beta zD_q(zD_q(I_{m_N}f(z))}{(1-\beta)I_{m_N}f(z)+\beta zD_q(I_{m_N}f(z))}\right] \prec H(z,t)$$

and

$$(1-\alpha)\left[\beta\frac{D_q(wD_q(I_{m_N}g(w)))}{D_q(I_{m_N}g(w))}+(1-\beta)\frac{wD_q(I_{m_N}g(w))}{I_{m_N}g(w)}\right]+\alpha\left[\frac{(1-\beta)wD_q(I_{m_N}g(w))+\beta wD_q(wD_q(I_{m_N}g(w)))}{(1-\beta)I_{m_N}g(w)+\beta wD_q(I_{m_N}g(w))}\right] \prec H(w,t)$$

where the function $g = f^{-1}$ is given by (1.2).

Using the suitably fixed parameters α and β, we state the following subclasses :

Definition 1.5 For $\alpha = 0$, $0 \leq \beta \leq 1$, $t \in (0,1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{N}_\Sigma(\beta, m_N, t, q)$ if it satisfies the subordination

$$\left[\beta\frac{D_q(zD_q(I_{m_N}f(z)))}{D_q(I_{m_N}f(z))}+(1-\beta)\frac{zD_q(I_{m_N}f(z))}{I_{m_N}f(z)}\right] \prec \frac{1}{1-2tz+z^2} =: H(z,t)$$

and

$$\left[\beta\frac{D_q(wD_q(I_{m_N}g(w)))}{D_q(I_{m_N}g(w))}+(1-\beta)\frac{wD_q(I_{m_N}g(w))}{I_{m_N}g(w)}\right] \prec \frac{1}{1-2tw+w^2} =: H(w,t)$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 1.6 For $\alpha = 1$, $0 \leq \beta \leq 1$, , $t \in (0,1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{B}_\Sigma(\beta, m_N, t, q)$ if it satisfies the subordination

$$\left[\frac{(1-\beta)zD_q(I_{m_N}f(z))+\beta zD_q(zD_q(I_{m_N}f(z)))}{(1-\beta)I_{m_N}f(z)+\beta zD_q(I_{m_N}f(z))}\right] \prec \frac{1}{1-2tz+z^2} =: H(z,t)$$

and

$$\left[\frac{(1-\beta)wD_q(I_{m_N}g(w))+\beta wD_q(wD_q(I_{m_N}g(w)))}{(1-\beta)I_{m_N}g(w)+\beta wD_q(I_{m_N}g(w))}\right] \prec \frac{1}{1-2tw+w^2} =: H(w,t).$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 1.7 For $\alpha = 0$ or 1, $\beta = 0$, $t \in (0,1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{S}_\Sigma(m_N, t, q)$ if it satisfies the subordinations

$$\left[\frac{zD_q(I_{m_N} f(z))}{I_{m_N} f(z)}\right] \prec \frac{1}{1 - 2tz + z^2} =: H(z, t)$$

and

$$\left[\frac{wD_q(I_{m_N} g(w))}{I_{m_N} g(w)}\right] \prec \frac{1}{1 - 2tw + w^2} =: H(w, t).$$

where the function $g = f^{-1}$ is given by (1.2)

Definition 1.8 For $\alpha = 0$ or 1, $\beta = 1$, $t \in (0,1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{K}_\Sigma(m_N, t, q)$ if it satisfies the subordinations

$$\left[\frac{D_q(zD_q(I_{m_N} f(z)))}{D_q(I_{m_N} f(z))}\right] \prec \frac{1}{1 - 2tz + z^2} =: H(z, t)$$

and

$$\left[\frac{D_q(wD_q(I_{m_N} g(w)))}{D_q(I_{m_N} g(w))}\right] \prec \frac{1}{1 - 2tw + w^2} =: H(w, t)$$

where the function $g = f^{-1}$ is given by (1.2).

2. Main Results

In this section, we determine the certain coefficient estimates and the Fekete–Szegö-type inequalities for functions in the class $\mathfrak{N}_\Sigma(\alpha, \beta, m_N, t, q)$.

Theorem 2.1 *For $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $t \in (0,1]$ and $0 < q < 1$, let $f \in \mathfrak{N}_\Sigma(\alpha, \beta, m_N, t, q)$. Then*

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\left\{\frac{m_N^2}{2} e^{-m_N}(q + q^2)[1 + \beta(q + q^2)] - \varphi(m_N, q)\right\} 4t^2 + q^2 m_N^2 e^{-m_N^2}[1 + \beta q]^2\right|}}$$

and

$$|a_3| \le \frac{4t}{m_N^2 e^{-m_N}(q + q^2)[1 + \beta(q + q^2)]} + \frac{8t^2}{m_N^2 e^{-m_N^2} q^2 [1 + \beta q]^2}.$$

where

$$\varphi(m_N, q) = q m_N^2 e^{-m_N^2}\left((1 - \alpha)[1 + \beta(q^2 + 2q)] + (\alpha + q)(1 + \beta q)^2\right) \tag{2.1}$$

Proof: Let $f \in \mathfrak{N}_\Sigma(\alpha, \beta, m_N t, q)$. Then there are two analytic functions $u, v : \mathbb{D} \to \mathbb{D}$ such that such that

$$(1-\alpha)\left[\beta\frac{D_q(zD_q(I_{m_N} f(z)))}{D_q(I_{m_N} f(z))} + (1 - \beta)\frac{zD_q(I_{m_N} f(z))}{I_{m_N} f(z)}\right] + \alpha\left[\frac{(1 - \beta)zD_q(I_{m_N} f(z)) + \beta zD_q(zD_q(I_{m_N} f(z)))}{(1 - \beta)I_{m_N} f(z) + \beta zD_q(I_{m_N} f(z))}\right] = H(u(z), t)$$

and

$$(1 - \alpha)\left[\beta\frac{D_q(wD_q(I_{m_N} g(w)))}{D_q(I_{m_N} g(w))} + (1 - \beta)\frac{wD_q(I_{m_N} g(w))}{I_{m_N} g(w)}\right] + \alpha\left[\frac{(1 - \beta)wD_q(I_{m_N} g(w)) + \beta wD_q(wD_q(I_{m_N} g(w)))}{(1 - \beta)I_{m_N} g(w) + \beta wD_q(I_{m_N} g(w))}\right] = H(v(w), t),$$

where the analytic functions u and v are given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + ..., \tag{2.2}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + ..., \tag{2.3}$$

with $u(0) = v(0) = 0$ and $\max\{|u(z)|, |v(w)|\} < 1$ for $z, w \in \mathbb{D}$.

Or, equivalently, that

$$(1 - \alpha)\left[\beta\frac{D_q(zD_q(I_{m_N} f(z)))}{D_q(I_{m_N} f(z))} + (1 - \beta)\frac{zD_q(I_{m_N} f(z))}{I_{m_N} f(z)}\right] + \alpha\left[\frac{(1 - \beta)zD_q(I_{m_N} f(z)) + \beta zD_q(zD_q(I_{m_N} f(z)))}{(1 - \beta)I_{m_N} f(z) + \beta zD_q(I_{m_N} f(z))}\right] = 1 + U_1(t)u(z) + U_2(t)u^2(z) + ... \tag{2.4}$$

$$(1-\alpha)\left[\beta\frac{D_q(wD_q(I_{m_N}g(w)))}{D_q(I_{m_N}g(w))}+(1-\beta)\frac{wD_q(I_{m_N}g(w))}{I_{m_N}g(w)}\right]+\alpha\left[\frac{(1-\beta)wD_q(I_{m_N}g(w))+\beta wD_q(wD_q(I_{m_N}g(w)))}{(1-\beta)I_{m_N}g(w)+\beta wD_q(I_{m_N}g(w))}\right]$$
$$=1+U_1(t)v(w)+U_2(t)v^2(w)+\ldots \tag{2.5}$$

From (2.2), (2.3), (2.4), and (2.5), we have

$$\begin{aligned}(1-\alpha)&\left[\beta\frac{D_q(zD_q(I_{m_N}f(z)))}{D_q(I_{m_N}f(z))}+(1-\beta)\frac{zD_q(I_{m_N}f(z))}{I_{m_N}f(z)}\right]\\ +\alpha&\left[\frac{(1-\beta)zD_q(I_{m_N}f(z))+\beta zD_q(zD_q(I_{m_N}f(z)))}{(1-\beta)I_{m_N}f(z)+\beta zD_q(I_{m_N}f(z))}\right]\\ &=1+U_1(t)u_1z+[U_1(t)u_2+U_2(t)u_1^2]z^2\ldots\end{aligned} \tag{2.6}$$

and

$$(1-\alpha)\left[\beta\frac{D_q(wD_q(I_{m_N}g(w)))}{D_q(I_{m_N}g(w))}+(1-\beta)\frac{wD_q(I_{m_N}g(w))}{I_{m_N}g(w)}\right]+\alpha\left[\frac{(1-\beta)wD_q(I_{m_N}g(w))+\beta wD_q(wD_q(I_{m_N}g(w)))}{(1-\beta)I_{m_N}g(w)+\beta wD_q(I_{m_N}g(w))}\right]$$
$$=1+U_1(t)v_1w+[U_1(t)v_2+U_2(t)v_1^2]w^2+\ldots \tag{2.7}$$

It is well known that, if

$$\max\{|u(z)|,|v(w)|\}<1,\quad z,w\in\mathbb{D},$$

then

$$|u_j|\leq 1\quad and\quad |v_j|\leq 1\quad \forall j\in\mathbb{N} \tag{2.8}$$

Now, by comparing the corresponding coefficients in (2.6) and (2.7) and after some simplification, we get

$$m_Ne^{-m_N}q[1+\beta q]a_2=U_1(t)u_1, \tag{2.9}$$

$$q\frac{m_N^2}{2}e^{-m_N}\left\{1+\beta(q+q^2)\right\}[2]_qa_3-qm_N^2e^{-m_N^2}\left\{(1-\alpha)(1+\beta(q^2+2q))+\alpha(1+\beta q)^2\right\}a_2^2$$
$$=U_1(t)u_2+U_2(t)u_1^2, \tag{2.10}$$

$$-m_Ne^{-m_N}q[1+\beta q]a_2=U_1(t)v_1 \tag{2.11}$$

$$q\frac{m_N^2}{2}e^{-m_N}\left[1+\beta(q+q^2)\right][2]_q(2a_2^2-a_3)-qm_N^2e^{-m_N^2}\left\{(1-\alpha)(1+\beta(q^2+2q))+\alpha(1+\beta q)^2\right\}a_2^2$$
$$=U_1(t)v_2+U_2(t)v_1^2. \tag{2.12}$$

It follows from (2.9) and (2.11)

$$u_1=-v_1 \tag{2.13}$$

and

$$2m_N^2e^{-m_N^2}q^2[1+\beta q]^2a_2^2=(U_1(t))^2(u_1^2+v_1^2). \tag{2.14}$$

If we add (2.10) and (2.12), we get

$$2q\left\{\frac{m_N^2}{2!}e^{-m_N}[1+\beta(q+q^2)][2]_q-m_N^2e^{-m_N^2}\left\{(1-\alpha)[1+\beta(q^2+2q)]+\alpha(1+\beta q)^2\right\}\right\}a_2^2$$
$$=U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2). \tag{2.15}$$

We substitute the value of $u_1^2+v_1^2$ from (2.14) into the right-hand side of (2.15), it reduce the following relation

$$2q\left\{\frac{m_N^2}{2!}e^{-m_N}[1+\beta(q+q^2)][2]_q-m_N^2e^{-m_N^2}\left\{(1-\alpha)[1+\beta(q^2+2q)]+\alpha(1+\beta q)^2\right\}\right\}(U_1(t))^2a_2^2$$
$$-2m_N^2e^{-m_N^2}q^2(1+\beta q)^2U_2(t)a_2^2=(U_1(t))^3(u_2+v_2),$$

it is equivalent to

$$a_2^2 = \left[\frac{(U_1(t))^3(u_2+v_2)}{2q\left\{\frac{m_N^2}{2}e^{-m_N}[1+\beta(q+q^2)][2]_q - m_N^2 e^{-m_N^2}\left((1-\alpha)[1+\beta(q^2+2q)]+\alpha[1+\beta q]^2\right)\right\}(U_1(t))^2 - 2q^2 m_N^2 e^{-m_N^2}[1+\beta q]^2 U_2(t)}\right] \tag{2.16}$$

From (2.16), we use the relations (1.6) and (2.8) and obtain

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\left\{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)] - \varphi(m_N,q)\right\}4t^2 + q^2 m_N^2 e^{-m_N^2}[1+\beta q]^2\right|}}$$

where $\varphi(m_N,q)$ is given by (2.1). Now, we subtract (2.12) from (2.10) and get the following relation

$$m_N^2 e^{-m_N}(q+q^2)[1+\beta(q+q^2)](a_3-a_2^2) = U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{2.17}$$

From (2.17), we use the relations that (2.13) and (2.14) and obtain

$$a_3 = \frac{U_1(t)(u_2-v_2)}{m_N^2 e^{-m_N}(q+q^2)[1+\beta(q+q^2)]} + \frac{(U_1(t))^2(u_1^2+v_1^2)}{m_N^2 e^{-m_N^2}q^2[1+\beta q]^2}.$$

Thus by applying (1.6), we obtain

$$|a_3| \le \frac{4t}{m_N^2 e^{-m_N}(q+q^2)[1+\beta(q+q^2)]} + \frac{8t^2}{m_N^2 e^{-m_N^2}q^2[1+\beta q]^2}.$$

□

In the next theorem, we present the Fekete-Szegö inequality for the class $\mathfrak{N}_\Sigma(\alpha,\beta,m_N,t,q)$.

Theorem 2.2 *For $0\le\alpha\le1$, $0\le\beta\le1$, $t\in(0,1]$, $0<q<1$ and $\mu\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathfrak{N}_\Sigma(\alpha,\beta,m_N,t,q)$. Then*

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2t}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}; & |\mu-1|\le\aleph \\ \frac{8t^3|\mu-1|}{\left|\left\{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]-\varphi(m_N,q)\right\}4t^2+q^2m_N^2e^{-m_N^2}[1+\beta q]^2\right|}; & |\mu-1|\ge\aleph. \end{cases}$$

where

$$\aleph = \frac{\left|\frac{m_N^2 e^{-m_N^2}q^2[1+\beta q]^2}{4t^2} + \frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)] - \varphi(m_N,q)\right|}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}$$

Proof: It follows from (2.16) and (2.17) that

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{U_1(t)(u_2-v_2)}{m_N^2 e^{-m_N}(q+q^2)[1+\beta(q+q^2)]} + (1-\mu)a_2^2 \\ &= \frac{U_1(t)(u_2-v_2)}{m_N^2 e^{-m_N}(q+q^2)[1+\beta(q+q^2)]} \\ &+ \left[\frac{(U_1(t))^3(u_2+v_2)(1-\mu)}{2q\left\{\frac{m_N^2}{2}e^{-m_N}[1+\beta(q+q^2)][2]_q - m_N^2e^{-m_N^2}\left((1-\alpha)[1+\beta(q^2+2q)]+\alpha[1+\beta q]^2\right)\right\}(U_1(t))^2 - 2q^2m_N^2e^{-m_N^2}[1+\beta q]^2U_2(t)}\right] \\ &= \frac{U_1(t)}{2}\left[\left(\vartheta(\mu,t)+\frac{1}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}\right)u_2 \right. \\ &\left. + \left(\vartheta(\mu,t)-\frac{1}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}\right)v_2\right], \end{aligned}$$

where

$$\vartheta(\mu,t)=\frac{(1-\mu)4t^2}{\left\{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]-\varphi(m_N,q)\right\}4t^2+q^2m_N^2e^{-m_N^2}[1+\beta q]^2}.$$

Then, in view (1.6), we conclude that

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2t}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}; & 0\leq\vartheta(\mu,t)|\leq\frac{1}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}\\ 2t|\vartheta(\mu,t)|; & |\vartheta(\mu,t)|\geq\frac{1}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}.\end{cases}$$

□

Taking $\mu=1$ in Theorem 2.2, we led to the following corollary.

Corollary 2.1 *For $0\leq\alpha\leq1$, $0\leq\beta\leq1$, $t\in(0,1]$, $0<q<1$ and $\mu\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathfrak{N}_\Sigma(\alpha,\beta,m_N,t,q)$. Then*

$$|a_3-a_2^2|\leq\frac{2t}{\frac{m_N^2}{2}e^{-m_N}(q+q^2)[1+\beta(q+q^2)]}.$$

3. Concluding Remarks and Observations

In the present paper, we obtained the upper bounds of initial Taylor coefficients of a new class of bi-univalent functions connected with the Neutrosophic Poisson distribution and Chebyshev polynomials. Also, we discussed the Fekete-Szegö inequality. Further, by fixing $\alpha=0$, $\alpha=1$, $\beta=0$, and $\beta=1$, we can state the above results for function classes given by the definitions 1.5, 1.6, 1.7, and 1.8.

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