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Solution of Differential Equations via Chebyshev Operational Matrix of Integration

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ABSTRACT: In this paper, operational matrices for integration and product operations of the fourth kind Chebyshev wavelets are constructed. These matrices are utilized to obtain solutions to linear differential equations, which indicate that these matrices are applicable for fourth kind Chebyshev wavelets. Solutions of the differential equations considered in this paper, resemble with their exact solutions. The characteristics of fourth kind Chebyshev wavelets are utilized to transform differential equations to the systems of algebraic equations, which are solved very efficiently using appropriate methods.

Keywords: Chebyshev wavelets (CW), operational matrix of integration (OMI), product operation matrix (POM), Newton's law of cooling differential equation, Bessel's differential equation, third order singular differential equation.

Contents

1. Introduction

Wavelets are compact or short-lived waves. Rather than continuing to oscillate indefinitely, they return to zero. Wavelets allow for the precise representation of various functions. They are considered as basis functions $\psi_{i,j}(t)$ in continuous time. A distinct characteristic of the wavelet basis is that each function $\psi_{i,j}(t)$ is derived from $\psi(t)$, which is known as mother wavelet. Typically, a collection of linearly independent (L.I.) functions is generated through the translation and dilation of the mother wavelet.
The theory of wavelets is relatively a new and developing domain in the field of mathematical research. It integrates into a variety of scientific and engineering disciplines. It's application spans a wide range of technological fields [28], especially in signal analysis [25], time-frequency analysis, and efficient algorithms for straight forward implementation [2]. Linear differential equations arise when we model a real world phenomenon. These models appear widely in science, engineering, economics, medicine and many other fields, for examples, motion with resistence, electrical systems, fluid mechanics, quantum mechanics etc.

In numerous instances, obtaining analytical solutions for linear initial value problems is not feasible. For these situations, we employ numerical method like operational matrix of integrations (OMI). The operational matrix of integration (OMI) is a highly effective numerical method for solving linear differential equations. This technique is based on converting differential equations into integral equations using

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operational matrix of integration by eliminating the integral operator in order to reduce the problem into a system of algebraic equations, which is further solved by suitable methods.

Employing OMI of Haar wavelets to address differential equations, a limitation arises due to a jump discontinuity at $x = 1/2$. Therefore, exploring new approaches to solve and examine differential equations has become a fascinating topic within the realm of wavelets.

Since Chebyshev wavelets are very useful wavelet methods, therefore, in this study, we develop the operational matrix of integration (OMI) and the product operation matrix (POM) for fourth kind Chebyshev wavelets and use this technique to solve some of the most important linear differential equations. The suggested technique for solving these differential equations utilizes a limited number of bases and takes advantage of the orthogonality of Chebyshev wavelets of fourth kind to transform the linear differential equations into a straightforward system of algebraic equations. One can find more details on orthogonal functions and polynomials in [3,5,7,16,26].

Further, in case of linear differential equations, we consider Newton's law of cooling, Bessel's differential equations and third order singular differential equation, which are solved using operational matrix of integration for $k = 3$ and $M = 4$ and product operational matrix. Newton's law of cooling differential equation has applications in modeling velocity decay of objects moving through viscous media in the field of motion with resistance; charging of capacitors in the field of electrical systems, while Bessel's differential equation has applications in laminar flow in pipes and cylindrical geometries in the field of fluid mechanics; solutions to the Schrödinger equation in cylindrical or spherical potentials in the field of quantum mechanics. Our third order singular differential equation has applications in models physical systems with time-dependent or spatially varying parameters, such as unsteady fluid flow or mechanical oscillations and its analysis provides insights into variable-coefficient dynamical systems frequently encountered in engineering and applied sciences.

It can be noted that in recent past, particular emphesis has been placed on the use of Legendre wavelets [7,6,11,27] and Hybrid functions [9,10,13,12]. Some most recent work in the context of present paper can also be found in [1,15,17,18,19,20,21,22,23,24,4,7,27,29].

The organization of this paper is outlined as follows: section 2 contains key definitions pertinent to the current study. In section 3, we present a format of operational matrices related to integration and the product operation matrix (POM) for Chebyshev wavelets (CW) of the fourth kind. In section 4, we find solutions of Newton's law of cooling differential equation, Bessel's differential equation and third order singular differential equation using proposed method and compare these solutions with their exact solutions. In section 5, conclusion is given.

2. Preliminaries

2.1. Wavelets and Chebyshev wavelets (CW)

Wavelets consist of a set of functions obtained from the dilation and translation of a particular function referred to as the mother wavelet. By continuously varying the parameters for dilation and translation, represented by a and b, we generate a continuous family of wavelets.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R},$$

where $\psi_{a,b}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$ provided $a \neq 0$ [14]. We consider the parameters a and b as discrete values $a = a_0^{-n}$, $b = mb_0a_0^{-n}, a_0 > 1, b_0 > 0$, where n and m are positive integers. Specifically, when $a_0 = 2$ and $b_0 = 1$, the set $\{\psi_{n,m}(t)\}$ constitutes an orthonormal basis.
The Chebyshev wavelets denoted as $\psi_{n,m}(t) = \psi(k, n, m, t)$ rely on four parameters. Here, n takes the values $1, 2, 3, ..., 2^{k-1}$, where k belongs to positive integer, t is time and $m = 0, 1, ..., M - 1$, which is the degree of Chebyshev polynomials (C.P).
Now, we define $\psi_{n,m}(t)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{1}{\pi}}\, W_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where m and n are as defined above.
In equation (2.1), the factor $\sqrt{\frac{1}{\pi}}$ is utilized to maintain orthonormality. The polynomials $W_m(t)$ denote the Chebyshev polynomials of fourth kind with degree m, which exhibit orthogonality concerning the weight function $w(t) = \sqrt{\frac{1-t}{1+t}}$ over the interval [-1,1], and they follow this recursive relationship:

$$W_m(t) = 2tW_{m-1}(t) - W_{m-2}(t); \qquad m = 2, 3, ...$$

We note that

$$W_0(t) = 1, W_1(t) = 2t + 1, W_2 = 4t^2 + 2t - 1, W_3(t) = 8t^3 + 4t^2 - 4t - 1, ...$$

It is noted that when working with Chebyshev wavelets, the weight function $w(t)$ must undergo dilation and translation, given by $w_n(t) = w(2^k t - 2n + 1)$, in order to obtain orthogonal wavelets.

2.2. Approximation of a function using Chebyshev wavelets.

A function $f \in L^2_{\tilde{w}}[0, 1)$, where $\tilde{w}(t) = w(2t - 1)$, can be expended as the following wavelet series

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{2.2}$$

where $c_{nm} = \langle f(t), \psi_{nm}\rangle$. $\langle , \rangle$ denotes the inner product with respect the weight function $w(t)$. After truncating (2.2), we have

$$\begin{aligned} f(t) &\simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \\ &= C^T\psi(t), \end{aligned} \tag{2.3}$$

where

$$C = [c_{1,0},\ c_{1,1}\ c_{1,2}, ..., c_{1,M},\ c_{2,0}, ..., c_{2,M},\ ...c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \tag{2.4}$$

and

$$\psi(t) = [\psi_{1,0}, \psi_{1,1}..., \psi_{1,M}, \psi_{2,0}, ...\psi_{2,M}, ..., \psi_{2,0}, ..., \psi_{2^{k-1},M-1}]^T. \tag{2.5}$$

In (2.4) and (2.5), the order of matrices is $2^{k-1}M \times 1$ and T stands for transposition.

3. Operational Matrices of Integration (OMI)

In this section, we give structures of OMI of fourth kind Chebyshev wavelets.

3.1. Operational matrix of integration of fourth-order Chebyshev wavelets for $k = 3$ and $M = 4$:

Here, we introduce the structure of OMI for fourth-kind Chebyshev wavelets, in particular for $k = 3$ and $M = 4$. The following is an analysis of sixteen basis functions defined on the interval [0,1):

$$\left.\begin{aligned} \psi_{1,0}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \psi_{1,1}(t) &= 2\sqrt{\tfrac{2}{\pi}}(16t - 1), \\ \psi_{1,2}(t) &= 2\sqrt{\tfrac{2}{\pi}}(256t^2 - 48t + 1), \\ \psi_{1,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3 - 1280t^2 + 96t - 1), \end{aligned}\right\} 0 \le t < \frac{1}{4}; \tag{3.1a}$$

$$\left.\begin{aligned} \psi_{2,0}(t) &= 2\sqrt{\tfrac{2}{\pi}}, \\ \psi_{2,1}(t) &= 2\sqrt{\tfrac{2}{\pi}}(16t - 5), \\ \psi_{2,2}(t) &= 2\sqrt{\tfrac{2}{\pi}}(256t^2 - 176t + 29), \\ \psi_{2,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3 - 4352t^2 + 1504t - 169), \end{aligned}\right\} \frac{1}{4} \le t < \frac{1}{2}; \tag{3.1b}$$

$$\left.\begin{aligned}\psi_{3,0}(t) &= 2\sqrt{\tfrac{2}{\pi}},\\ \psi_{3,1}(t) &= 2\sqrt{\tfrac{2}{\pi}}(16t-9),\\ \psi_{3,2}(t) &= 2\sqrt{\tfrac{2}{\pi}}(256t^2-304t+89),\\ \psi_{3,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3-7424t^2+4448t-881),\end{aligned}\right\}\frac{1}{2}\le t<\frac{3}{4}; \tag{3.1c}$$

$$\left.\begin{aligned}\psi_{4,0}(t) &= 2\sqrt{\tfrac{2}{\pi}},\\ \psi_{4,1}(t) &= 2\sqrt{\tfrac{2}{\pi}}(16t-13),\\ \psi_{4,2}(t) &= 2\sqrt{\tfrac{2}{\pi}}(256t^2-432t+181),\\ \psi_{4,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3-10496t^2+8928t-2521),\end{aligned}\right\}\frac{3}{4}\le t<1. \tag{3.1d}$$

Let

$$\psi_{16}(t) = [\psi_{1,0}(t)\ \psi_{1,1}(t)\ \psi_{1,2}(t)\ \psi_{1,3}(t)\ ...\psi_{4,0}(t)\ \psi_{4,1}(t)\ \psi_{4,2}(t)\ \psi_{4,3}]^T. \tag{3.2}$$

By integrating the first basis function between 0 and t, we get

$$\int_0^t \psi_{1,0}(t')dt' = \begin{cases} 2\sqrt{\frac{2}{\pi}}t, & 0\le t<\frac{1}{4};\\ \frac{1}{\sqrt{2\pi}}, & \frac{1}{4}\le t<\frac{1}{2};\\ \frac{1}{\sqrt{2\pi}}, & \frac{1}{2}\le t<\frac{3}{4};\\ \frac{1}{\sqrt{2\pi}}, & \frac{3}{4}\le t<1. \end{cases} \tag{3.3}$$

Expanding L.H.S of (3.3) in form of the basis function, we have

$$\begin{aligned}\int_0^t \psi_{1,0}(t')\,dt' = {}& a_{1,0}\psi_{1,0} + a_{1,1}\psi_{1,1} + a_{1,2}\psi_{1,2} + a_{1,3}\psi_{1,3} + a_{2,0}\psi_{2,0} + a_{2,1}\psi_{2,1} + a_{2,2}\psi_{2,2}\\ & + a_{2,3}\psi_{2,3} + a_{3,0}\psi_{3,0} + a_{3,1}\psi_{3,1} + a_{3,2}\psi_{3,2} + a_{3,3}\psi_{3,3}\\ & + a_{4,0}\psi_{4,0} + a_{4,1}\psi_{4,1} + a_{4,2}\psi_{4,2} + a_{4,3}\psi_{4,3},\end{aligned} \tag{3.4}$$

where the first coefficient is

$$\begin{aligned} a_{1,0} &= \left\langle \int_0^t \psi_{1,0}(t')dt', \psi_{1,0}(t)\right\rangle_{w_n}\\ &= \frac{1}{16}.\end{aligned}$$

Other coefficients of (3.4) can be calculated in the same manner, which are as follows: $a_{1,1} = \frac{1}{16}, a_{1,2} = a_{1,3} = 0, a_{2,0} = \frac{1}{4}, a_{2,1} = a_{2,2} = a_{2,3} = 0, a_{3,0} = \frac{1}{4}, a_{3,1} = a_{3,2} = a_{3,3} = 0,$ $a_{4,0} = \frac{1}{4}, a_{4,1} = a_{4,2} = a_{4,3} = 0.$
Thus, (3.4) is defined as

$$\begin{aligned}\int_0^t \psi_{1,0}(t')dt' &= \frac{1}{16}\psi_{1,0}(t) + \frac{1}{16}\psi_{1,1}(t) + \frac{1}{4}\psi_{2,0}(t) + \frac{1}{4}\psi_{3,0}(t) + \frac{1}{4}\psi_{2,0}(t)\\ &= \left[\frac{1}{16}\ \frac{1}{16}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\psi_{16}(t).\end{aligned} \tag{3.5}$$

By integrating the second basis function from 0 to t, we get

$$\int_0^t \psi_{1,1}(t')dt' = \begin{cases} 2\sqrt{\frac{2}{\pi}}(8t^2-t), & 0\le t<\frac{1}{4};\\ \frac{1}{\sqrt{2\pi}}, & \frac{1}{4}\le t<\frac{1}{2};\\ \frac{1}{\sqrt{2\pi}}, & \frac{1}{2}\le t<\frac{3}{4};\\ \frac{1}{\sqrt{2\pi}}, & \frac{3}{4}\le t<1. \end{cases} \tag{3.6}$$

Expanding L.H.S of (3.6) in form of the basis function, we have

$$\begin{aligned}\int_0^t \psi_{1,1}(t)dt' &= \frac{1}{32}\psi_{1,1} + \frac{1}{32}\psi_{1,2} + \frac{1}{4}\psi_{2,0} + \frac{1}{4}\psi_{3,0} + \frac{1}{4}\psi_{4,0} \\ &= \begin{bmatrix} 0 & \frac{1}{32} & \frac{1}{32} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}\psi_{16}(t). \end{aligned} \tag{3.7}$$

Adopting the similar procedure, we obtain

$$\begin{aligned}
\int_0^t \psi_{1,2}(t')dt' &= \begin{bmatrix} \frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{1,3}(t')dt' &= \begin{bmatrix} \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{2,0}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{2,1}(t')dt &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{2,2}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-1}{48} & \frac{-1}{32} & \frac{-1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{2,3}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{3,0}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{3,1}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{3,2}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{48} & 0 & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{3,3}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & 0 & \frac{1}{12} & 0 & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{4,0}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{4,1}(t')dt' &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{4,2}(t')dt &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} \end{bmatrix} \psi_{16}(t). \\
\int_0^t \psi_{4,3}(t')dt &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} \end{bmatrix} \psi_{16}(t).
\end{aligned}$$

Now, we write

$$\int_0^t \psi_{16}(t')dt' = P_{16\times 16}\psi_{16}(t), \tag{3.8}$$

where $P_{16\times16}$ is an operational matrix of integration (OMI), given by

$$P_{16\times16}=\left[\begin{array}{cccccccccccccccc}
\frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{32} & \frac{1}{32} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
\frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\
\frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 5 & 6 & 6 \\
0 & 0 & 0 & 0 & \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192} & \frac{1}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{1}{32} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{48} & \frac{-1}{32} & \frac{1}{96} & \frac{1}{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{96} & 0 & \frac{-1}{48} & \frac{1}{192}
\end{array}\right]_{16\times16}. \tag{3.9}$$

4. Solution of Linear Differential Equations

In this section, we obtain numerical solutions of three very famous linear differential equations and compare their results with the exact solutions.

4.1. Solution of Newton's law of cooling differential equation

Example 4.1 *Solution of Newton's law of cooling differential equation.*

$$y'(t)+2y(t)=50; \tag{4.1}$$

$$y(0)=0. \tag{4.2}$$

Exact solution of (4.1) with (4.2) is $y(t)=25(1-e^{-2t})$.
Here, we solve above initial value problem using Chebyshev wavelets of fourth kind for $k=3$ and $M=4$. Consider

$$y(t)=C^T\psi(t). \tag{4.3}$$

In equation (4.3),

$$C=[c_{1,0}\ c_{1,1}\ c_{1,2}\ c_{1,3}\ c_{2,0}\ c_{2,1}\ c_{2,2}\ c_{2,3}\ c_{3,0}\ c_{3,1}\ c_{3,2}\ c_{3,3}\ c_{4,0}\ c_{4,1}\ c_{4,2}\ c_{4,3}] \tag{4.4}$$

and

$$\psi(t)=[\psi_{1,0}\ \psi_{1,1}\ \psi_{1,2}\ \psi_{1,3}\ \psi_{2,0}\ \psi_{2,1}\ \psi_{2,2}\ \psi_{2,3}\ \psi_{3,0}\ \psi_{3,1}\ \psi_{3,2}\ \psi_{3,3}\ \psi_{4,0}\ \psi_{4,1}\ \psi_{4,2}\ \psi_{4,3}]. \tag{4.5}$$

The components of $\psi(t)$ are provided in (3.1a) to (3.1d). Now, we also express

$$50=\left[25\sqrt{\frac{\pi}{2}}\ 0\ 0\ 0\ 25\sqrt{\frac{\pi}{2}}\ 0\ 0\ 0\ 25\sqrt{\frac{\pi}{2}}\ 0\ 0\ 0\ 25\sqrt{\frac{\pi}{2}}\ 0\ 0\ 0\right]=E^T\psi(t). \tag{4.6}$$

By integrating equation (4.1) from 0 to t and applying equations (4.2) and (4.6), we derive

$$C^T\psi(t)+2C^T\int_0^t\psi(t')dt'=\int_0^t E^T\psi(t)dt. \tag{4.7}$$

Using (3.8), we have

$$\begin{aligned}C^T+2C^TP&=E^TP\\(I+2P^T)C&=P^TE.\end{aligned} \tag{4.8}$$

Now, (4.8) can be written as

$$DC=F, \tag{4.9}$$

where $D = (I + 2P^T)$ and $F = P^T E$.
Equation (4.9) is a set of 16 algebraic equations with 16 unknowns. We solve (4.9) for C and obtain the following matrix:

$$C = \begin{bmatrix} 1.73694138204544 \\ 1.6328920982191 \\ -0.0998049990719927 \\ 0.00411567006482444 \\ 7.21775456585815 \\ 0.990400554216232 \\ -0.0605348794952581 \\ 0.00249628369052611 \\ 10.5420406106764 \\ 0.60070917047221 \\ -0.0367163135071231 \\ 0.00151407478379889 \\ 12.5583249351288 \\ 0.364349056503685 \\ -0.022269602067333 \\ 0.000918334105869401 \end{bmatrix}_{16\times 1}. \tag{4.10}$$

Now, using the value of C, we obtain the value of $y(t)$ in Example 4.1.
In Table 1, a comparison is presented between the estimated solution obtained using the proposed method and the exact solution at various points within the interval [0, 1).

Table 1: Estimated and exact solutions for $y(t)$ in Example 4.1

t	**Estimated value of** $y(t)$ **using CW**	**Exact value of** $y(t)$	**Absolute error**
0.0	0.0002052321	0.0000000000	0.00020523
0.1	4.5319294136	4.5317311731	0.00019824
0.2	8.2415998451	8.241998849	0.00039900
0.3	11.279502544	11.279709098	0.00020655
0.4	13.766604466	13.766775897	0.00017143
0.5	15.803220765	15.803013971	0.00020679
0.6	17.470276085	17.470144702	0.00013138
0.7	18.835028902	18.835075901	0.00004699
0.8	19.952681339	19.952587050	0.00009429
0.9	20.867575535	20.867527794	0.00004774

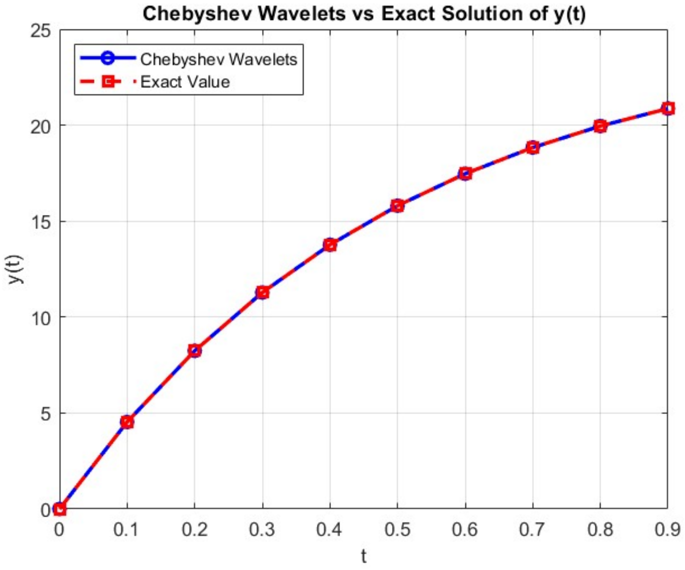

Figure 1: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 4.1.

4.2. Solution of Bessel's differential equation

Example 4.2 *Consider the Bessel's differential equation of order zero.*

$$ty''(t) + y'(t) + ty(t) = 0; \tag{4.11}$$

$$y(0) = 1, y'(0) = 0. \tag{4.12}$$

Bessel's function of first kind of order zero, denoted by $J_0(t)$, is the solution to the differential equation stated above. It can be defined as follows:

$$J_0(t) = \sum_{q=0}^{\infty} \frac{(-1)^q}{(q!)^2} \left(\frac{t}{2}\right)^{2q}. \tag{4.13}$$

Here, we solve the problem using Chebyshev wavelets of fourth kind for $k = 3$ and $M = 4$. Initially, The unknown function $y(t)$ is regarded as

$$y''(t) = C^T \psi(t), \tag{4.14}$$

where C and $\psi(t)$ are given by (4.4) and (4.5) respectively and the elements of $\psi(t)$ are given in (3.1a) to (3.1d).
Integrating (4.14) ranging from 0 to t and using (3.8) and (4.12), we obtain

$$\begin{aligned} y'(t) &\simeq C^T P\psi(t) + y'(0) \\ &= C^T P\psi(t) + A^T\psi(t) \\ y'(t) &= C^T P\psi(t) \end{aligned} \tag{4.15}$$

Now, integrating (4.15) ranging from 0 to t and using (3.8) and (4.12), we obtain

$$\begin{aligned} y(t) &\simeq C^T P^2\psi(t) + ty'(0) + y(0) \\ y(t) &= C^T P^2\psi(t) + B^T\psi(t), \end{aligned} \tag{4.16}$$

where

$$B^T = \left[\frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ \ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{2\sqrt{2}}\ 0\ 0\ 0\right] \tag{4.17}$$

and P is the 16×16 fourth kind Chebyshev wavelets OMI given by (3.9).
Also, we express t as $F^T\psi(t)$ in the following manner

$$\begin{aligned} t &= \begin{bmatrix} \frac{\sqrt{\pi}}{32\sqrt{2}} & \frac{\sqrt{\pi}}{32\sqrt{2}} & 0 \ 0 & \frac{5\sqrt{\pi}}{32\sqrt{2}} & \frac{\sqrt{\pi}}{32\sqrt{2}} & 0 & 0 & \frac{9\sqrt{\pi}}{32\sqrt{2}} & \frac{\sqrt{\pi}}{32\sqrt{2}} & 0 \ 0 & \frac{13\sqrt{\pi}}{32\sqrt{2}} & \frac{\sqrt{\pi}}{32\sqrt{2}} & 0 \ 0 \end{bmatrix} \\ &= F^T\psi(t). \end{aligned} \tag{4.18}$$

Substituting (4.14) to (4.18) in (4.11), we get

$$F^T\psi(t)\psi^T(t)C + \psi^T(t)P^TC + F^T\psi(t)\psi^T(t)P^{2T}C + F^T\psi(t)\psi^T(t)B = 0. \tag{4.19}$$

We shall also make use of the following property concerning the product of two vectors of the fourth kind Chebyshev wavelets function.

$$F^T\psi(t)\psi^T(t) = \psi^T(t)\tilde{F}, \tag{4.20}$$

where $\psi(t)\psi^T(t)$, a product operation matrix of order 16×16, is given by

$$\psi\psi^T(t) = \begin{bmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \cdots & \cdots & \cdots & \psi_{10}\psi_{42} & \psi_{10}\psi_{43} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \cdots & \cdots & \cdots & \psi_{11}\psi_{12} & \psi_{11}\psi_{43} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \cdots & \cdots & \cdots & \psi_{12}\psi_{12} & \psi_{12}\psi_{43} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{42}\psi_{10} & \psi_{42}\psi_{11} & \cdots & \cdots & \cdots & \psi_{42}\psi_{12} & \psi_{42}\psi_{43} \\ \psi_{43}\psi_{10} & \psi_{43}\psi_{11} & \cdots & \cdots & \cdots & \psi_{43}\psi_{12} & \psi_{43}\psi_{43} \end{bmatrix}_{16\times16} \tag{4.21}$$

and

$$\tilde{F} = \begin{bmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix}_{16\times16,} \tag{4.22}$$

where $H_i, i = 1, 2, 3, 4$ are 4×4 matrices.
Here, by computation, we get

$$\tilde{F} = \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16\sqrt{2}} & \frac{1}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{5}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{5}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{5}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{9}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{9}{16} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{9}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{16} & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{13}{16} & \frac{1}{16\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{13}{16} & \frac{1}{16\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{13}{16} \end{bmatrix}_{16\times16} \tag{4.23}$$

Now, using (4.20) in (4.19), we get

$$(\tilde{F} + P^T + \tilde{F}P^{2T})C + \tilde{F}B = 0. \tag{4.24}$$

Equation (4.24) gives a set of 16 algebraic equations with 16 unknowns. We solve (4.24) for C and obtain the following matrix

$$C = \begin{bmatrix} 0.625435433869856 \\ -0.00183061920602593 \\ -0.000608252932943494 \\ 6.0801809463276 \times 10^{-7} \\ 0.610866264832839 \\ -0.00662538970020933 \\ -0.000583188282479087 \\ 3.13689212680988 \times 10^{-6} \\ 0.577518325260104 \\ -0.0111140285540483 \\ -0.00052991367375305 \\ 5.29018401761515e \times 10^{-6} \\ 0.526938365230667 \\ -0.0150858207876289 \\ -0.000452434799967362 \\ 7.16999632822132e \times 10^{-6} \end{bmatrix}_{16\times1.} \tag{4.25}$$

Now, using the value of C, we obtain the value of $y(t)$ in Example 4.2.
In Table 2, a comparison is made between the estimated solution obtained using proposed method together with the exact solution at different points in the interval [0,1).

Table 2: Estimated and exact solutions for $y(t)$ in Example 4.2

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.0	1.0000000	1.000000	0.0000000
0.1	0.9974983	0.997501	0.0000027
0.2	0.9900193	0.990024	0.0000047
0.3	0.9776217	0.977626	0.0000043
0.4	0.9604165	0.960398	0.0000185
0.5	0.9384895	0.938469	0.0000205
0.6	0.9120127	0.912004	0.0000087
0.7	0.8812052	0.881200	0.0000052
0.8	0.8462632	0.846287	0.0000238
0.9	0.8075177	0.807523	0.0000053

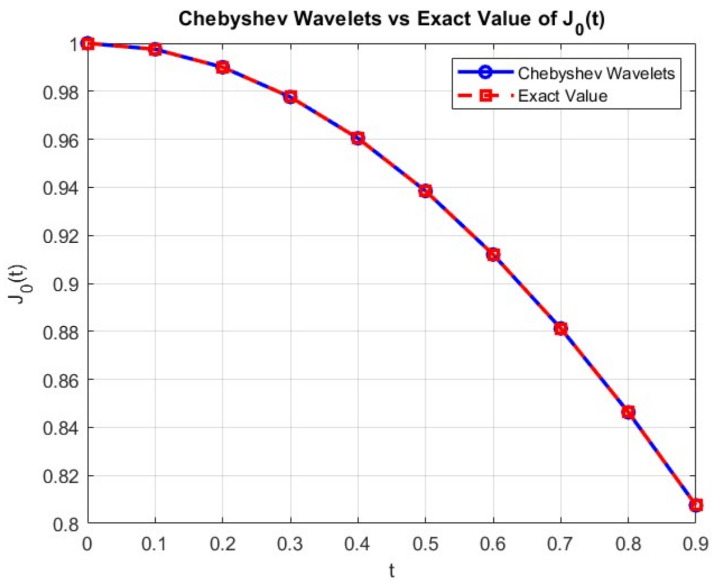

Figure 2: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 4.2.

4.3. Solution of singular differential equation of third order

Example 4.3 *Consider the singular differential equation of third order.*

$$y'''(t) - \frac{2}{t}y(t) - y'(t) = 0; \tag{4.26}$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2. \tag{4.27}$$

The exact solution of (4.26) with (4.27) is $y(t) = te^t$.
Here, we solve the problem using Chebyshev wavelets of fourth kind for $k = 3$ and $M = 4$. Initially, The unknown function $y(t)$ is regarded as

$$y'''(t) = C^T\psi(t), \tag{4.28}$$

where C and $\psi(t)$ are given by (4.4) and (4.5) respectively and the elements of $\psi(t)$ are given in (3.1a) to (3.1d).
Integrating (4.28) ranging from 0 to t and using (3.8) and (4.27), we obtain

$$\begin{aligned} y''(t) &\simeq C^T P\psi(t) + y''(0) \\ &= C^T P\psi(t) + 2B^T\psi(t) \end{aligned} \tag{4.29}$$

Now, integrating (4.29) ranging from 0 to t and using (3.8) and (4.27), we obtain

$$\begin{aligned} y'(t) &\simeq C^T P^2\psi(t) + 2B^T P\psi(t) + y'(0) \\ y'(t) &= C^T P^2\psi(t) + 2B^T P\psi(t) + B^T\psi(t), \end{aligned} \tag{4.30}$$

Now, integrating (4.30) ranging from 0 to t and using (3.8) and (4.27), we obtain

$$\begin{aligned} y(t) &\simeq C^T P^3\psi(t) + 2B^T P^2\psi(t) + B^T P\psi(t) + y(0) \\ y(t) &= C^T P^3\psi(t) + 2B^T P^2\psi(t) + B^T P\psi(t), \end{aligned} \tag{4.31}$$

Substituting (4.28) to (4.31) in (4.26), we get

$$\begin{aligned} &F^T\psi(t)\psi^T(t)C - 2\psi^T(t)P^{3T}C - 2\psi^T(t)P^{2T}B - \psi^T(t)P^TB - F^T\psi(t)\psi^T(t)P^{2T}C \\ &\qquad -2F^T\psi(t)\psi^T(t)P^TB - F^T\psi(t)\psi^T(t)B = 0. \end{aligned} \tag{4.32}$$

Now using (4.17), (4.18) and (4.20) in (4.32), we get

$$(\tilde{F} - 2P^{3T} - \tilde{F}P^{2T})C = (2P^{T} + 4P^{2T} + \tilde{F} + 2\tilde{F}P^{T})B. \tag{4.33}$$

Equation (4.33) gives a set of 16 algebraic equations with 16 unknowns. We solve (4.33) for C and obtain the following matrix

$$C = \begin{bmatrix} 0.0444679337684086 \\ 0.047241305418149 \\ 0.002857605754005 \\ 8.67408286980065e \times 10^{-5} \\ 0.271593462935781 \\ 0.074481874940148 \\ 0.00411859293252955 \\ 0.000123302778143351 \\ 0.624303384699505 \\ 0.113474830835664 \\ 0.00585770180573594 \\ 0.000169933363234506 \\ 1.15585867820509 \\ 0.16863812936225 \\ 0.00824606550279476 \\ 0.000233299037031645 \end{bmatrix}_{16\times 1}. \tag{4.34}$$

Now, using the value of C, we obtain the value of $y(t)$ in Example 4.3.
In Table 3, a comparison is made between the estimated solution obtained using proposed method together with the exact solution at different points in the interval [0,1).

Table 3: Estimated and exact solutions for $y(t)$ in Example 4.3

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.0	0.00013	0.00000	0.00013
0.1	0.11053	0.11052	0.00001
0.2	0.24428	0.24428	0.00000
0.3	0.40485	0.40496	0.00011
0.4	0.59659	0.59673	0.00014
0.5	0.82424	0.82436	0.00012
0.6	1.09327	1.09327	0.00000
0.7	1.40989	1.40963	0.00026
0.8	1.78108	1.78043	0.00065
0.9	2.21486	2.21364	0.00122

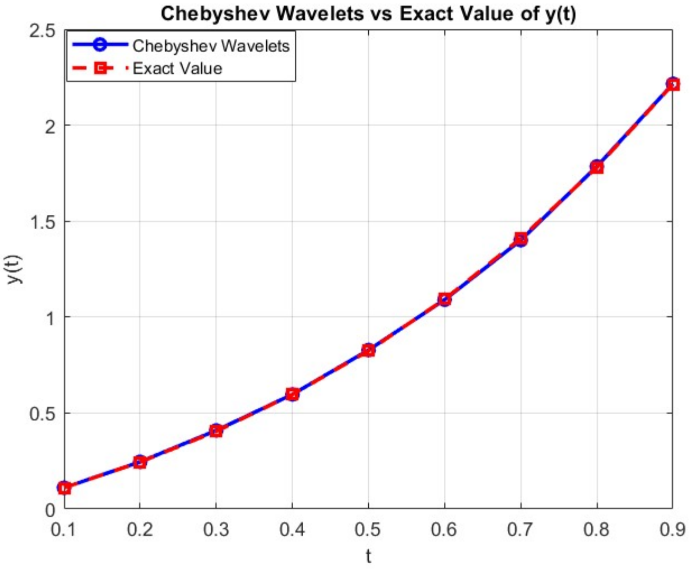

Figure 3: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 4.3.

5. Conclusion

In this work, an efficient and accurate method for solving linear differential equations has been developed. Fourth kind Chebyshev wavelets operational matrices for integration along with a matrix for product operations, have been generated. Firstly, we dealt with linear differential equations namely, Newton's law of cooling differential equation (ordinary) and Bessel's differential equation of zero order (singular) and third order singular differential equation (singular). The differential equations were addressed using the operational matrix of fourth kind Chebyshev wavelets derived for $k = 3$ and $M = 4$. The estimated solutions obtained using present approach, have been compared with their exact solutions. Finally, it is observed that these estimated solutions resemble with their exact solutions.

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