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# Identification of the Source Term of an Ill-Posed Inverse Fractional Parabolic Problem

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ABSTRACT: In this paper, we investigate an ill-posed inverse fractional problem in the sense of Hadamard. Our objective is to identify the source term using Tikhonov regularization. Since the initial condition is missing, the no-regret control approach is employed to solve the regularization problem. The source term is characterized through an associated optimality system.

Key Words: Inverse fractional problem, Tikhonov regularization, identification of the source term, optimality system.

## Contents



## 1. Introduction

Recent advances in fractional calculus have provided powerful tools for modeling various physical and engineering phenomena described by integro-differential and parabolic equations. Several numerical and analytical methods have been developed to address fractional and variable-order systems, including efficient algorithms for solving linear and nonlinear Volterra integro-differential equations [1] and fractional differential equations involving conformable operators [2]. The influence of stochastic perturbations and fractional-order effects on nonlinear dynamical systems has also been analyzed to improve stability and accuracy [3,4]. Moreover, fractional diffusion and parabolic models with integral or over-determination conditions have been extensively investigated to study the existence, uniqueness, and solvability of inverse and direct problems [5,6,7,8,9,10,11,12]. These works demonstrate the increasing importance of fractional modeling techniques in understanding complex diffusion, reaction–diffusion, and control systems, as well as their applications in engineering and applied sciences [13].

The modeling of wave equations has drawn significant attention from researchers due to its wide range of applications in physics, mechanics, and engineering problems (see [14,15] and their references). Many studies have addressed various aspects of the wave equation. The direct problem of the wave equation consists of determining the field when all relevant data are known. If some of this information is missing, the problem becomes an inverse wave problem (see [16]). The classification of inverse wave problems according to the type of missing data is discussed in [17]: if the initial condition is missing, the problem is referred to as a retrospective problem; if the boundary condition is unknown, it is termed a boundary problem; and when the source term is unknown, the problem is known as an inverse source problem.

Inverse wave problems have numerous practical applications and play a vital role in predicting earthquakes, exploring oil and gas reserves, and advancing medical imaging technologies. This research focuses

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on the inverse source problem of electromagnetic waves, which involves identifying the electric current density from the tangential trace of the electric field obtained through boundary measurements. However, the considered inverse fractional wave problem is ill-posed in the sense of Hadamard. To address this challenge, we employ Tikhonov regularization to identify the unknown source term. Several earlier studies have successfully used the Tikhonov regularization method to handle similar ill-posed problems (see [18,19]).

The no-regret control approach, initially introduced by Lions [20] and later extended by Nakoulima [21,22], has been applied by several authors (see [16,23]) and is adopted here to deal with the absence of wave speed information. The unknown source term is characterized through an optimality system. This paper is organized as follows. We begin with some fundamental definitions and concepts from fractional calculus. Next, we present the Tikhonov regularization technique for the ill-posed fractional problem. Since our problem involves missing data, both the no-regret and low-regret control methods are applied, and their corresponding characterizations are provided.

## 2. Preliminaries

In this section, we present fundamental concepts on fractional differentiation and integration that are essential for our research ( [24,25]).

**Definition 1.** *For $\alpha \in \mathbb{R}_+$, we define the fractional integral of a function $f$ belonging to $L^1([0,T],X)$ with respect to $\alpha$ as follows:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

*where the function $\Gamma$ is the Gamma function.*

**Definition 2.** *Let $1 < \alpha < 2$. We define:*

1. *The left Caputo derivative:*

$$^{c}D_t^\alpha f = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f^{(2)}(s)\, ds,$$

2. *The right Caputo derivative:*

$$_t^cD^\alpha f = \frac{1}{\Gamma(2-\alpha)} \int_t^T (s-t)^{1-\alpha} f^{(2)}(s)\, ds,$$

3. *The left Riemann–Liouville derivative:*

$$^{R}D_t^\alpha f = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} f(s)\, ds,$$

4. *The right Riemann–Liouville derivative:*

$$_t^RD^\alpha f = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^T (s-t)^{1-\alpha} f(s)\, ds.$$

*The right Caputo and Riemann–Liouville derivatives are connected by the following relationship:*

$$^{R}D_t^\alpha f = {}^{c}D_t^\alpha f + \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(0)t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

If $f(0) = 0$ and $f'(0) = 0$, then $^{R}D_t^\alpha$ and $^{c}D_t^\alpha$ coincide, i.e.,

$$^{R}D_t^\alpha f = {}^{c}D_t^\alpha f.$$

**Definition 3.** *Let us define the semi-norms for any value of $\sigma > 0$ as follows:*

$$
\begin{aligned}
|f|^2_{{}^{l}H^\sigma(\Omega)} &= \|{}^{R}D^\sigma_t f\|^2_{L^2(\Omega)}, \\
|f|^2_{{}^{r}H^\sigma(\Omega)} &= \|{}^{R}_{t}D^\sigma f\|^2_{L^2(\Omega)}, \\
|f|^2_{{}^{c}H^\sigma(\Omega)} &= \left| \frac{\left({}^{R}D^\sigma_t f, {}^{R}_{t}D^\sigma f\right)^2_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{\frac{1}{2}}.
\end{aligned}
$$

*and the norms*

$$\|f\|^2_{{}^{l}H^\sigma(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{{}^{l}H^\sigma(\Omega)}\right)^{\frac{1}{2}},$$

$$\|f\|^2_{{}^{r}H^\sigma(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{{}^{r}H^\sigma(\Omega)}\right)^{\frac{1}{2}},$$

*and*

$$\|f\|^2_{{}^{c}H^\sigma(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{{}^{c}H^\sigma(\Omega)}\right)^{\frac{1}{2}}.$$

*The spaces ${}^{l}H^\sigma_0(\Omega)$ and ${}^{r}H^\sigma_0(\Omega)$ can be defined as the closure spaces of $C^\infty_0(\Omega)$ with respect to the norms $\|\cdot\|^2_{{}^{l}H^\sigma(\Omega)}$ and $\|\cdot\|^2_{{}^{r}H^\sigma(\Omega)}$, respectively.*

**Lemma 1.** *If $f$ belongs to ${}^{l}H^\sigma(\Omega)$ and $g \in C^\infty_0(\Omega)$, for any real $\sigma \in \mathbb{R}_+$, then*

$$\left({}^{R}D^\sigma_t f, g\right)_{L^2(\Omega)} = \left(f, {}^{R}_{t}D^\sigma g\right)_{L^2(\Omega)}.$$

**Lemma 2.** *For $0 < \sigma < 2$, $\sigma \neq 1$, and $f \in H^{\frac{\sigma}{2}}_0(\Omega)$, we have*

$${}^{R}D^\sigma_t f = {}^{R}D^{\frac{\sigma}{2}}_t\, {}^{R}D^{\frac{\sigma}{2}}_t f.$$

**Lemma 3.** *For $\sigma \in \mathbb{R}_+$ such that $\sigma \neq n + \frac{1}{2}$, the semi-norms $|\cdot|_{{}^{l}H^\sigma(\Omega)}$, $|\cdot|_{{}^{r}H^\sigma(\Omega)}$, and $|\cdot|_{{}^{c}H^\sigma(\Omega)}$ are equivalent. Hence, we introduce the following notation:*

$$|\cdot|_{{}^{l}H^\sigma(\Omega)} \cong |\cdot|_{{}^{r}H^\sigma(\Omega)} \cong |\cdot|_{{}^{c}H^\sigma(\Omega)}.$$

**Lemma 4.** *For $\sigma > 0$, the space ${}^{r}H^\sigma(\Omega)$ is complete under the norm $\|\cdot\|_{{}^{r}H^\sigma(\Omega)}$.*

**Definition 4.** *Let $\phi \in C([0,T], X)$ be such that $D^\alpha \phi \in L^2((0,T), X)$, and let $y$ be such that $D^\alpha_t y \in L^2((0,T), X)$ and $y(0) \in X$. Then*

$$
\begin{aligned}
\int_0^T \int_\Omega {}^{c}D^\alpha_t y(x,t)\, \phi(x,t)\, dx\, dt = &\int_\Omega \frac{\partial}{\partial t} y(x,T)\, I^{2-\alpha}\phi(x,T)\, dx - \int_\Omega \frac{\partial}{\partial t} y(x,0)\, I^{2-\alpha}\phi(x,0)\, dx \\
&- \int_\Omega y(x,T)\, \frac{\partial}{\partial t} I^{2-\alpha}\phi(x,T)\, dx + \int_\Omega y(x,0)\, \frac{\partial}{\partial t} I^{2-\alpha}\phi(x,0)\, dx \\
&+ \int_0^T \int_\Omega y(x,t)\, D^\alpha_{RL}\phi(x,t)\, dx\, dt.
\end{aligned}
$$

## 3. Description of the Problem

This section concerns the inverse source fractional problem associated with an ill-posed wave equation. The objective is to determine the source function using the Tikhonov regularization method. We consider the following fractional wave equation:

$$
\begin{cases}
\partial^\beta_{0+} y(x,t) - \Delta y(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\
y(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \\
y(x,0) = g_1(x), \quad x \in \Omega, \\
y_t(x,0) = g_2(x), \quad x \in \Omega.
\end{cases}
\tag{3.1}
$$

Here, $y$ represents the propagation of the wave equation, $f$ is the unknown source term representing the density of electric current, and $(g_1, g_2)$ are the missing speed components of the wave equation. The operator $\partial_{0^+}^{\beta}$ denotes the Caputo fractional derivative of order $\beta$.

**Proposition 1** ( [23]). *If $f \in L^2(Q)$ and $(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, then the system (3.1) has a unique continuous solution denoted by $y(f, g_1, g_2) = y(t, x, f, g_1, g_2) \in L^2(Q)$.*

Problem (3.1) is classified as an ill-posed problem in the sense of Hadamard. This implies that the solution does not depend continuously on the given data. However, by applying a regularization method, we can obtain an identification of the source term based on the final boundary observation

$$y(x, T) = h(x).$$

The Tikhonov regularization problem is to find $f \in L^2(Q)$ satisfying

$$\inf_{f \in L^2(Q)} J(f, g_1, g_2), \tag{3.2}$$

where $J$ is defined by

$$J(f, g_1, g_2) = \|y(T, f, g_1, g_2) - h\|_{L^2(Q)}^2 + \lambda \|f\|_{L^2(Q)}^2, \tag{3.3}$$

and $\lambda$ is the regularization parameter. Thus, the Tikhonov regularization problem becomes an optimal control problem with incomplete data. Since the functions $(g_1, g_2)$ are unknown, problem (3.2) has no sense in its current form. Therefore, to solve our Tikhonov regularization problem, *we use the no-regret control approach.*

**Remark 1.** *We choose the control function as the source term $f$. Solving the optimal control problem with incomplete data provides an identification of the source term.*

## 4. No-Regret and Low-Regret Controls Method

As discussed previously, the Tikhonov regularization problem becomes an optimal control problem with incomplete data. For this reason, we apply the method of no-regret and low-regret controls. We aim to solve the following infsup problem:

$$\inf_{f \in L^2(Q)} \sup_{(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(\Omega)} J(f, g_1, g_2). \tag{4.1}$$

In this situation, we have $\sup_{(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(\Omega)} J(f, g_1, g_2)$ is equal to infinity. Therefore, we only consider the functions $f$ satisfying

$$J(f, g_1, g_2) \leq J(0, g_1, g_2). \tag{4.2}$$

Then, we solve the following problem:

$$\inf_{f \in L^2(Q)} \sup_{(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(\Omega)} \left[ J(f, g_1, g_2) - J(0, g_1, g_2) \right]. \tag{4.3}$$

**Definition 5.** *A function $w$ is called a no-regret control for problems (3.1) and (3.3) if and only if it solves problem (4.3).*

**Lemma 5.** *For every $f \in L^2(Q)$, problem (4.3) is equivalent to*

$$\inf_{f \in L^2(Q)} \left[ J(f, 0, 0) - J(0, 0, 0) + 2 \sup_{(g_1, g_2) \in H_0^1(\Omega) \times H_0^1(\Omega)} \left[ (I^{2-\alpha}\xi(0), g_1)_{L^2(\Omega)} - \left( \frac{\partial}{\partial t} I^{2-\alpha}\xi(0), g_2 \right)_{L^2(\Omega)} \right] \right],$$

*where $\xi = \xi(f, t, x)$ is the solution of*

$$\begin{cases} D_{RL}^{\alpha}\xi - \Delta\xi = y(f, 0), & (x, t) \in \Omega \times (0, T), \\ \xi = 0, & (x, t) \in \Gamma \times (0, T), \\ I^{2-\alpha}\xi(T) = 0, & x \in \Omega, \\ \dfrac{\partial}{\partial t} I^{2-\alpha}\xi(T) = 0, & x \in \Omega. \end{cases} \tag{4.4}$$

*Proof.* By the linearity of the state $y$, we can write

$$y(f, g_1, g_2) = y(f, 0, 0) + y(0, g_1, g_2).$$

Using this relation, we easily obtain

$$J(f, g_1, g_2) - J(0, g_1, g_2) = J(f, 0, 0) - J(0, 0, 0) + 2\left(y(f, 0, 0), y(0, g_1, g_2)\right)_{L^2(Q)}.$$

By introducing the adjoint state $\xi$ that satisfies (4.4), and applying integration by parts, we get

$$J(f, g_1, g_2) - J(0, g_1, g_2) = J(f, 0, 0) - J(0, 0, 0) + 2\left[(I^{2-\alpha}\xi(0), g_1)_{L^2(\Omega)} - \left(\frac{\partial}{\partial t} I^{2-\alpha}\xi(0), g_2\right)_{L^2(\Omega)}\right].$$

Since the no-regret control exists only on the following subset:

$$K = \left\{\forall f \in L^2(Q) \text{ such that } (I^{2-\alpha}\xi(0), g_1)_{L^2(\Omega)} = 0,\ \left(\frac{\partial}{\partial t} I^{2-\alpha}\xi(0), g_2\right)_{L^2(\Omega)} = 0\right\}.$$

Constructing this subset is difficult. To overcome this problem, we relax our problem by introducing a quadratic perturbation to (4.2), obtaining

$$J(f, g_1, g_2) \leq J(0, g_1, g_2) + \gamma\left(\|g_1\|^2 + \|g_2\|^2\right).$$

The infsup problem then becomes

$$\begin{aligned}
&\inf_{f\in L^2(Q)} \sup_{(g_1,g_2)\in H_0^1(\Omega)\times H_0^1(\Omega)} \left[J(f, g_1, g_2) - J(0, g_1, g_2) - \gamma\left(\|g_1\|^2 + \|g_2\|^2\right)\right] \\
&\quad = \inf_{f\in L^2(Q)} \Bigg[J(f, 0, 0) - J(0, 0, 0) + \sup_{(g_1,g_2)\in H_0^1(\Omega)\times H_0^1(\Omega)} \Bigg[(2I^{2-\alpha}\xi(0), g_1)_{L^2(\Omega)} \\
&\qquad - \left(2\frac{\partial}{\partial t} I^{2-\alpha}\xi(0), g_2\right)_{L^2(\Omega)} - \gamma\left(\|g_1\|^2 + \|g_2\|^2\right)\Bigg]\Bigg].
\end{aligned}$$

This means that

$$\inf_{f\in L^2(Q)} J_\gamma(f), \tag{4.5}$$

where

$$J^\gamma(f) = J(f, 0, 0) - J(0, 0, 0) + \frac{1}{\gamma}\left(\|I^{2-\alpha}\xi(0)\|^2 + \left\|\frac{\partial}{\partial t} I^{2-\alpha}\xi(0)\right\|^2\right). \tag{4.6}$$

Hence, the problem becomes a classical optimal control problem. □

### 4.1. Low-regret control (definition, existence, uniqueness, and characterization)

This section is divided into two parts. First, we prove the existence and uniqueness of the low-regret control. Then, using the first-order Euler–Lagrange optimality condition, we obtain the characterization of the low-regret control.

**Definition 6.** *A function $w_\gamma$ is called a low-regret control for problems (3.1) and (4.6) if and only if it solves problem (4.5).*

*4.1.1. Existence and uniqueness of a low-regret control.* The problem admits a unique low-regret control. From (4.6), it can be deduced that

$$J^\gamma(f) \geq -J(0,0,0) = \text{constant}.$$

This means that $\inf_{f\in U} J^\gamma(f)$ exists, and we denote it by $d_\gamma$. Let $\{f_n^\gamma\}$ be a minimizing sequence satisfying

$$\lim_{n\to\infty} J^\gamma(f_n^\gamma) = \inf_f J^\gamma(f) = d_\gamma.$$

We have

$$\begin{aligned} -J(0,0,0) \leq J^\gamma(f_n^\gamma) &= J(f_n^\gamma,0,0) - J(0,0,0) + \frac{1}{\gamma}\left(\|I^{2-\alpha}\xi(0)\|^2 + \left\|\frac{\partial}{\partial t}I^{2-\alpha}\xi(0)\right\|^2\right) \\ &\leq d_\gamma + 1, \end{aligned}$$

which implies that

$$\|y(T,f_n^\gamma,0,0)-h\|^2 + \lambda\|f_n^\gamma\|^2 + \frac{1}{\gamma}\left(\|I^{2-\alpha}\xi_n(0)\|^2 + \left\|\frac{\partial}{\partial t}I^{2-\alpha}\xi_n(0)\right\|^2\right) \leq d_\gamma + J(0,0,0) + 1 = C_\gamma.$$

Considering that the constant $C_\gamma$ is independent of $n$, we deduce that

$$\begin{aligned} \|f_n^\gamma\|_{L^2(Q)} &\leq C_\gamma, \\ \|y(T,f_n^\gamma,0,0)\|_{L^2(Q)} &\leq C_\gamma, \\ \|I^{2-\alpha}\xi_n(0)\|_{L^2(Q)} &\leq \sqrt{\gamma}\,C_\gamma, \\ \left\|\frac{\partial}{\partial t}I^{2-\alpha}\xi_n(0)\right\|_{L^2(Q)} &\leq \sqrt{\gamma}\,C_\gamma. \end{aligned}$$

We have that $y_n^\gamma = y(T,f_n^\gamma,0,0)$ is the solution of

$$\begin{cases} {}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma = f_n^\gamma, & (x,t)\in\Omega\times(0,T), \\ y_n^\gamma(x,t) = 0, & (x,t)\in\Gamma\times(0,T), \\ y_n^\gamma(x,0) = 0, & x\in\Omega, \\ \dfrac{\partial}{\partial t}y_n^\gamma(x,0) = 0, & x\in\Omega. \end{cases} \tag{4.7}$$

Since $y_n^\gamma(x,0)=0$ and $\dfrac{\partial}{\partial t}y_n^\gamma(x,0)=0$, the Caputo and Riemann–Liouville derivatives coincide:

$${}^C\mathcal{D}^\alpha = D^\alpha_{RL}.$$

Hence, we can apply the previous lemma. By introducing a multiplier $y_n^\gamma$, we multiply the first equation in (4.7) by $y_n^\gamma$ and integrate over $Q$:

$$\int_Q \left({}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma\right) y_n^\gamma\,dx\,dt = \int_Q f_n^\gamma y_n^\gamma\,dx\,dt.$$

and

$$\begin{aligned} \int_Q ({}^C\mathcal{D}^\alpha y_n^\gamma) y_n^\gamma\,dx\,dt &= \left({}^C_0\mathcal{D}_t^{\frac{\alpha}{2}}\,{}^C_0\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma,\, y_n^\gamma\right)_{L^2(Q)} \\ &= \left({}^C_0\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma,\, {}^C_t\mathcal{D}_T^{\frac{\alpha}{2}} y_n^\gamma\right)_{L^2(Q)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right)|y_n^\gamma|^2_{{}^cH^{\frac{\alpha}{2}}(Q)} \simeq \cos\left(\frac{\alpha\pi}{2}\right)|y_n^\gamma|^2_{{}^lH^{\frac{\alpha}{2}}(Q)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^C_0\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|^2_{L^2(Q)}. \end{aligned}$$

Moreover, we have

$$-\int_Q \Delta y_n^\gamma\, y_n^\gamma\, dx\, dt = \|\nabla y_n^\gamma\|^2_{L^2(Q)}.$$

By adding the two equalities, we obtain

$$\cos\Big(\frac{\alpha\pi}{2}\Big)\left\|{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|^2_{L^2(Q)} + \|\nabla y_n^\gamma\|^2_{L^2(Q)} = \int_Q f_n^\gamma y_n^\gamma\, dx\, dt.$$

Using the Cauchy inequality, we have

$$2\cos\Big(\frac{\alpha\pi}{2}\Big)\left\|{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|^2_{L^2(Q)} + 2\,\|\nabla y_n^\gamma\|^2_{L^2(Q)} \le \|f_n^\gamma\|^2_{L^2(Q)} + \|y_n^\gamma\|^2_{L^2(Q)}.$$

Applying the Poincaré inequality, we obtain

$$2\cos\Big(\frac{\alpha\pi}{2}\Big)\left\|{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|^2_{L^2(Q)} + (2-c)\,\|\nabla y_n^\gamma\|^2_{L^2(Q)} \le \|f_n^\gamma\|^2_{L^2(Q)}.$$

Finally, we have

$$\left\|{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|^2_{L^2(Q)} + \|\nabla y_n^\gamma\|^2_{L^2(Q)} \le K\,\|f_n^\gamma\|^2_{L^2(Q)}, \quad K = \frac{1}{\min\left\{2\cos\left(\frac{\alpha\pi}{2}\right), (2-c)\right\}}.$$

Therefore, we can deduce that

$$\left\|{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right\|_{L^2(Q)} \le C^\gamma,$$
$$\|\nabla y_n^\gamma\|_{L^2(Q)} \le C^\gamma.$$

We also deduce that

$$\left\|{}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma\right\|_{L^2(Q)} \le C^\gamma.$$

Then there exists a subsequence, still denoted by $(f_n^\gamma)$, $\left({}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma\right)$, and $(y_n^\gamma)$, such that, when $n \to +\infty$,

$$\begin{aligned}
f_n^\gamma &\rightharpoonup w^\gamma && \text{weakly in } L^2(Q),\\
y_n^\gamma &\rightharpoonup y^\gamma && \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big),\\
{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma &\rightharpoonup f && \text{weakly in } L^2(Q),\\
{}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma &\rightharpoonup g && \text{weakly in } L^2(Q).
\end{aligned}$$

Because of the continuity of the fractional and spatial derivatives, we obtain

$$\begin{aligned}
{}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y_n^\gamma &\rightharpoonup {}_0^C\mathcal{D}_t^{\frac{\alpha}{2}} y^\gamma && \text{weakly in } L^2(Q),\\
{}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma &\rightharpoonup {}^C\mathcal{D}^\alpha y^\gamma - \Delta y^\gamma && \text{weakly in } L^2(Q).
\end{aligned}$$

According to the uniqueness of the limit, we obtain

$${}^C\mathcal{D}^\alpha y^\gamma - \Delta y^\gamma = w^\gamma \quad \text{in } Q.$$

From $y_n^\gamma \rightharpoonup y^\gamma$ weakly in $L^2(Q)$, we have

$$y^\gamma(x,0) = 0, \qquad \frac{\partial}{\partial t} y^\gamma(x,0) = 0.$$

It remains to prove the boundary condition. Multiplying the first equality in (4.7) by a test function $\phi \in D(Q)$ such that

$$I^{2-\alpha}\phi(T) = 0, \qquad \frac{\partial}{\partial t} I^{2-\alpha}\phi(T) = 0 \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \Sigma,$$

and integrating over $Q$, we get

$$\int_Q \left({}^C\mathcal{D}^\alpha y_n^\gamma - \Delta y_n^\gamma\right)\phi\, dx\, dt = \int_Q f_n^\gamma\, \phi\, dx\, dt.$$

Integrating by parts and using Green's formula, we obtain

$$\int_Q \left(D_{RL}^\alpha \phi - \Delta\phi\right) y_n^\gamma\, dx\, dt = \int_Q f_n^\gamma\, \phi\, dx\, dt.$$

Passing to the limit as $n \to \infty$, and using the weak convergences established above, we get

$$\int_Q \left(D_{RL}^\alpha \phi - \Delta\phi\right) y^\gamma\, dx\, dt = \int_Q w^\gamma\, \phi\, dx\, dt.$$

Integrating once more, we have

$$\int_Q \left({}^C\mathcal{D}^\alpha y^\gamma - \Delta y^\gamma\right)\phi\, dx\, dt - \int_\Sigma y^\gamma\, \frac{\partial\phi}{\partial\nu} = \int_Q w^\gamma\, \phi\, dx\, dt,$$

so

$$-\int_\Sigma y^\gamma\, \frac{\partial\phi}{\partial\nu} = 0,$$

hence

$$y^\gamma = 0 \quad \text{on } \Sigma.$$

On the other hand, we know that $\xi_n = \xi(f_n)$ is the solution of

$$\begin{cases} D_{RL}^\alpha \xi_n - \Delta\xi_n = y_n, & (x,t) \in \Omega \times (0,T), \\ \xi_n = 0, & (x,t) \in \Gamma \times (0,T), \\ I^{2-\alpha}\xi_n(T) = 0, & x \in \Omega, \\ \dfrac{\partial}{\partial t} I^{2-\alpha}\xi_n(T) = 0, & x \in \Omega. \end{cases} \tag{4.8}$$

Using the same approach as in the previous state equation for $y_n$, we obtain the following energy inequality:

$$\left\| {}_0^{RL}\mathcal{D}_t^{\frac{\alpha}{2}}\xi_n \right\|_{L^2(Q)}^2 + \|\nabla\xi_n\|_{L^2(Q)}^2 \le C\,\|y_n\|_{L^2(Q)}^2\,, \quad C = \frac{1}{\min\left\{2\cos\left(\frac{\alpha\pi}{2}\right), (2-c')\right\}}.$$

and

$$\left\| {}_0^{RL}\mathcal{D}_t^\alpha \xi_n - \Delta\xi_n \right\|_{L^2(Q)} \le C^\gamma.$$

We conclude that

$$\begin{aligned} &\xi_n \rightharpoonup \xi^\gamma && \text{weakly in } L^2(Q), \\ &\xi_n = 0 && \text{on } \Sigma, \end{aligned}$$

which implies that

$$\xi_n \rightharpoonup \xi^\gamma \quad \text{weakly in } \mathcal{D}'(Q).$$

From [23], we deduce that the fractional integral $I^{2-\alpha}$ is continuous from $L^2$ to $L^2$. Hence, we obtain

$$I^{2-\alpha}\xi_n \rightharpoonup I^{2-\alpha}\xi^\gamma \quad \text{weakly in } L^2(Q).$$

Therefore, we have

$$I^{2-\alpha}\xi^\gamma(T) = 0, \quad x \in \Omega, \qquad \frac{\partial}{\partial t} I^{2-\alpha}\xi^\gamma(T) = 0, \quad x \in \Omega.$$

It follows that $\xi^\gamma$ satisfies

$$\begin{cases} D^\alpha_{RL}\xi^\gamma - \Delta\xi^\gamma = y^\gamma, & (x,t)\in\Omega\times(0,T),\\ \xi^\gamma = 0, & (x,t)\in\Gamma\times(0,T),\\ I^{2-\alpha}\xi^\gamma(T) = 0, & x\in\Omega,\\ \dfrac{\partial}{\partial t}I^{2-\alpha}\xi^\gamma(T) = 0, & x\in\Omega. \end{cases} \tag{4.9}$$

The functional $J^\gamma$ is lower semi-continuous; therefore, we have

$$J^\gamma(w_\gamma) \le \lim_{n\to\infty}\inf_{f\in U} J^\gamma(f_n) = \inf_{f\in U} J^\gamma(f) = d_\gamma.$$

Since $J^\gamma$ is also strictly convex, we deduce that $w_\gamma$ is the unique minimizer.

*4.1.2. Determining the characterization of the low-regret control.* For all $w_\gamma$, we have

$$J'_\gamma(w_\gamma)(f-w_\gamma) = (y^\gamma - y_d,\, y(f-w_\gamma))_{L^2} + \lambda(w_\gamma,\, f-w_\gamma)_{L^2}$$

$$+\frac{1}{\gamma}\Big[\big(I^{2-\alpha}\xi^\gamma(0),\, I^{2-\alpha}\xi(f-w_\gamma)(0)\big)_{L^2} + \big(\tfrac{\partial}{\partial t}I^{2-\alpha}\xi^\gamma(0),\, \tfrac{\partial}{\partial t}I^{2-\alpha}\xi(f-w_\gamma)(0)\big)_{L^2}\Big] = 0.$$

We introduce the state $\rho^\gamma$, which is the solution of

$$\begin{cases} {}^C\mathcal{D}^\alpha\rho^\gamma - \Delta\rho^\gamma = 0, & (x,t)\in\Omega\times(0,T),\\ \rho^\gamma = 0, & (x,t)\in\Gamma\times(0,T),\\ \rho^\gamma(0) = -\dfrac{1}{\gamma}\dfrac{\partial}{\partial t}I^{2-\alpha}\xi^\gamma(0), & x\in\Omega,\\ \dfrac{\partial}{\partial t}\rho^\gamma(0) = \dfrac{1}{\gamma}I^{2-\alpha}\xi^\gamma(T), & x\in\Omega. \end{cases} \tag{4.10}$$

By integrating by parts, we obtain

$$\begin{aligned}\int_Q \big({}^C\mathcal{D}^\alpha\rho^\gamma - \Delta\rho^\gamma\big)\,\xi(f-w_\gamma)\,dx\,dt &= \int_Q \rho^\gamma\, y(f-w_\gamma)\,dx\,dt - \int_Q \frac{\partial\rho^\gamma(0)}{\partial t}\, I^{2-\alpha}\xi(0)(f-w_\gamma)\,dx\,dt\\ &\quad + \int_Q \rho^\gamma(0)\,\frac{\partial}{\partial t}I^{2-\alpha}\xi(0)(f-w_\gamma)\,dx\,dt\\ &= 0.\end{aligned}$$

We now define another adjoint state $p^\gamma$ as

$$\begin{cases} D^\alpha_{RL}p^\gamma - \Delta p^\gamma = y^\gamma - h + \rho^\gamma, & (x,t)\in\Omega\times(0,T),\\ p^\gamma = 0, & (x,t)\in\Gamma\times(0,T),\\ I^{2-\alpha}p^\gamma(T) = 0, & x\in\Omega,\\ \dfrac{\partial}{\partial t}I^{2-\alpha}p^\gamma(T) = 0, & x\in\Omega. \end{cases} \tag{4.11}$$

Thus, we have

$$J'_\gamma(w_\gamma)(f-w_\gamma) = (p^\gamma + \lambda w_\gamma,\, f-w_\gamma)_{L^2} = 0,$$

and consequently,

$$p^\gamma + \lambda w_\gamma = 0 \quad \text{a.e. in } Q.$$

## 5. Finding the Optimality System of the No-Regret Control

This section is devoted to the full characterization of the no-regret control. We establish the convergence of the low-regret control sequence to the unique no-regret control. Based on this hypothesis, we then derive the corresponding optimality system of the no-regret control.

**Proposition 2.** *The optimality system characterizing the no-regret control $w = \lim_{\gamma\to 0} w_\gamma$ is defined by the solution of the following coupled problems involving the state variables $\{y, \rho\}$ and the adjoint variables $\{\xi, p\}$:*

$$\begin{cases} {}^C\mathcal{D}^\alpha y(x,t) - \Delta y(x,t) = w, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \Gamma \times (0,T), \\ y(x,0) = 0, \quad y_t(x,0) = 0, & x \in \Omega. \end{cases}$$

$$\begin{cases} D^\alpha_{RL}\xi(x,t) - \Delta\xi(x,t) = y(f,0), & (x,t) \in \Omega \times (0,T), \\ \xi = 0, & (x,t) \in \Gamma \times (0,T), \\ I^{2-\alpha}\xi(T) = 0, \quad \dfrac{\partial}{\partial t} I^{2-\alpha}\xi(T) = 0, & x \in \Omega. \end{cases}$$

$$\begin{cases} {}^C\mathcal{D}^\alpha \rho(x,t) - \Delta\rho(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ \rho = 0, & (x,t) \in \Gamma \times (0,T), \\ \rho(0) = -\dfrac{1}{\gamma}\dfrac{\partial}{\partial t} I^{2-\alpha}\xi(0) = 0, & x \in \Omega, \\ \dfrac{\partial}{\partial t}\rho(0) = \dfrac{1}{\gamma} I^{2-\alpha}\xi(T), & x \in \Omega. \end{cases}$$

$$\begin{cases} D^\alpha_{RL} p(x,t) - \Delta p(x,t) = y - h + \rho, & (x,t) \in \Omega \times (0,T), \\ p = 0, & (x,t) \in \Gamma \times (0,T), \\ I^{2-\alpha} p(T) = 0, \quad \dfrac{\partial}{\partial t} I^{2-\alpha} p(T) = 0, & x \in \Omega. \end{cases}$$

*The associated optimality condition is given by*

$$p + \lambda w = 0 \quad a.e. \ in \ Q.$$

*Proof.* We begin by establishing the convergence of the sequence of low-regret controls towards the unique no-regret control. Since $w_\gamma$ minimizes $J^\gamma$, we have

$$J^\gamma(w_\gamma) \le J^\gamma(0) = K,$$

where $K$ is a positive constant independent of $\gamma$. This yields the following bounds:

$$\begin{aligned} \|w_\gamma\|_{L^2(Q)} &\le K, \\ \|y(T, w_\gamma, 0, 0)\|_{L^2(Q)} &\le K, \\ \|I^{2-\alpha}\xi_\gamma(0)\|_{L^2(Q)} &\le \sqrt{\gamma}\, K, \quad (5.1) \\ \left\| \frac{\partial}{\partial t} I^{2-\alpha}\xi_\gamma(0) \right\|_{L^2(Q)} &\le \sqrt{\gamma}\, K. \quad (5.2) \end{aligned}$$

Here, $y(T, w_\gamma, 0, 0)$ denotes the solution of

$$\begin{cases} {}^C\mathcal{D}^\alpha y_\gamma(x,t) - \Delta y_\gamma(x,t) = w_\gamma, & (x,t) \in \Omega \times (0,T), \\ y_\gamma(x,t) = 0, & (x,t) \in \Gamma \times (0,T), \\ y_\gamma(x,0) = 0, \quad \dfrac{\partial}{\partial t} y_\gamma(x,0) = 0, & x \in \Omega. \end{cases}$$

We note that by following the same steps as in Section 4.1.1, we easily obtain

$$\begin{aligned} w_\gamma &\rightharpoonup w \quad \text{weakly in } L^2(Q),\\ y_\gamma &\rightharpoonup y \quad \text{weakly in } L^2(Q),\\ {}^C\mathcal{D}^\alpha y_\gamma - \Delta y_\gamma &\rightharpoonup {}^C\mathcal{D}^\alpha y - \Delta y \quad \text{weakly in } L^2(Q),\\ y_\gamma(x,t) &\to y(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T),\\ y_\gamma(x,0) &\to y(x,0) = 0, \quad \frac{\partial}{\partial t} y_\gamma(x,0) \to \frac{\partial}{\partial t} y(x,0) = 0, \quad x \in \Omega. \end{aligned}$$

Furthermore, we obtain that $\xi = \xi(x,t,w)$ is the solution of

$$\begin{cases} D^\alpha_{RL}\xi(x,t) - \Delta\xi(x,t) = y(f,0), & (x,t) \in \Omega \times (0,T),\\ \xi(x,t) = 0, & (x,t) \in \Gamma \times (0,T),\\ I^{2-\alpha}\xi(T) = 0, \quad \dfrac{\partial}{\partial t} I^{2-\alpha}\xi(T) = 0, & x \in \Omega. \end{cases}$$

From (5.1) and (5.2), when $\gamma \to 0$ we obtain

$$I^{2-\alpha}\xi_\gamma(T) \to 0 \quad \text{strongly in } L^2(Q),$$

$$\frac{\partial}{\partial t} I^{2-\alpha}\xi_\gamma(T) \to 0 \quad \text{strongly in } L^2(Q).$$

This implies that

$$\left(I^{2-\alpha}\xi(0), g_1\right)_{L^2(\Omega)} = \left(\frac{\partial}{\partial t} I^{2-\alpha}\xi(0), g_0\right)_{L^2(\Omega)} = 0,$$

which proves that $w$ is indeed the no-regret control.

Next, we continue the characterization of the no-regret control. We have

$$\begin{cases} {}^C\mathcal{D}^\alpha \rho^\gamma - \Delta\rho^\gamma = 0, & (x,t) \in \Omega \times (0,T),\\ \rho^\gamma = 0, & (x,t) \in \Gamma \times (0,T),\\ \rho^\gamma(0) = -\dfrac{1}{\gamma}\dfrac{\partial}{\partial t} I^{2-\alpha}\xi^\gamma(0), & x \in \Omega,\\ \dfrac{\partial}{\partial t}\rho^\gamma(0) = \dfrac{1}{\gamma} I^{2-\alpha}\xi^\gamma(0), & x \in \Omega. \end{cases}$$

Since

$$\rho^\gamma(0) \neq 0, \qquad \frac{\partial}{\partial t}\rho^\gamma(0) \neq 0,$$

we introduce a new function

$$\sigma^\gamma = \rho^\gamma - U,$$

where

$$U(0) = -\frac{1}{\gamma}\frac{\partial}{\partial t} I^{2-\alpha}\xi^\gamma(0), \qquad \frac{\partial}{\partial t} U(0) = \frac{1}{\gamma} I^{2-\alpha}\xi^\gamma(0).$$

Hence, we obtain

$$\begin{cases} {}^C\mathcal{D}^\alpha \sigma^\gamma - \Delta\sigma^\gamma = \tilde{f}, & (x,t) \in \Omega \times (0,T),\\ \sigma^\gamma = 0, & (x,t) \in \Gamma \times (0,T),\\ \sigma^\gamma(0) = 0, & x \in \Omega,\\ \dfrac{\partial}{\partial t}\sigma^\gamma(0) = 0, & x \in \Omega, \end{cases}$$

where

$$\tilde{f} = \Delta U - {}^C\mathcal{D}^\alpha U.$$

It follows that ${}^{C}\mathcal{D}^{\alpha} = D^{\alpha}$, so it is straightforward to obtain

$$\left\| {}_{0}^{C}\mathcal{D}_{t}^{\frac{\alpha}{2}} \sigma^{\gamma} \right\|_{L^2(Q)}^{2} + \|\nabla \sigma^{\gamma}\|_{L^2(Q)}^{2} \leq K' \left\| \tilde{f} \right\|_{L^2(Q)}^{2}, \qquad K' = \frac{1}{\min\left\{2\cos\left(\frac{\alpha\pi}{2}\right), (2-c')\right\}}.$$

Hence, we deduce the following convergence:

$$\sigma^{\gamma} \rightharpoonup \sigma \quad \text{weakly in } L^2\left(0,T;H_0^1(\Omega)\right).$$

Thanks to the continuous embedding between $L^2\left(0,T;H_0^1(\Omega)\right)$ and $L^2\left(0,T;L^2(\Omega)\right)$, we obtain

$$\sigma^{\gamma} \rightharpoonup \sigma \quad \text{weakly in } L^2(Q).$$

On the other hand, we know that

$$\rho^{\gamma} = \sigma^{\gamma} + U \rightharpoonup \sigma + U = \rho \quad \text{weakly in } L^2(Q).$$

Moreover, from (5.1) and (5.2), we have

$$\begin{aligned} -\frac{1}{\gamma}\frac{\partial}{\partial t} I^{2-\alpha}\xi^{\gamma}(0) &\rightharpoonup \lambda \quad \text{weakly in } L^2(\Omega), \\ \frac{1}{\gamma} I^{2-\alpha}\xi^{\gamma}(0) &\rightharpoonup \delta \quad \text{weakly in } L^2(\Omega). \end{aligned}$$

By passing to the limit in the state equation governing $\rho^{\gamma}$, we obtain

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\rho - \Delta\rho = 0, & (x,t) \in \Omega \times (0,T), \\ \rho = 0, & (x,t) \in \Gamma \times (0,T), \\ \rho(0) = \lambda, & x \in \Omega, \\ \dfrac{\partial}{\partial t}\rho(0) = \delta, & x \in \Omega. \end{cases}$$

On the other hand, we have

$$\begin{cases} D_{RL}^{\alpha} p^{\gamma} - \Delta p^{\gamma} = y_{\gamma} - h + \rho^{\gamma}, & (x,t) \in \Omega \times (0,T), \\ p^{\gamma} = 0, & (x,t) \in \Gamma \times (0,T), \\ I^{2-\alpha} p^{\gamma}(T) = 0, & x \in \Omega, \\ \dfrac{\partial}{\partial t} I^{2-\alpha} p^{\gamma}(T) = 0, & x \in \Omega. \end{cases}$$

Since

$$y_{\gamma} - h + \rho^{\gamma} \in L^2(Q) \quad \text{and} \quad p^{\gamma} \rightharpoonup p \text{ weakly in } L^2(Q),$$

by the same argument, we obtain

$$\begin{cases} D_{RL}^{\alpha} p - \Delta p = y - h + \rho, & (x,t) \in \Omega \times (0,T), \\ p = 0, & (x,t) \in \Gamma \times (0,T), \\ I^{2-\alpha} p(T) = 0, & x \in \Omega, \\ \dfrac{\partial}{\partial t} I^{2-\alpha} p(T) = 0, & x \in \Omega. \end{cases}$$

□

## Conclusion

In this paper, we investigated an ill-posed inverse fractional parabolic problem in the sense of Hadamard, where the aim was to identify the unknown source term. To address the lack of initial data, we introduced a regularization strategy based on Tikhonov's method combined with a no-regret control approach. The low-regret control problem was first analyzed to establish existence, uniqueness, and characterization through an optimality system derived from the first-order optimality condition. Then, by passing to the limit as the regularization parameter $\gamma \to 0$, we proved the convergence of the low-regret controls to the unique no-regret control. The resulting optimality system, consisting of the state and adjoint fractional equations, provides a complete characterization of the no-regret control and ensures the stability of the reconstructed source. The theoretical results confirm that the proposed framework is mathematically well-posed and suitable for further numerical implementation in fractional inverse problems.

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