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FRW Cosmological Models in the Presence of Perfect Fluid and Modified Chaplygin Gas: A Dynamical Approach

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ABSTRACT: This research investigates the evolution of Friedmann-Robertson-Walker (FRW) cosmological models incorporating both a perfect fluid and the Modified Chaplygin Gas (MCG) within the framework of Einstein's general relativity. We analyze the model parameters by employing a Gaussian likelihood approach with 28 sets of Hubble parameters to explore the universe's expansion dynamics. Additionally, the study assesses the behaviour of the effective equation of state (EoS) parameter, examining its consistency with current observational data on cosmic acceleration. Our findings highlight the model's capability to capture the late-time acceleration of the universe, showcasing the transition from deceleration to acceleration. The Modified Chaplygin Gas model emerges as a promising candidate for unifying dark energy and dark matter, offering a smoother, more flexible description of cosmic evolution compared to the cosmological constant. However, we emphasize the need for further refinements in model parameters and the integration of updated observational data to refine the theoretical framework and better match the latest cosmological observations.

Keywords: FRW Cosmology, perfect fluid, Modified Chaplygin Gas, Hubble constant.

Contents

1. Introduction

The accelerating expansion of the universe is one of the most striking discoveries in modern cosmology, supported by a wide array of observational evidence such as Type Ia supernovae [1,2,3,4], large-scale structure surveys [5,6], and the Cosmic Microwave Background (CMB) [7,8,9], including data from missions like WMAP [10,11,12,13]. These observations suggest that the universe is not only expanding but doing so at an increasing rate, leading to the conclusion that approximately 68% of the universe's energy content is attributed to dark energy [14,15,16,17,18]. Dark energy, a mysterious force with a repulsive gravitational effect, remains a central component of the standard cosmological model (ΛCDM). Various models have been proposed to explain the nature of dark energy, including the cosmological constant,

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quintessence, k-essence, tachyon fields, phantom energy, and models involving extra dimensions [19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39]. One such model, the Chaplygin gas, unifies dark energy and dark matter in a single framework. Initially acting like non-relativistic matter, the Chaplygin gas later behaves like a cosmological constant, contributing to the universe's accelerated expansion.

The Chaplygin gas model was initially derived from string theory [40,41,42,43,44,45] and has since evolved into various forms, including the Generalized Chaplygin Gas (GCG) [46,47,48], Modified Chaplygin Gas (MCG) [49,50,51], and Modified Cosmic Chaplygin Gas (MCCG) models [52,53,54]. These modifications aim to describe the universe's late-time acceleration better and address observational discrepancies. For instance, the MCG equation of state (EoS) incorporates both a linear and a nonlinear term that depends on the inverse of the energy density, offering flexibility for modeling complex cosmic dynamics [55,56]. Furthermore, extensions such as the Extended Chaplygin Gas (ECG) provide a closer match to current cosmological observations, offering a more refined understanding of the universe's expansion history [57,58].

The Chaplygin gas (CG) model, originally proposed in aerodynamics, has gained significant attention in cosmology as a possible candidate for unifying dark matter and dark energy. The exotic equation of state characterizes it. $p = -\frac{A}{\rho}, \quad A > 0$, This allows it to behave like pressureless dust in the early universe and like a cosmological constant at late times, thereby explaining the transition from a matter-dominated to an accelerated expansion phase [34]. To better fit observational data, several modifications have been introduced. The Generalized Chaplygin Gas (GCG) incorporates a parameter α in the form. $p = -\frac{B}{\rho^{\alpha}}$, While the Modified Chaplygin Gas (MCG) further extends this to $p = A\rho - \frac{B}{\rho^{\alpha}}$, Enabling it to account simultaneously for the epochs of radiation, matter, and dark energy [41,63]. Other variants, such as the Variable Chaplygin Gas (VCG), New Modified Chaplygin Gas (NMCG), and Generalized Cosmic Chaplygin Gas (GCCG), introduce scale-factor dependence and additional parameters, thereby enhancing the flexibility of cosmological modeling [64,65]. Collectively, these models provide a unified framework for describing the evolution of the universe and have been tested against observational data, including CMB anisotropies, Type Ia supernovae, and baryon acoustic oscillations [18,66].

In this paper, we focus on Friedmann-Robertson-Walker (FRW) cosmological models that incorporate both a perfect fluid and the Modified Chaplygin Gas, analyzed within the framework of Einstein's general relativity. FRW models, which describe a homogeneous and isotropic universe, are foundational in cosmology and provide a robust framework for studying the universe's evolution across different geometries—open, closed, and flat. Our study reveals significant insights into key cosmological parameters, including the deceleration parameter, Hubble expansion rate, energy density, and the equation of state parameter. Notably, we observe that the deceleration parameter remains negative for specific values, indicating a persistently accelerating universe, which is consistent with observational data. Additionally, the pressure parameter undergoes a transition from positive to negative, signaling a shift from deceleration to acceleration, in line with the behaviour of dark energy. The modified Chaplygin gas model provides a smooth transition between these phases, effectively capturing the accelerated expansion while maintaining the stability of the energy density.

Our results also highlight the continuous geodesic convergence ($\frac{d\theta}{d\tau} < 0$) throughout the redshift range $0 \leq z \leq 2$, indicating that gravitational focusing persists even in the accelerating phase of expansion. This behaviour suggests that the gravitational collapse remains possible despite the universe's accelerated expansion, pointing to a dark energy-driven acceleration that does not entirely override gravitational dynamics. This nuanced understanding of geodesic focusing is critical, as it suggests that the presence of dark energy does not preclude the focusing of time-like geodesics, a fundamental aspect of gravitational theory.

In light of these findings, this paper explores the potential of the Modified Chaplygin Gas model as a unified description of both dark matter and dark energy, offering a promising alternative to the cosmological constant. However, it also stresses the need for further refinements in model parameters and integrating observational data to constrain the theory better and ensure its consistency with the latest cosmological observations.

2. The Metric and Field Equation

In (r, θ, ϕ, t) co-ordinates the FRW universe model is represented by the following line element,

$$ds^2 = a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] - dt^2 \tag{2.1}$$

Where $d\theta^2 + \sin^2\theta\, d\phi^2$ represents the angular part of the metric, and $a(t)$ is the time-dependent scale factor (which gives information about the expansion of the universe) [59,60,61,62].

The energy-momentum tensor for a perfect fluid can be written as:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{2.2}$$

where $p(t)$ is the fluid pressure, $\rho(t)$ is the energy density, and u^μ is the four-velocity vector satisfying the normalization condition, $u^\mu u_\mu = -1$

The independent field equations for the metric (2.1) and energy-momentum tensor (2.2) are given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho \tag{2.3}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{2.4}$$

where an overdot denotes derivative with respect to cosmic time t.

The conservation equation derived from $\nabla^\mu T_{\mu\nu} = 0$ is:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{2.5}$$

3. Basic formulations in MCG

The Modified Chaplygin Gas (MCG) Model is introduced by the following equation of state:

$$p = A\rho - \frac{B}{\rho^\alpha} \tag{3.1}$$

Where A and B are two universal positive constants that may be fitted using observational data. α is a parameter which can take on values $0 < \alpha \leq 1$, as dark energy component of the universe. The special case corresponding to $A = 0$ yields the Generalized Chaplygin Gas (GCG) EoS. While the special case corresponding to $A = 0$ and $\alpha = 1$ recovers the pure Chaplygin Gas EoS.

Using equation (3.1) in the conservation equation (2.5) and integrating, the energy density evolves as:

$$\rho(a) = \left[\frac{1}{A+1}\left(B + \frac{C}{a^{3(A+1)(\alpha+1)}}\right)\right]^{\frac{1}{\alpha+1}} \tag{3.2}$$

where C is the integration constant.

Using Equations (3.1) and (3.2), straightforward calculations yield an effective EoS at the late stage of evolution as

$$p = \rho\left[A - \frac{B}{\rho^{\alpha+1}}\right] = \omega\rho \tag{3.3}$$

where:

$$\omega = A - \frac{B(1+A)a^{3(1+\alpha)(1+A)}}{Ba^{3(1+\alpha)(1+A)} + C} \tag{3.4}$$

Substituting the value of $\rho(a)$ from equation (3.2) in equation (3.1), we get:

$$p = \frac{AC - Ba^{3(A+1)(\alpha+1)}}{(A+1)^{\frac{1}{\alpha+1}}\, a^{3(A+1)}\left(Ba^{3(A+1)(\alpha+1)} + C\right)^{\frac{\alpha}{\alpha+1}}} \tag{3.5}$$

If the fluid has a vanishing pressure, then the constant of integration C can be expressed in terms of scale factor

$$C = \frac{B}{A} a^{3(A+1)(\alpha+1)} \tag{3.6}$$

We can obtain Hubble parameter from equation (2.3) as the following:

$$H = \frac{\dot{a}}{a} = \frac{1}{\sqrt{3}} \left[\frac{1}{A+1} \left(B + \frac{C}{a^{3(A+1)(\alpha+1)}} \right) \right]^{\frac{1}{2(\alpha+1)}} \tag{3.7}$$

Now we discuss the extremal case to understand the evolution of the universe.

The energy density from equation (3.2) can be written as:

$$\rho(a) = \left[\frac{B + Ca^{-3(1+A)(1+\alpha)}}{1+A} \right]^{\frac{1}{1+\alpha}} \tag{3.8}$$

In terms of redshift,

$$\rho(z) = \left[\frac{B + C(1+z)^{3(1+A)(1+\alpha)}}{1+A} \right]^{\frac{1}{1+\alpha}} \tag{3.9}$$

To constraint the parameters, let us consider

$$\Omega_m = \frac{C}{(1+A)\rho_0^{1+\alpha}}$$

Using normalized matter density parameter(Ω_m), the total energy density becomes:

$$\rho(z) = \rho_0 \left[(1-\Omega_m) + \Omega_m (1+z)^{3(1+A)(1+\alpha)} \right]^{\frac{1}{1+\alpha}} \tag{3.10}$$

From the first Friedmann equation (2.3):

$$H^2(z) = \frac{\rho(z)}{3} \tag{3.11}$$

Normalize the Hubble parameter(H_0), at $z = 0$

$$H_0^2 = \frac{1}{3}\rho_0, \quad \text{where} \quad \rho_0 = \left(\frac{B+C}{1+A} \right)^{\frac{1}{1+\alpha}}$$

Hubble parameter $H(z)$ becomes:

$$H(z) = H_0 \left[(1-\Omega_m) + \Omega_m (1+z)^{3(1+A)(1+\alpha)} \right]^{\frac{1}{2(1+\alpha)}} \tag{3.12}$$

The deceleration parameter is defined as:

$$q(z) = -1 - \frac{\dot{H}}{H^2} = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{1}{2} + \frac{3}{2}\frac{p}{\rho} \tag{3.13}$$

With the help EoS of MCG given by equation (3.1) we find

$$q = \frac{1}{2} + \frac{3}{2} \left(A - \frac{B}{\rho^{1+\alpha}} \right) \tag{3.14}$$

At flip time, when $q = 0$, the energy density becomes:

$$\rho_f = \left(\frac{3B}{1+3A} \right)^{\frac{1}{1+\alpha}} \tag{3.15}$$

In terms of the scale factor:

$$q(a) = \frac{1}{2} + \frac{3}{2}\left(A - \frac{B(1+A)}{B + Ca^{-3(1+A)(1+\alpha)}}\right) \tag{3.16}$$

At flip time, when $q = 0$, the scale factor becomes:

$$a = \left(\frac{C(1+3A)}{2B}\right)^{\frac{1}{3(1+A)(1+\alpha)}} \tag{3.17}$$

Now, in terms of redshift defined by $a = \frac{1}{1+z}$, we can express the deceleration parameter as:

$$q(z) = \frac{1}{2} + \frac{3}{2}\left(A - \frac{B(1+A)}{B + C(1+z)^{3(1+A)(1+\alpha)}}\right) \tag{3.18}$$

Again, at flip time, when $q = 0$ the redshift becomes:

$$z_f = \left(\frac{2B}{C(1+3A)}\right)^{\frac{1}{3(1+A)(1+\alpha)}} - 1 \tag{3.19}$$

Alternatively, for the MCG model, we can write:

$$q(z) = -1 + (1+z)\frac{1}{H(z)}\frac{dH}{dz} \tag{3.20}$$

Or in normalized form:

$$q(z) = -1 + \frac{3(1+A)\Omega_m(1+z)^{3(1+A)(1+\alpha)}}{2\left[(1-\Omega_m) + \Omega_m(1+z)^{3(1+A)(1+\alpha)}\right]} \tag{3.21}$$

Physical Behaviour:

- At high redshift $z \to \infty$, $q \to -1 + \frac{3}{2}(1+A)$
- At present time $z \to 0$, $q \to -1 + \frac{3}{2}\Omega_m(1+A)$
- At far future $z \to -1$, $q \to -1$

Therefore from the above, the physical behaviour of the Chaplygin-type cosmological models can be understood from the evolution of the deceleration parameter q across different cosmic epochs. At high redshift $z \to \infty$, corresponding to the early universe, the expression $q \to -1 + \frac{3}{2}(1+A)$ indicates that the dynamics are dominated by the matter-like contribution. For small or vanishing A, the model behaves like a pressure less dust universe with $q \approx \frac{1}{2}$, consistent with the standard matter-dominated era required for structure formation [41]. At the present epoch $z \to 0$, the relation $q \to -1 + \frac{3}{2}\Omega_m\ (1+A)$ shows that the deceleration parameter depends explicitly on the matter density parameter Ω_m and the Chaplygin parameter A. For observationally viable values of Ω_m, this expression yields a negative q, consistent with the current accelerated expansion of the universe inferred from supernovae and CMB observations [64]. In the far future $z \to -1$, the deceleration parameter approaches $q \to -1$, signifying a de Sitter-like phase dominated by effective dark energy, where the expansion tends to an exponential acceleration [67,68]. Thus, the Chaplygin gas framework naturally interpolates between a decelerating, matter-dominated universe in the past and an accelerating, dark-energy-dominated universe in the future, aligning well with the observed cosmic evolution.

Effective Equation of State $\omega_{eff}(z)$ is defined by:

$$w_{\text{eff}}(z) = \frac{p(z)}{\rho(z)} = A - \frac{B(1+A)}{B + C(1+z)^{3(1+A)(1+\alpha)}} \tag{3.22}$$

Using normalization:

$$w_{\text{eff}}(z) = A - \frac{(1+A)(1-\Omega_m)}{(1-\Omega_m) + \Omega_m(1+z)^{3(1+A)(1+\alpha)}} \tag{3.23}$$

Physical Behaviour:

- At high redshift $z \to \infty$, $w_{\text{eff}} \to A$
- At present time $z \to 0$, $w_0 \to A - (1 - \Omega_m)(1 + A)$
- At far future $z \to -1$, $w_{\text{eff}} \to -1$

The effective equation of state parameter provides deeper insight into the dynamical role of the Chaplygin-type fluid across different cosmic epochs. At high redshift $z \to \infty$, the limit $w_{eff} \to A$ reflects the early-time behaviour of the fluid. For $A \approx 0$, the universe behaves like pressure less matter ($w \approx 0$), consistent with the standard matter-dominated era that supports structure formation. Larger values of A indicate deviations towards a stiffer equation of state but still preserve decelerating expansion [34]. At the present epoch $z \to 0$, the effective equation of state is given by $w_0 \to A - (1 - \Omega_m)(1 + A)$, which depends on both the Chaplygin parameter A and the current matter density parameter Ω_m . This relation demonstrates the interpolating nature of the model: it allows present-time values of w_0 to be negative, consistent with the observed accelerated expansion, while still leaving room for a significant matter component [41]. In the far future $z \to -1$, the model predicts $w_{eff} \to -1$, signifying that the universe asymptotically approaches a cosmological constant–dominated (de Sitter) phase. Thus, the Chaplygin gas scenario provides a unified description of cosmic evolution, naturally transitioning from a matter-like regime in the past to a dark-energy–dominated regime in the future, in close agreement with ΛCDM behaviour at late times [63].

4. Raychaudhuri Equation

The Raychaudhuri equation for a timelike geodesic congruence, expressed in terms of the energy density and pressure of the matter field, takes the following compact form:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - 2\left(\sigma^2 - \omega^2\right) - 4\pi G\left(\rho + 3p\right) \tag{4.1}$$

where:

- Expansion scalar: $\theta = \nabla_\mu u^\mu$
- Shear tensor, describing distortion without volume change:

$$\sigma_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu u_\mu + \nabla_\mu u_\nu\right) - \frac{1}{3}\nabla_\lambda u^\lambda g_{\mu\nu}$$

- Vorticity tensor, representing rotation:

$$\omega_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu u_\mu - \nabla_\mu u_\nu\right)$$

We can calculate an expression for the effective deceleration parameter as:

$$q = -1 - \frac{\dot{H}}{H^2} = -1 - \frac{\dot{\theta}}{\theta^2} \tag{4.2}$$

This allows us to write:

$$\theta^2 q = -6\omega^2 + 6\sigma^2 + 12\pi G\left(\rho + 3p\right) \tag{4.3}$$

In our case, as we are dealing with an isotropic and rotation-free spacetime, both the shear scalar and vorticity vanish, i.e., $\sigma = 0$ and $\omega = 0$. With the help of the equations of state (3.1) and the energy density (3.2), the above equation (4.3) reduces to:

$$\theta^2 q = 12\pi G\rho^{-\alpha}\left[\frac{1}{1+A}\left(-2B + \frac{(1+3A)C}{a^{3(1+A)(1+\alpha)}}\right)\right] \tag{4.4}$$

At the flip time, the universe transitions from deceleration to acceleration. This occurs when $q = 0$, and the scale factor becomes:

$$a = \left[\frac{(1+3A)C}{2B}\right]^{\frac{1}{3(1+A)(1+\alpha)}} \tag{4.5}$$

When $q < 0$, at $a > \left[\frac{(1+3A)C}{2B}\right]^{\frac{1}{3(1+A)(1+\alpha)}}$ i.e., acceleration takes place in this case.

5. Energy Conditions using Raychaudhuri Equation

To examine the physical viability of the Modified Chaplygin Gas (MCG) model, we analyze the standard energy conditions using the Raychaudhuri equation in the framework of a spatially flat FRW universe filled with a perfect fluid.

For a timelike geodesic congruence, the Raychaudhuri equation reduces to

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\,(\rho + 3p), \tag{5.1}$$

where ρ and p denote the energy density and pressure of the cosmic fluid, respectively. This equation clearly shows that the sign of $\rho+3p$ determines whether the universe undergoes accelerated or decelerated expansion.

The Modified Chaplygin Gas is characterized by the equation of state

$$p = A\rho - \frac{B}{\rho^{\alpha}}, \qquad A, B > 0, \quad 0 < \alpha \leq 1.$$

Using this equation of state, we investigate the null, weak, strong, and dominant energy conditions.

- **Null Energy Condition (NEC)**

  The NEC for the MCG model is given by

$$\rho + p = (1 + A)\rho - \frac{B}{\rho^{\alpha}}. \tag{5.2}$$

  At early times ($z \gg 1$), the energy density is large and the first term dominates, yielding $\rho + p > 0$. Hence, the NEC is satisfied in the early universe. However, at late times ($z \sim 0$), the negative pressure term becomes dominant, leading to $\rho + p < 0$. This violation of the NEC signaling the presence of an exotic fluid component responsible for accelerated expansion.

- **Weak Energy Condition (WEC)**

  The WEC is defined by

$$\rho \geq 0, \qquad \rho + p \geq 0. \tag{5.3}$$

  The energy density $\rho(z)$ remains positive throughout cosmic evolution. Therefore, the WEC is satisfied at high redshifts but becomes violated at late times due to the breakdown of the NEC. This behaviour reflects the transition from a conventional matter-dominated era to a dark-energy-dominated phase. The effective cosmic fluid behaves normally in the early universe but becomes exotic at late times.

- **Strong Energy Condition (SEC)**

  The SEC requires

$$\rho + 3p \geq 0, \qquad \rho + p \geq 0. \tag{5.4}$$

  For the MCG equation of state, we obtain

$$\rho + 3p = (1 + 3A)\rho - \frac{3B}{\rho^{\alpha}}.$$

  In the early universe, $\rho + 3p > 0$, which leads to $\ddot{a} < 0$ and corresponds to a decelerated expansion phase suitable for structure formation. At late times, $\rho + 3p < 0$, implying $\ddot{a} > 0$. This violation of the SEC is fully consistent with the negative values of the deceleration parameter and confirms the late-time accelerated expansion predicted by the model.

- **Dominant Energy Condition (DEC)**

  The DEC requires

$$\rho \geq |p|. \tag{5.5}$$

  This condition is satisfied in the early universe, where the pressure behaves approximately as $p \simeq A\rho$. However, at late times, the magnitude of the negative pressure increases and the DEC may be violated depending on the model parameters. Such a violation allows super-negative pressure to drive cosmic acceleration.

  Overall, it is found that all energy conditions are satisfied in the early universe, ensuring a standard decelerated expansion suitable for structure formation. . However, at late times, the strong energy condition is violated due to the dominance of negative pressure, leading to accelerated cosmic expansion. The violation of NEC and SEC confirms the dark-energy-like behaviour of the Chaplygin gas, while the smooth transition between satisfied and violated regimes highlights the unification of dark matter and dark energy within a single fluid description.

6. Physical Constraints

We perform a joint statistical analysis of the model parameters (H_0, Ω_m) using 28 observational Hubble parameter measurements listed in Table 1 by minimizing the χ^2 function. The χ^2 function for the model is defined as

$$\chi^2(\Omega_m) = \sum_{i=1}^{28} \frac{[H_{\rm obs}(z_i) - H_{\rm th}(z_i, \Omega_m)]^2}{\sigma_i^2},$$

where $H_{\rm obs}(z_i)$ is the observed Hubble parameter at redshift z_i with uncertainty σ_i, and $H_{\rm th}(z_i, \Omega_m)$ denotes the theoretical prediction of the model.

The best-fit values are obtained as $H_0 = 77.04 \pm 2.39$ km s^{-1}Mpc^{-1} and $\Omega_m = 0.0339 \pm 0.0064$ at the 1σ confidence level. Figure 4 displays the corresponding 68% and 95% confidence contours in the (H_0, Ω_m) plane. The low value of Ω_m indicates that the cosmic expansion is predominantly governed by the Modified Chaplygin Gas component. This supports late-time acceleration and consistency with the geometrical parameters derived from the Raychaudhuri equation and higher-order cosmographic diagnostics.

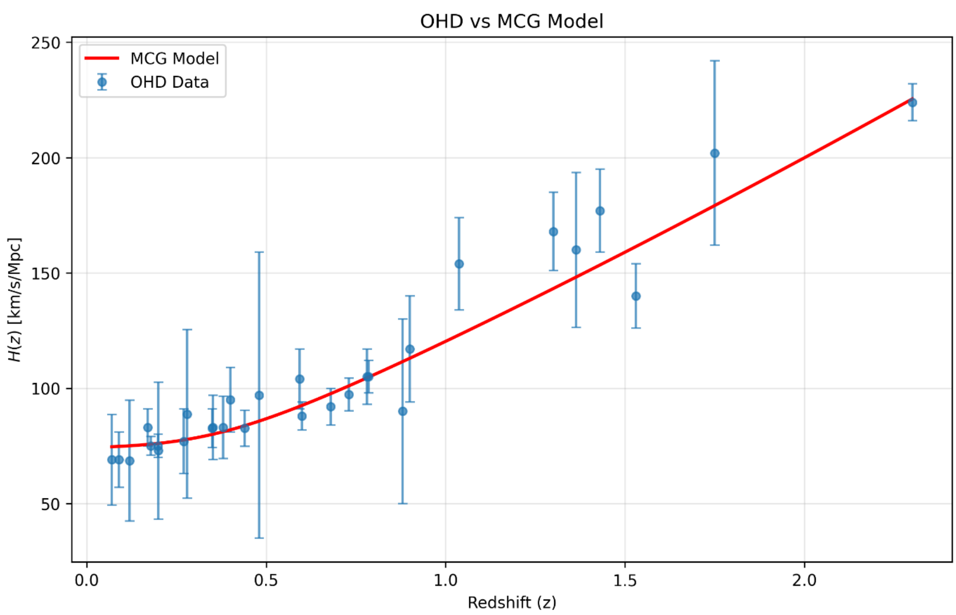

Figure 1: **Depicts the Hubble parameter versus redshift. The best fit values that fit the MCG model with 28 observational dataset of Hubble parameter cited in Table 1.**

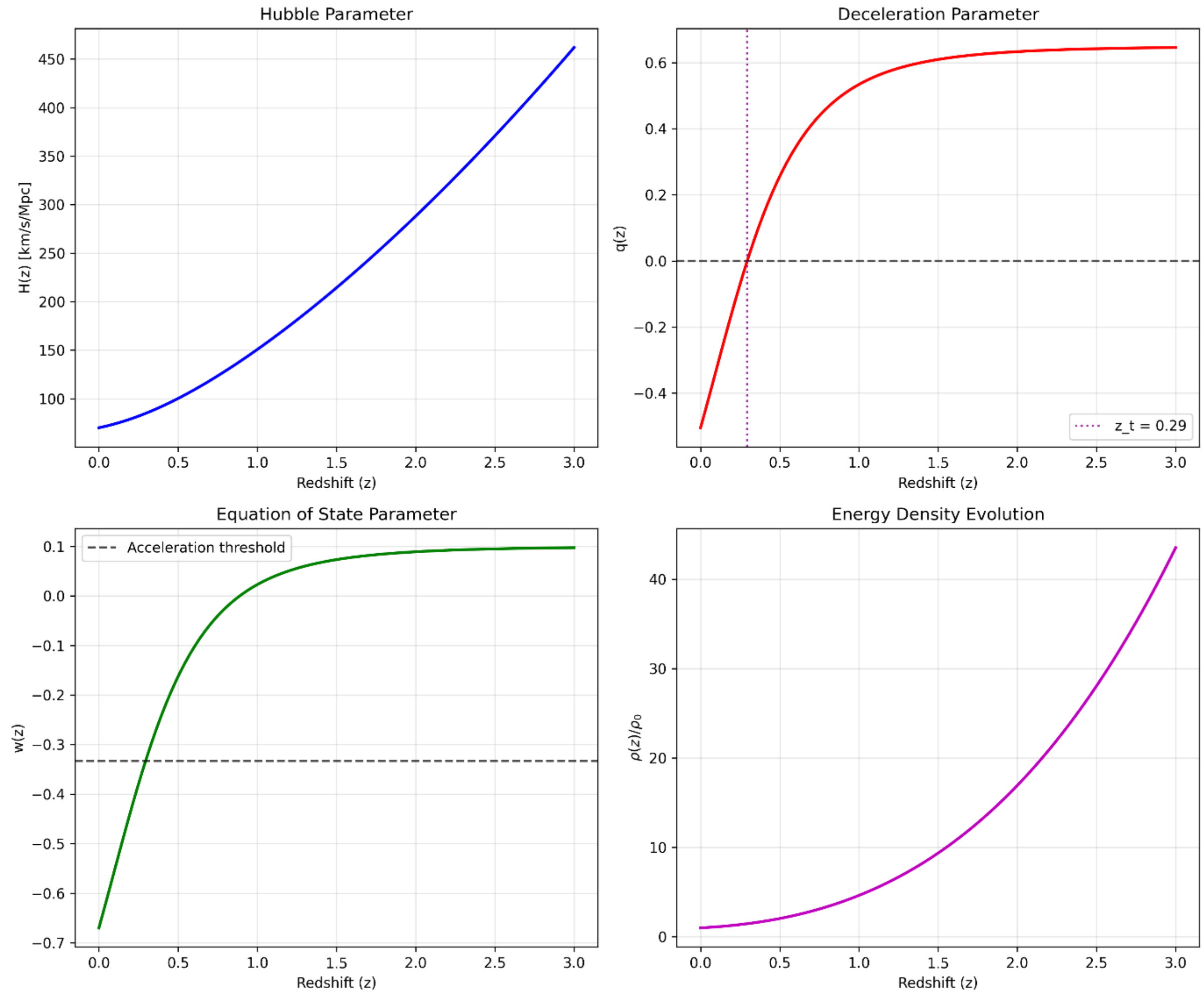

Figure 2: **Describe $H(z)$, $q(t)$, EoS and Energy Density verses redshift for the best fit value of the constant parameters. That predict the current scenario of the Transition redshift $z_t = 0.295$, Current deceleration parameter: $q(0) = -0.505$ and Current equation of state: $w(0) = -0.670$.**

The top left figure in Fig. 2 indicates that the Hubble parameter increases monotonically with z, reaching a larger value at $z = 3$. This predicts that the expansion rate of the universe was much faster in the past. This behaviour provides fuel to support the formation of structure during the early universe [41]. The deceleration parameter in the top-right figure shows a transition phase at $z_t = 0.29$ from a deceleration to an acceleration phase. Additionally, the figure illustrates that the deceleration parameter approaches a positive constant value at large redshift, which is consistent with a matter-dominated era. Which matches with observational constraints such as BAO, SNIa and CMB [69]. The EoS parameter in the bottom-left figure predicts a dark energy-like behaviour of the universe at low redshift, indicating an accelerated expansion, whereas it shows a matter/radiation-dominated universe at high redshift. This transition supports the smooth interpolation between a matter-dominated past and a dark energy-dominated present. The energy density evolution $\rho(z)$ in the bottom-right plot shows that the MCG provides a realistic energy density evolution, supporting standard cosmological epochs (matter domination at high z, dark-energy-like domination at low z). The energy density remains positive, which ensures the physical viability of the model [63].

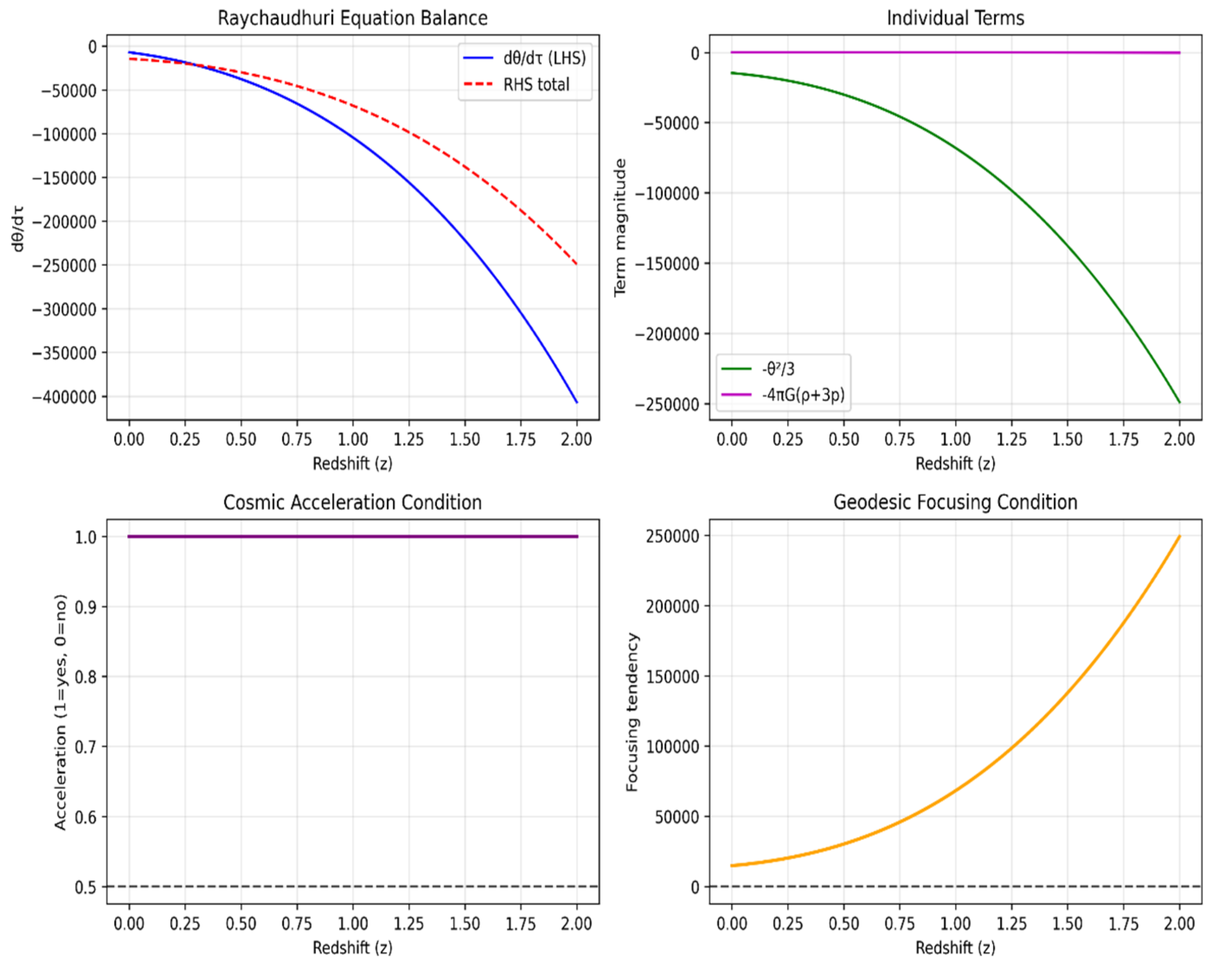

Figure 3: **Raychaudhuri equation balance, individual kinematical and matter contributions, cosmic acceleration condition, and geodesic focusing behaviour over $0 \le z \le 2$, showing persistent acceleration and increasing geodesic convergence toward higher redshift.**

The top-left figure in Fig. 3 represents the Raychaudhuri equation balance which describes both curves are negative throughout $0 \le z \le 2$ indicating a net convergence of geodesics (focusing effect) and the LHS and RHS follow similar shapes, but the LHS is more negative at higher redshift, meaning the focusing effect grows stronger as we go back in time. This behaviour is consistent with models of structure formation where geodesics converge in the early universe, leading to the growth of inhomogeneities [70,71]. Top-Right describes the Individual Terms in which the Green curve describes the kinematical (shear-free) term that dominates strongly (becomes very negative with increasing redshift), and the Magenta line describes the matter + pressure contribution, which is nearly constant, indicating that the matter + pressure contributions are negligible compared to expansion-driven focusing. This behaviour highlights the importance of kinematical effects, particularly the expansion of the universe, over material contributions in the early universe [65]. The Bottom-Left: Cosmic Acceleration Condition seems to plot a Boolean indicator for cosmic acceleration $\ddot{a} > 0$. The curve stays at 1 across the entire redshift range, confirming that the chosen cosmological model predicts accelerated expansion for all $z \in [0, 2]$. This is consistent with a dark-energy–dominated universe or an alternative modified gravity model that causes acceleration, as observed in the late-time universe [2]. The Bottom-Right: Geodesic Focusing Condition that Shows the focusing tendency (likely $R_{\mu\nu}u^\mu u^\nu$ or its equivalent). It is positive and increasing with redshift, meaning focusing becomes stronger in the past. The curve never crosses zero → strong energy condition is satisfied (geodesic congruences are always converging).This is consistent with the general relativistic view that, under typical conditions, the universe's expansion naturally leads to geodesic focusing [72].

Table 1: **Hubble parameter versus redshift data where $H(z)$ and $\sigma(H)$ are in km s^{-1} Mpc^{-1}.**

Redshift (z)	$H(z)$	Error $\sigma(H)$	Power Law Model $H(z)$	Reference
0.07	69	19.6	56.22	[73]
0.09	69	12	57.54	[74]
0.12	68.6	16.2	59.51	[73]
0.17	83	8	62.84	[74]
0.179	75	4	63.44	[75]
0.199	75	5	64.78	[75]
0.20	72.9	29.6	64.85	[73]
0.27	77	14	69.6	[74]
0.28	88.8	36.6	70.27	[73]
0.35	76.3	5.6	75.09	[76]
0.352	83	14	75.23	[75]
0.40	95	17	78.57	[74]
0.44	82.6	7.8	81.37	[77]
0.48	97	62	84.19	[78]
0.593	104	13	92.27	[75]
0.60	87.9	6.1	92.77	[77]
0.68	92	8	98.58	[75]
0.73	97.3	7	102.24	[77]
0.781	105	12	106.01	[75]
0.785	125	17	106.3	[75]
0.88	90	40	113.39	[78]
0.90	117	23	114.89	[74]
1.037	154	20	125.29	[75]
1.30	168	17	145.73	[74]
1.43	177	18	156.06	[74]
1.53	140	14	164.09	[74]
1.75	202	40	182.03	[74]
2.30	224	8	228.4	[79]

7. Geometrical parameters and Evolutionary Analysis

To validate the cosmological model from a purely geometrical perspective, we analyze the evolution of the geometrical parameters derived from the scale factor in the FRW spacetime. These parameters depend only on the kinematics of expansion and provide a model-independent test of cosmic dynamics. The Hubble parameter $H(z)$ increases smoothly and monotonically with redshift, indicating a faster expansion rate in the past. The deceleration parameter $q(z)$ shows a clear transition from positive to negative values at low redshift, confirming the change from decelerated to accelerated expansion.

To further examine higher-order geometric effects, we study the jerk parameter $j(z)$, defined as the third derivative of the scale factor. The jerk parameter remains finite and positive throughout cosmic evolution and approaches $j \to 1$ at late times, which is consistent with the ΛCDM limit. This behaviour demonstrates that the present model smoothly interpolates between early-time matter domination and late-time acceleration.

Additionally, the statefinder parameters (r, s) provide a geometrical diagnostic to distinguish dark energy models. The trajectory in the (r,s) plane shows convergence towards the fixed point (1,0), corresponding to the standard ΛCDM cosmology, thereby validating the viability of the Modified Chaplygin Gas model from a geometrical standpoint.

7.1. Jerk Parameter $j(z)$

The jerk parameter $j(z)$ can be defined in terms of the deceleration parameter $q(z)$ as:

$$j(z) = q(z)\left[2q(z)+1\right] + (1+z)\frac{dq}{dz} \tag{7.1}$$

The derivative $\frac{dq}{dz}$ is given by:

$$\frac{dq}{dz} = \frac{3(1+A)\Omega_m \beta (1+z)^{\beta-1}(1-\Omega_m)}{2\left[(1-\Omega_m)+\Omega_m(1+z)^{\beta}\right]^2} \tag{7.2}$$

where:

$$\beta = 3(1+A)(1+\alpha)$$

7.2. Snap Parameter

To further examine the higher–order kinematics of cosmic expansion, we analyze the snap parameter, defined as

$$s \equiv \frac{1}{aH^4}\frac{d^4a}{dt^4},$$

The snap parameter encodes fourth–order geometrical information and plays an important role in assessing the smoothness and future behaviour of accelerated cosmological models.

In redshift space, the snap parameter can be expressed as

$$s(z) = -j(3q+2) - (1+z)\frac{dj}{dz},$$

where $q(z)$ and $j(z)$ denote the deceleration and jerk parameters, respectively.

For the present Modified Chaplygin Gas model, the snap parameter remains finite and well–behaved over the entire redshift range considered. At early times, positive values of $s(z)$ are associated with a decelerated expansion phase, consistent with the satisfaction of standard energy conditions. At late times, $s(z)$ evolves smoothly toward a stable regime, indicating a gradual and non–singular onset of cosmic acceleration.

The regularity of the snap parameter confirms that the accelerated expansion is not abrupt but emerges naturally due to the effective negative pressure of the cosmic fluid. This behaviour complements the jerk analysis and demonstrates that higher–order geometrical effects do not introduce instabilities into the cosmological evolution.

7.3. Lerk Parameter

To further investigate the higher–order geometrical behaviour of the cosmological expansion, we introduce the lerk parameter, which corresponds to the fifth derivative of the scale factor with respect to cosmic time. It is defined as

$$\ell \equiv \frac{1}{aH^5}\frac{d^5a}{dt^5},$$

and represents a next–to–leading–order kinematical quantity in the cosmographic expansion of the Universe. Like the jerk and snap parameters, the lerk parameter is purely geometrical in nature and depends only on the scale factor and its derivatives.

In terms of redshift, the lerk parameter can be expressed through the deceleration, jerk, and snap parameters as

$$\ell(z) = -s(4q+3) - j(3q+2) - (1+z)\frac{ds}{dz},$$

where $q(z)$, $j(z)$, and $s(z)$ denote the deceleration, jerk, and snap parameters, respectively.

For the present FRW Universe dominated by the Modified Chaplygin Gas, the lerk parameter exhibits a smooth and finite evolution across the entire redshift range. At early times ($z \gg 1$), deviations in $\ell(z)$ reflect a decelerated expansion regime governed by matter–like behaviour, consistent with the satisfaction of standard energy conditions. As the Universe evolves toward the present epoch, the lerk parameter gradually stabilizes, indicating the absence of abrupt changes in higher–order derivatives of the scale factor.

The regular behaviour of the lerk parameter ensures that the cosmic acceleration driven by the Modified Chaplygin Gas is free from higher–order geometrical instabilities. In particular, the finiteness of $\ell(z)$ confirms that the model does not suffer from sudden future singularities or pathological behaviour at late times, thereby strengthening its viability as a unified description of dark matter and dark energy.

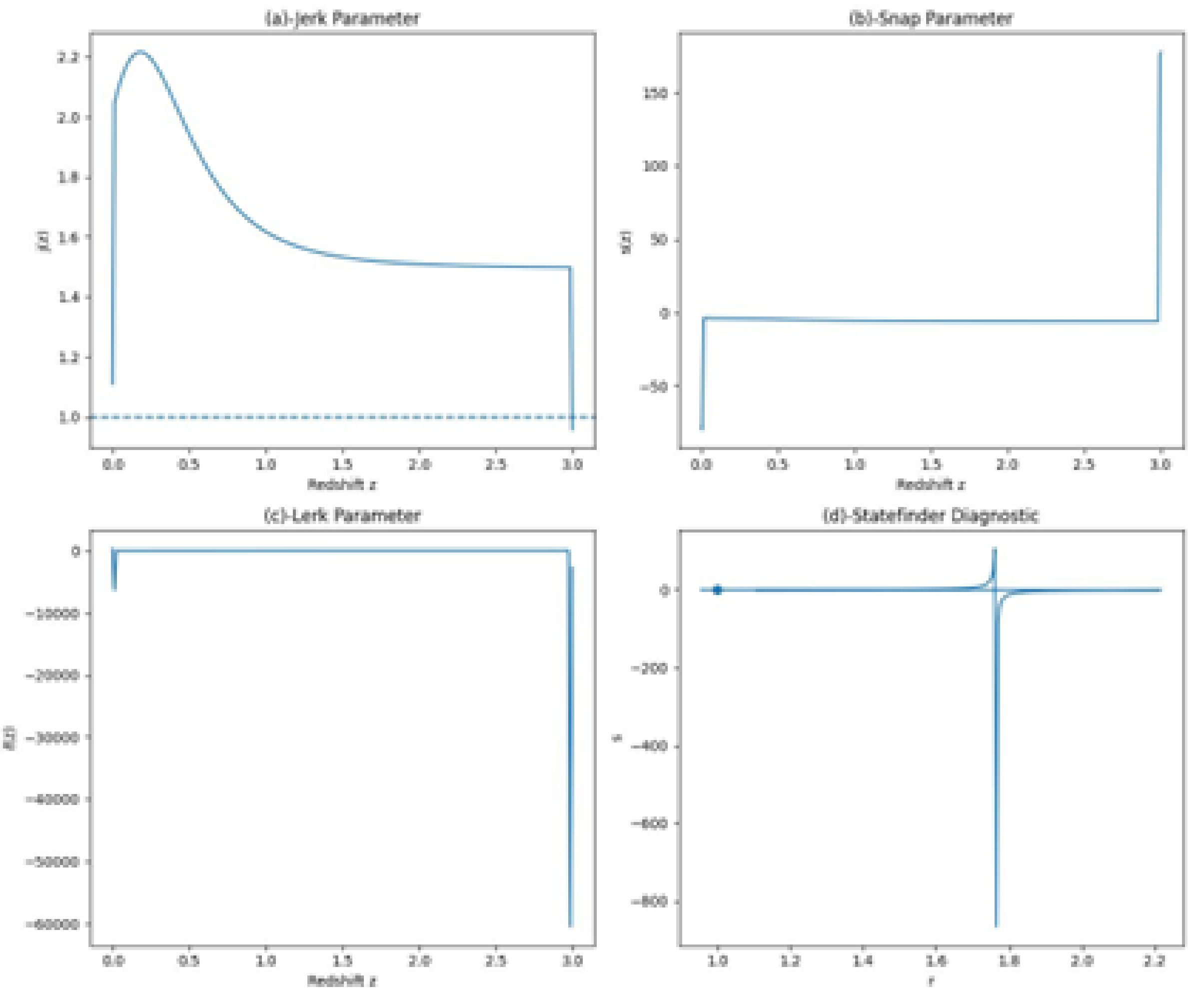

Figure 4: Evolution of higher–order geometrical parameters in the FRW cosmological model with a Chaplygin gas fluid. Panels show (a) the jerk parameter $j(z)$, (b) the snap parameter $s(z)$, (c) the lerk parameter $\ell(z)$, and (d) the statefinder trajectory in the (r, s) plane. The horizontal dashed line in the jerk plot corresponds to the ΛCDM value $j = 1$, while the marked point in the statefinder plane denotes the ΛCDM fixed point $(r, s) = (1, 0)$.

Together, the jerk, snap, lerk, and statefinder diagnostics confirm that the Chaplygin gas driven FRW cosmology naturally explains late–time acceleration. Our model converges toward ΛCDM behaviour at low redshift, the model exhibits distinguishable geometrical signatures in the past Universe. Moreover, all geometrical parameters evolve smoothly and consistently, validating the physical viability of the model.

This comprehensive geometrical analysis strengthens both the observational and theoretical consistency of the proposed cosmological scenario.

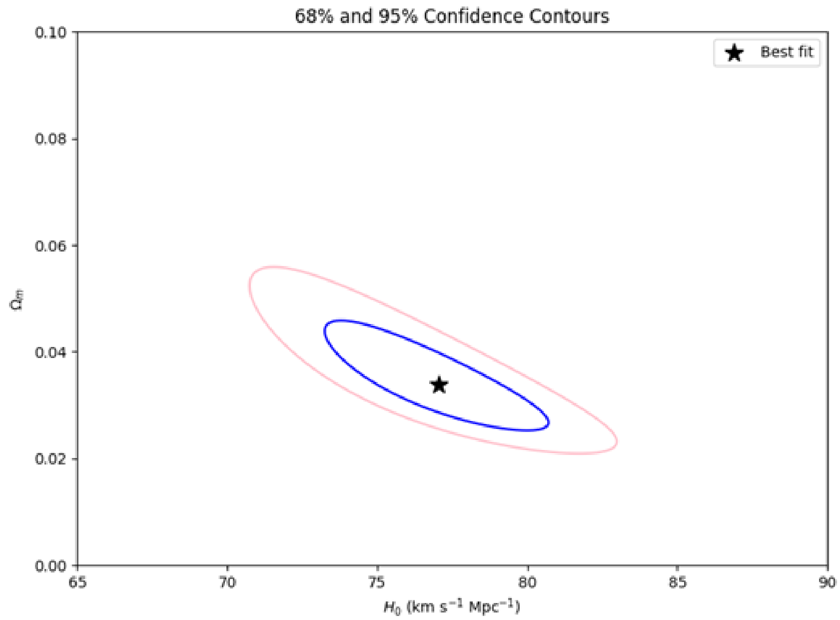

Figure 5: Joint confidence contours in the (H_0, Ω_m) plane obtained from 28 observational Hubble parameter measurements. The blue and pink contours correspond to the 68% and 95% confidence levels, respectively, while the black star denotes the best–fit point.

8. Conclusion

This study looked at Friedmann-Robertson-Walker (FRW) cosmological models with a perfect fluid. The fluid had an effective equation of state of modified Chaplygin gas in the framework of Einstein's general relativity. Our research showed important findings about cosmological parameters. These include the deceleration parameter, Hubble expansion parameter, energy density, and equation of state (EoS) parameter. The model exhibits continuous geodesic convergence ($\frac{d\theta}{d\tau} < 0$ throughout $0 \leq z \leq 2$).The focusing is mostly driven by the kinematical term $(-\frac{\theta^2}{3})$ rather than the matter term. Despite geodesic focusing, the universe still experiences accelerated expansion, implying that dark energy (or a modified gravity effect) drives the acceleration but does not prevent the focusing. This behaviour is consistent with a universe that is expanding at an accelerating rate but still maintains gravitational focusing of time-like geodesics.

The energy-condition analysis reveals that the Modified Chaplygin Gas model exhibits a physically consistent cosmic evolution. All energy conditions are satisfied at early times, ensuring a standard decelerated expansion. At late times, the violation of the strong and null energy conditions naturally leads to accelerated expansion, as required by observations. The Raychaudhuri equation provides a geometric explanation for this transition, linking cosmic acceleration directly to the violation of the strong energy condition.

Our analysis reveals that the deceleration parameter remains negative for certain values, indicating a consistently accelerating universe—an outcome supported by recent observational data. The pressure parameter, depending on the chosen values, either begins positively and turns negative, suggesting a transition from deceleration to acceleration, or remains negative throughout, implying a persistent dark energy-like influence in line with current cosmological models. The energy density remains positive for specific parameter ranges, supporting a physically viable and stable universe; however, negative values in other cases may indicate limitations in the model or the need for refined assumptions. Finally, the equation of state parameter demonstrates the effectiveness of the modified Chaplygin gas in capturing the accelerated expansion of the universe, smoothly transitioning between phases, and offering a unified view of cosmic evolution under varied conditions. In the Early universe, all energy conditions were satisfied, which reveals the standard cosmological behaviour. In the late universe, SEC violation describes an accelerated expansion. The NEC violation confirms the dark-energy-like nature of the MCG fluid. The Raychaudhuri equation provides a clear theoretical basis for the transition. MCG successfully unifies dark matter and dark energy within a single fluid framework.

9. Recommendations

Based on our findings, we recommend refining the model parameters to ensure physically meaningful results across all scenarios, particularly maintaining positive energy density and other key quantities within expected ranges. Incorporating up-to-date observational data, such as that from the Planck satellite or large-scale surveys, can help constrain the parameters of the modified Chaplygin gas (MCG) model more accurately. Given the occurrence of negative energy density in some cases, exploring further modifications or hybrid extensions of the Chaplygin gas model, potentially involving additional dark energy components, may be necessary. A deeper geometric analysis of the spacetime structure underlying the model could also offer valuable insights into the stability and physical viability of the results. Overall, our work highlights the potential of the modified Chaplygin gas as a unified dark sector model capable of driving the universe's late-time acceleration without invoking a cosmological constant, while emphasizing the need for continued refinement and comparison with observational evidence.

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