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Some Fixed Points Theorems for Various Mappings Involving Control Functions in Perturbed Metric Spaces

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ABSTRACT: In this paper, we prove common fixed point theorems for non-compatible mappings along with reciprocal continuous mappings and weakly compatible along with variants of $R-$ weak commutative, weakly compatible maps along with property (E.A.) and common limit range property.

Keywords: Perturbed metric space, non-compatible maps, weakly commuting mappings and its variants, weakly compatible mappings, property (E.A.), common limit range property (CLR Property).

Contents

1. Introduction

The measurement of the distance between two points is not always exact. During measurement, some errors may occur. These errors may be slightly positive, slightly negative, or sometimes zero. If error is zero, then it corresponds to the metric. To account for these, a positive error is subtracted and a negative error is added during determining the exact value of the distance function. These errors may play a significant role during measurement.
In order to overcome the difficulty, whenever error is added in metric, Mohamed Jleli and Bessem Samet [7] gave the notion of a perturbed metric space. Perturbed metric spaces represent a useful and practical improvement over the metric spaces. The significance of perturbed metric spaces lies across a wide range of mathematical and applied disciplines.
Even though for small positive errors, the structure of these spaces still retains the properties of metric spaces. In this way, perturbed metric spaces help to bridge the gap between the mathematical models and real-world situations, where exact distance are not measurable.

In 2025, Mohamed Jleli and Bessem Samet [7] introduced a more general form of distance function, known as perturbed metric space as follows :

Definition 1.1. Let $D, P : X \times X \to [0, \infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function

$$D - P : X \times X \to \mathbb{R},$$

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defined by the relation

$$(D-P)(x,y) = D(x,y) - P(x,y),$$

for all $x, y, z \in X$, is a exact metric on X, i.e., for all $x, y, z \in X$, it satisfies the following conditions

(i) $(D-P)(x,y) \geq 0$;

(ii) $(D-P)(x,y) = 0$ if and only if $x = y$;

(iii) $(D-P)(x,y) = (D-P)(y,x)$;

(iv) $(D-P)(x,y) \leq (D-P)(x,z) + (D-P)(z,y)$.

P is called a *perturbing function* and $D = d + P$ be an *perturbed metric.*
The set X endowed with D and *perturbed function* P denoted by (X, D, P) is known as *perturbed metric spaces.*

Notice that a perturbed metric on X is not necessarily a metric on X. But a metric is always perturbed metric when perturbed error is zero.

Example 1.1. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x,y) = |x-y| + x^2y^4, \ \textit{for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x,y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x,y) = D(x,y) - P(x,y) \ , \ \textit{where}$$

$$d(x,y) = |x-y|, \quad x, y \in \mathbb{R}.$$

Here we note that D is not necessarily a metric, because $D(1,1) = 1 \neq 0$ as $x = y$, but D is perturbed metric on X with respect to perturbed function P.

We now introduce topological structure in perturbed metric space.

The topological structure of the perturbed metric space (X, D, P) corresponds to the balls in metric spaces and is induced by the exact metric $d = D - P$. That is, the topology $\tau_{D,P}$ on X is defined as:

$$\tau_{D,P} := \tau_d = \{U \subseteq X \mid \forall x \in U, \exists r > 0 \text{ such that } B_d(x,r) \subseteq U\},$$

where the open ball with respect to d is given by:

$$B_d(x,r) = \{y \in X \mid d(x,y) = D(x,y) - P(x,y) < r\}.$$

Definition 1.2. Let (X, D, P) be a perturbed metric space with perturbed function P. A sequence $\{x_n\}$ in X is said to be

(i) *perturbed convergent sequence*, if $\{x_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.

(ii) *perturbed Cauchy sequence*, if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

A mapping T defined on (X, D, P) is a *perturbed continuous mapping*, if T is continuous with respect to the exact metric d.

We recall some elementary properties of perturbed metric spaces [7] .

Proposition 1.1. [7] Let $D, P, Q : X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.

(i) If (X, D, P) and (X, D, Q) be two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space.

Here for the convenience of readers, we provide the proof of the proposition 1.1.

Proof.

(i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}[(D - P) + (D - Q)] = D - \frac{P + Q}{2}$$

is a metric on X, which proves (i).

(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D - P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii).

2. Preliminaries

We first recall some notions of weakly commuting, compatible, and related mappings that are useful in the development of our main results in perturbed metric spaces.

Definition 2.1. [5] Two self-mappings f and g be of a perturbed metric space (X, D, P) are said to be weakly commuting if

$$D(fgx, gfx) \le D(gx, fx) \quad \text{for all } x \in X.$$

Definition 2.2. [6] Let S and T be two mappings of a perturbed metric space (X, D, P) into itself. Then S and T are called compatible if and only if

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Example 2.1. [6] Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^4, \ \textit{for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

Let $S, T : X \to X$ be defined by $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all $x \in X$, where $X = [0, \infty)$. Taking the sequence $\{x_n\}$ as $x_n = \frac{1}{n}, n > 0$, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X,$$

then S and T are said to be compatible

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0.$$

Definition 2.3. [4] A pair of maps A and S is called a weakly compatible pair if they commute at coincidence points.
Remark 2.1. [6] Weakly compatible maps need not be compatible.

Example 2.2. Let $X = [2, 20]$ and $D : \mathbb{R} \times \mathbb{R} \to [2, 20]$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \ \textit{for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [2, 20]$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Defining $S, T : X \to X$ as below:

$$Sx = \begin{cases} 2 & \text{if } x = 2 \textit{ or} > 5 \\ 6 & \text{if } 2 < x \leq 5. \end{cases} \qquad Tx = \begin{cases} 12, & \text{if } 2 < x \leq 5 \\ x - 3, & \text{if } x > 5 \\ x, & \text{if } x = 2. \end{cases}$$

The mappings S and T are non-compatible since sequence $\{x_n\}$ defined by
$\{x_n\} = 5 + (\frac{1}{n}),\ n \geq 1$. Then $Tx_n \to 2$, $Sx_n \to 2$. But they are weakly compatible since they commute at coincidence point $x = 2$. But they are not compatible at that point.

Definition 2.4. [5] A pair of self-mappings (f, g) of a perturbed metric space (X, D, P) is said to be R-weakly commuting if there exists some $R > 0$ such that

$$D(fgx, gfx) \leq RD(fx, gx), \quad \text{for all } x \in X.$$

Definition 2.5. [5] *A pair of self-mappings (f, g) of a perturbed metric space (X, D, P) is said to be*

(i) *R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that*

$$D(gfx, ffx) \leq RD(fx, gx), \quad \text{for all } x \in X.$$

(ii) *R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that*

$$D(fgx, ggx) \leq RD(fx, gx), \quad \text{for all } x \in X.$$

Definition 2.6. [5] *Let (X, D, P) be a perturbed metric space and $f, g : X \to X$ be two self-mappings. The pair (f, g) is said to be R-weakly commuting of type (P), if there exists a constant $R > 0$ such that*

$$D(ffx, ggx) \leq RD(fx, gx) \quad \textit{for all } x \in X.$$

Remark 2.2. Now we gave an example which show that *R-weakly commuting mappings*, *R-weakly commuting of type (A_g)*, *R-weakly commuting of type (A_f)*, and *R-weakly commuting of type (P)* are independent to each other.

Example 2.3. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \text{ for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Let $f, g : X \to X$ defined by $fx = x$ and $gx = x^2$, for all x, where $X = [0, 1]$. Then, by a straightforward calculation, one can show that for all $x \in X$,

$$f(f(x)) = x, \quad g(f(x)) = x^2, \quad f(g(x)) = x^2, \quad g(g(x)) = x^4,$$

and

$$\begin{aligned} D(fg(x), gf(x)) &= D(x^2, x^2) \\ &= |x^2 - x^2| + x^4 \cdot x^4 = x^8, \\ D(f(x), g(x)) &= D(x, x^2) \\ &= |x - x^2| + x^2 \cdot x^4 = |x - x^2| + x^6, \end{aligned}$$

$$D(gf(x), ff(x)) = D(x^2, x) = |x^2 - x| + x^4,$$
$$D(fg(x), gg(x)) = D(x^2, x^4) = |x^2 - x^4| + x^4 \cdot x^8 = |x^2 - x^4| + x^{12}.$$

Therefore, we conclude as follows:

1. The pair (f, g) is R-weakly commuting for all positive real values of $R \geq 1$.
2. For $R = 2$, the pair (f, g) is R-weakly commuting of type (A_f), type (A_g), and of type (P).
3. For $R = 1$, the pair (f, g) is R-weakly commuting and type (A_f), but not of type (P) and type (A_g).

Definition 2.7. Two self-maps A and S of a perturbed metric space (X, D, P) are called pointwise R-weakly commuting on X if for a given $x \in X$ there exists $R > 0$ such that

$$D(ASx, SAx) \leq RD(Ax, Sx).$$

It is to be noted that compatible maps are necessarily pointwise R-weakly commuting in perturbed metric spaces (X, D, P). Since compatible maps commute at their coincident points, but the converse may not be true.

Example 2.4. Let $X = [2, 20]$ and $D : \mathbb{R} \times \mathbb{R} \to [2, 20]$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \text{ for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [2, 20]$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Defining $S, T : X \to X$ as below:

$$Sx = \begin{cases} 2 & \text{if } x = 2 \text{ } or > 5 \\ 6 & \text{if } 2 < x \leq 5. \end{cases} \quad Tx = \begin{cases} 12, & \text{if } 2 < x \leq 5 \\ x - 3, & \text{if } x > 5 \\ x, & \text{if } x = 2. \end{cases}$$

The mappings S and T are non-compatible since sequence $\{x_n\}$ defined by $\{x_n\} = 5 + (\frac{1}{n}),\ n \geq 1$. Then $Tx_n \to 2$, $Sx_n \to 2$. But they are pointwise $R-$ weakly commuting since they commute at coincidence point $x = 2$. But they are not compatible at that point.

In 1999, R.P. Pant [10] introduced the notion of reciprocally continuous in metric spaces as follows:
Definition 2.8. Let A and S be mappings from a metric space (X, d) into itself. Then A and S are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \in X.$$

Now we use this notion of reciprocally continuous in setting of perturbed metric spaces.
Definition 2.9. Let A and S be mappings from a perturbed metric space (X, D, P) into itself. Then A and S are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \in X.$$

Remark 2.3. Continuous mappings are reciprocally continuous on (X, D, P) but the converse may not be true.

Example 2.4. Let $X = [2, 20]$ and D be the perturbed metric on X. Define mappings $A, S : X \to X$ by

$$Ax = 2 \quad \text{if } x = 2, \qquad Sx = 2 \quad \text{if } x = 2,$$

$$Ax = 3 \quad \text{if } x > 2, \qquad Sx = 6 \quad \text{if } x > 2.$$

It is noted that A and S are reciprocally continuous mappings but they are not continuous.

Example 2.5. Let $X = [4, 30]$ and d be the perturbed metric on X. Define mappings $A, S : X \to X$ by

$$Ax = x \quad \text{if } x = 4, \qquad Sx = x \quad \text{if } x = 4,$$

$$Ax = 5 \quad \text{if } x > 4, \qquad Sx = 10 \quad \text{if } x > 4.$$

Here A and S are reciprocally continuous mappings but A and S are not continuous.
Aamri and El Moutawakil [2] introduced notion *property (E.A.)* and proved common fixed point theorems for the property (E.A.) along with weakly compatible maps. A major benefit of property (E.A.) is that it ensures convergence of desired sequences without completeness. Aamri and Moutawakil [2] introduced the notion of property (E.A.) as follows:

Definition 2.10. *Let S and T be two self-maps in a perturbed metric space (X, D, P). The pair (S, T) is said to satisfy property (E.A.), if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \textit{for some } t \in X.$$

Example 2.6. Let $X = [0, +\infty)$. Define $S, T : X \to X$ by $Tx = \dfrac{x}{2}$ and $Sx = \dfrac{3x}{5}$, for all $x \in X$. Consider the sequence $x_n = \dfrac{1}{n}$. Clearly,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0.$$

Then S and T satisfy property (E.A.).

Example 2.7. Let $X = [4, +\infty)$. Define $S, T : X \to X$ by $Tx = x + 2$ and $Sx = 3x + 2$, for all $x \in X$. Suppose that the property (E.A.) holds. Then, there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = z - 2 \quad \text{and} \quad \lim_{n\to\infty} x_n = \frac{z-2}{3}.$$

Thus, $z = 2$, which is a contradiction, since $2 \notin X$. Hence S and T do not satisfy property (E.A.).

Notice that weakly compatible and property (E.A.) are independent of each other in perturbed metric spaces.

Example 2.8. Let $X = \mathbb{R}^+$ and D be the perturbed metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 0, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \text{ or } x = 0 \end{cases} \quad \text{and} \quad gx = \lfloor x \rfloor,$$

the greatest integer less than or equal to x, for all $x \in X$.

Consider a sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$, $n \geq 2$ in $(1, 2)$, then we have

$$\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n.$$

Similarly, for the sequence $\{y_n\} = \left\{1 - \frac{1}{n}\right\}$, $n \geq 2$ in $(0, 1)$, we have

$$\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n.$$

Thus the pair (f, g) satisfies property (E.A.). However, f and g are not weakly compatible as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2, \ldots$ but none of these points are coincidence points of f and g.

Thus we can conclude that, property (E.A.) does not imply weak compatibility.
The notion of E.A. property was further generalized by Sintunavarat and Kumam [14] who brought out the notion of common limit in the range property (CLR property).

Maps f, g over a metric space (X, d) satisfy the common limit in the range of g property (X be any metric space or perturbed metric space) if

$$\lim_n fx_n = \lim_n gx_n = gt \quad \text{for some } t \in X.$$

We shall denote the common limit in the range of g property hence onwards by *CLRg property*.

The implication of the CLR property and E.A. property lies in the fact that:

(a) In both these properties, the hypothesis of continuity of involved maps is relaxed along with relaxation of the condition of containment of the range subspace of a map into the range subspaces of other maps. This relaxation is mostly needed for the construction of joint iterate sequences in results related to fixed points.

(b) For the E.A. property, the closed range subspace of mapping condition replaces the need for completeness of space (or range subspaces of involved maps), whereas the CLR property makes it possible to entirely relax and not replace by any other condition, the need of space completeness (or that of involved maps' range subspaces).

3. Fixed Point Theorems for Non-Compatible Mappings

In this section, we prove a common fixed point theorem for a larger class of mappings, extending the class of compatible continuous mappings to include non-compatible and discontinuous mappings. This result generalizes the theorem of Kumar and Chugh [12] in the setting of perturbed metric spaces.

D. Delbosco [3] considered the set S of all real continuous functions $g : [0, \infty)^3 \to [0, \infty)$ satisfying the following properties :
(i) $g(1, 1, 1) = h < 1$.
(ii) If $u, v \geq 0$ are such that

$$u \leq g(u, v, v) \quad \text{or} \quad u \leq g(v, u, v) \quad \text{or} \quad u \leq g(v, v, u), \quad \text{then} \quad u \leq hv.$$

But later on Constantin [1] considered the family G of all continuous functions g, where

$$g : [0, \infty)^5 \to [0, \infty)$$

satisfies the following properties:

(g_1) g is non-decreasing in the 4^{th} and 5^{th} variable.

(g_2) If $u, v \in [0, \infty)$ are such that

$$u \leq g(v, v, u, u + v, 0) \quad \text{or} \quad u \leq g(v, u, v, u + v, 0) \quad \text{or} \quad u \leq g(v, u, v, u + v, 0) \quad \text{or} \quad u \leq g(v, u, v, 0, u + v),$$

then $u \leq hv$ where $0 < h < 1$ is a given constant.

(g_3) If $u \in [0, \infty)$ is such that

$$u \leq g(u, 0, 0, u, u) \quad \text{or} \quad u \leq g(0, u, 0, u, u) \quad \text{or} \quad u \leq g(0, 0, u, u, u),$$

then $u = 0$.

Theorem 3.1. [12] Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions:
Let (A, S) and (B, T) be pointwise $R-$ weakly commuting pairs of self mappings of a complete metric space (X, d) such that

(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.2) $d(Ax, By) \leq g\big(d(Sx, Ty),\ d(Ax, Sx),\ d(By, Ty),\ d(Ax, Ty),\ d(By, Sx)\big)$ for all $x, y \in X$, where $g \in G$.

Suppose that (A, S) and (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Now we prove this theorem in the setting of perturbed metric spaces.

Let A, B, S and T be mappings from a perturbed metric space (X, D, P) into itself satisfying the following conditions:
Let (A, S) and (B, T) be pointwise $R-$ weakly commuting pairs of self mappings of a complete perturbed metric space (X, D, P) such that

(3.3) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.4) $D(Ax, By) \leq g\big(D(Sx, Ty),\ D(Ax, Sx),\ D(By, Ty),\ D(Ax, Ty),\ D(By, Sx)\big)$ for all $x, y \in X$, where $g \in G$.

Then for an arbitrary point x_0 in X, by (3.3), we choose a point x_1 such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},\ n = 0, 1, 2, 3, \ldots \tag{3.5}$$

Lemma 3.1. Let A, B, S and T be mappings from a complete perturbed metric space (X, D, P) into itself satisfying the conditions (3.3) and (3.4). Then the sequence $\{y_n\}$ defined by (3.5) is a Cauchy sequence in X.

Proof: From (3.4) we have

$$\begin{aligned} D(Ax_{2n}, Bx_{2n+1}) \leq g\big(&D(Sx_{2n}, Tx_{2n+1}),\ D(Ax_{2n}, Sx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\ &D(Ax_{2n}, Tx_{2n+1}),\ D(Bx_{2n+1}, Sx_{2n})\big). \end{aligned}$$

$$D(y_{2n}, y_{2n+1}) \leq g\big(D(y_{2n-1}, y_{2n}),\ D(y_{2n}, y_{2n-1}),\ D(y_{2n+1}, y_{2n}), D(y_{2n}, y_{2n}), D(y_{2n+1}, y_{2n-1}).$$

$$D(y_{2n}, y_{2n+1}) \leq g\big(D(y_{2n-1}, y_{2n}),\ D(y_{2n}, y_{2n-1}),\ D(y_{2n+1}, y_{2n}),\ 0,\ [D(y_{2n+1}, y_{2n}) + D(y_{2n}, y_{2n-1})]\big).$$

By (g_2), we obtain

$$D(y_{2n}, y_{2n+1}) \leq h\, D(y_{2n-1}, y_{2n}).$$

But

$$D(y_n, y_{n+1}) \leq hD(y_{n-1}, y_n) \leq \cdots \leq h^n D(y_0, y_1),$$

$$D(y_n, y_{n+1}) \leq h^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric, we deduce that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq h^n D(y_0, y_1) \quad , n \in \mathbb{N}.$$

Since,

$$d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}) + P(y_n, y_{n+1}).$$

Therefore,

$$d(y_n, y_{n+1}) \leq h^n D(y_0, y_1) \quad , n \in \mathbb{N}.$$

Moreover, for every integer $m > 0$, we get

$$\begin{aligned} d(y_n, y_{n+m}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+m-1}, y_{n+m})\\ &\leq h^n D(y_0, y_1) + h^{n+1} D(y_0, y_1) + \cdots + h^{n+m-1} D(y_0, y_1)\\ &= h^n D(y_0, y_1)\,(1 + h + h^2 + \cdots + h^{m-1}) \end{aligned}$$

$$d(y_n, y_{n+m}) \leq \frac{h^n}{1-h}\, D(y_0, y_1). \tag{3.6}$$

Therefore $\{y_n\}$ is a Cauchy sequence in metric space (X, d), so $\{y_n\}$ is also a perturbed Cauchy sequence in the perturbed metric space (X, D, P).

Taking limit as $n \to \infty$, we have $d(y_n, y_{n+m}) \to 0$. Therefore, $\{y_n\}$ is a perturbed Cauchy sequence in (X, D, P).

Theorem 3.2. Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of a complete perturbed metric space (X, D, P) satisfying (3.3) and (3.4).

Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Proof : By Lemma 3.1, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Then

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \quad \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Suppose (A, S) are compatible and reciprocally continuous. Then by reciprocal continuity

$$\lim_{n\to\infty} ASx_{2n} = Az, \quad \lim_{n\to\infty} SAx_{2n} = Sz.$$

Also, by compatibility of A and S, $Az = Sz$. Since $AX \subset TX$, there exists $v \in X$ such that $Az = Tv$.

Now

$$D(Az, Bv) \le g(D(Sz, Tv), D(Az, Sz), D(Bv, Tv), D(Az, Tv), D(Bv, Sz))$$

$$D(Az, Bv) \le g(D(Tv, Tv), D(Sz, Sz), D(Az, Bv), D(Az, Az), D(Az, Bv))$$

Substituting, we get

$$D(Az, Bv) \le g(0, 0, D(Az, Bv), 0, D(Az, Bv)).$$

By (g_3), we obtain $Az = Bv$.

Thus $Az = Sz = Tv = Bv$.

Since A and S are pointwise R-weakly commuting, there exists $R > 0$ such that $D(ASz, SAz) \le RD(Az, Sz) = 0$, which implies $ASz = SAz$ and $AAz = SAz = SSz$. Similarly, B and T are pointwise R-weakly commuting, so $BBv = BTv = TBv = TTv$.

From (3.4), we get

$$D(Az, AAz) = D(AAz, Bv) \le g(D(SAz, Tv), D(AAz, SAz), D(Bv, Tv), D(AAz, Tv), D(Bv, SAz)).$$

$$D(Az, AAz) \le g(D(AAz, Bv), 0, 0, D(AAz, Bv), D(AAz, Bv)).$$

By (g_3), $AAz = Az$. Hence Az is a common fixed point of A and S. Similarly, $Bv(= Az)$ is a common fixed point of B and T.

Finally, suppose Az and Aw are distinct common fixed points. Then by (3.4),

$$D(Az, Aw) = D(AAz, BAw) \le g(D(SAz, TAw),\ D(AAz, SAz),\ D(BAw, TAw),\ D(AAz, TAw),\ D(BAw, SAz))$$

$$= g(D(Az, Aw),\ D(Az, Az),\ D(Aw, Aw),\ D(Az, Aw),\ D(Aw, Az))$$

$$= g(D(Az, Aw),\ 0,\ 0,\ D(Az, Aw),\ D(Az, Aw)).$$

By (g_3), $Az = Aw$. Thus the common fixed point is unique. This completes the proof.

The following corollaries follow immediately from Theorem 3.2.

Corollary 3.2.1. Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of a complete perturbed metric space (X, D, P) satisfying (3.3), (3.4) and

$$D(Ax, By) \le hM(x, y), \quad 0 \le h < 1, \quad x, y \in X, \tag{3.7}$$

where

$$M(x, y) = \max\{D(Sx, Ty), D(Ax, Sx), D(By, Ty), \tfrac{1}{2}(D(Ax, Ty) + D(By, Sx))\}.$$

Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Proof: We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, \tfrac{1}{2}(x_4 + x_5)\}.$$

Since $g \in G$, we can apply Theorem 3.2 and deduce the Corollary.

Example 3.1. Let $X = [2, 20]$ and let (X, D, P) be a perturbed metric space defined by

$$D(x, y) = |x - y| + \phi(x, y), \quad x, y \in X,$$

where the perturbation function $\phi : X \times X \to [0, \infty)$ satisfies

$$\phi(x, x) = 0 \quad \text{for all } x \in X.$$

Define mappings $A, B, S, T : X \to X$ by

$$A(x) = \begin{cases} x, & x = 2, \\ 3, & x > 2, \end{cases} \quad S(x) = \begin{cases} x, & x = 2, \\ 6, & x > 2, \end{cases}$$

$$B(x) = \begin{cases} x, & x = 2 \text{ or } x > 5, \\ 6, & 2 < x \leq 5, \end{cases} \quad T(x) = \begin{cases} x, & x = 2, \\ 12, & 2 < x \leq 5, \\ x - 3, & x > 5. \end{cases}$$

Then A and S are reciprocally continuous but not continuous at the fixed point $x = 2$. The mappings B and T are non-compatible but pointwise R-weakly commuting. Thus A, B, S, T satisfy all conditions of Corollary 3.2.1 with $h = \frac{2}{3}$ and have a unique common fixed point $x = 2$.

Corollary 3.2.2 Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of a complete perturbed metric space (X, D, P) satisfying (3.3), (3.4) and

$$D(Ax, By) \leq h \max\{D(Ax, Sx), D(By, Ty), \tfrac{1}{2}D(Ax, Ty), \tfrac{1}{2}D(By, Sx), D(Sx, Ty)\}, \tag{3.8}$$

for all $x, y \in X$, where $0 \leq h < 1$. Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point in X.

Proof: We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by

$$g(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, \tfrac{1}{2}x_4, \tfrac{1}{2}x_5\}.$$

Since $g \in G$, we can apply Theorem 3.2 to obtain this corollary.

4. Fixed Point Theorems for Weakly Compatible Mappings

In this section, we prove a common fixed point theorem which extends the class of compatible continuous mappings to a larger class of weakly compatible mappings without appealing to the continuity of any map. This result generalizes the theorem of Chugh and Kumar [9] in the setting of perturbed metric spaces.

Theorem 4.1. [9] Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:

(4.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(4.2) $d(Ax, By) \leq \varphi\big(d(Sx, Ty),\ d(Ax, Sx),\ d(By, Ty),\ d(Ax, Ty),\ d(By, Sx)\big)$ for all $x, y \in X$, where $\varphi \in F$.

(4.3) (A, S) and (B, T) are weakly compatible pairs.

Then A, B, S and T have a unique common fixed point in X.

Now we prove this theorem in the setting of perturbed metric spaces.

Let R^+ denote the set of non-negative real numbers and F a family of all mappings $\varphi : (R^+)^5 \to R^+$ such that φ is upper semi-continuous, non-decreasing in each coordinate variable and for any $t > 0$,

$$\varphi(t, t, 0, \alpha t, 0) \leq \beta t, \qquad \varphi(t, t, 0, 0, \alpha t) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \varphi(t, t, \alpha_1 t, \alpha_2 t, \alpha_3 t) < t,$$

where $\gamma : R^+ \to R^+$ is a mapping and $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Lemma 4.1 [13]. For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n times composition of γ.

Let A, B, S and T be mappings from a perturbed metric space (X, D, P) into itself satisfying the following conditions:

(4.4) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(4.5) $D(Ax, By) \leq \varphi\big(D(Sx, Ty),\ D(Ax, Sx),\ D(By, Ty),\ D(Ax, Ty),\ D(By, Sx)\big)$ for all $x, y \in X$, where $\varphi \in F$.

Then for arbitrary point x_0 in X, by (4.4), we choose a point x_1 such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},\ n = 0, 1, 2, 3, \ldots \tag{4.6}$$

Lemma 4.2 $\lim_{n\to\infty} D(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence in X defined by (4.6).

Proof. Let $D_n = D(y_n, y_{n+1})$, $n = 0, 1, 2, \ldots$. Now, we shall prove the sequence $\{D_n\}$ is non-increasing in R^+, that is, $D_n \leq D_{n-1}$ for $n = 1, 2, 3, \ldots$. From (4.4), we have

$$\begin{aligned} D(Ax_{2n}, Bx_{2n+1}) \leq \varphi\big(&D(Sx_{2n}, Tx_{2n+1}),\ D(Ax_{2n}, Sx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\ &D(Ax_{2n}, Tx_{2n+1}),\ D(Bx_{2n+1}, Sx_{2n})\big). \end{aligned}$$

$$\begin{aligned} D(y_{2n}, y_{2n+1}) &\leq \varphi\big(D(y_{2n-1}, y_{2n}),\ D(y_{2n}, y_{2n-1}),\ D(y_{2n+1}, y_{2n}),\ D(y_{2n}, y_{2n}),\\ &\qquad D(y_{2n+1}, y_{2n-1})\big)\\ &= \varphi\big(D(y_{2n-1}, y_{2n}),\ D(y_{2n}, y_{2n-1}),\ D(y_{2n+1}, y_{2n}),\ 0,\ [D(y_{2n+1}, y_{2n})\\ &\qquad + D(y_{2n}, y_{2n-1})]\big)\\ &= \varphi(D_{2n-1}, D_{2n-1}, D_{2n}, 0, D_{2n} + D_{2n-1}). \end{aligned} \tag{4.7}$$

Suppose that $D_{n-1} < D_n$ for some n. Then, for some $\alpha < 2$, $D_{n-1} + D_n = \alpha D_n$. Since φ is non-increasing in each variable and $\beta < 1$ for some $\alpha < 2$. From (4.7), we have

$$D_{2n} \leq \varphi(D_{2n}, D_{2n}, D_{2n}, 0, \alpha D_{2n}) \leq \beta D_{2n} < D_{2n}.$$

Similarly, we have $D_{2n+1} < D_{2n+1}$. Hence, for every n, $D_n \leq \beta D_n < D_n$, which is a contradiction. Therefore, $\{D_n\}$ is a non-increasing sequence in R^+. Now, again by (4.4), we have

$$\begin{aligned} D_1 = D(y_1, y_2) = D(Ax_2, Bx_1) &\leq \varphi\big(D(Sx_2, Tx_1),\ D(Ax_2, Sx_2),\ D(Bx_1, Tx_1),\\ &\qquad D(Ax_2, Tx_1),\ D(Bx_1, Sx_2)\big)\\ &= \varphi\big(D(y_1, y_0),\ D(y_2, y_1),\ D(y_1, y_0),\ D(y_2, y_0),\ D(y_1, y_1)\big)\\ &= \varphi(D_0, D_1, D_0, D_0 + D_1, 0)\\ &\leq \varphi(D_0, D_0, D_0, 2D_0, D_0)\\ &= \gamma(D_0). \end{aligned}$$

In general, we have $D_n \leq \gamma^n(D_0)$, which implies that, if $D_0 > 0$, by Lemma 4.1,

$$\lim_{n\to\infty} D_n \leq \lim_{n\to\infty} \gamma^n(D_0) = 0.$$

Therefore, we have $\lim_{n\to\infty} D_n = 0$. For $D_0 = 0$, since $\{D_n\}$ is non-increasing, we have $\lim_{n\to\infty} D_n = 0$. This completes the proof.

Lemma 4.3. The sequence $\{y_n\}$ defined by (4.6) is a Cauchy in X.

Proof. Let $d = D - P$ be the exact metric. By definition, if $\{y_n\}$ is a Cauchy sequence in the metric space (X, d), that is, $\{y_n\}$ is a perturbed Cauchy sequence in the perturbed metric space (X, D, P). So here we prove sequence $\{y_n\}$ is cauchy sequence in (X, d).

By virtue of Lemma 4.2, it is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d\big(y_{2m(k)}, y_{2n(k)}\big) > \varepsilon. \tag{4.8}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (4.8), that is,

$$d\big(y_{2n(k)}, y_{2m(k)-2}\big) \leq \varepsilon \ \text{ and } \ d\big(y_{2n(k)}, y_{2m(k)}\big) > \varepsilon. \tag{4.9}$$

Then for each even integer $2k$, we have

$$\varepsilon \leq d\big(y_{2n(k)}, y_{2m(k)}\big) \leq d\big(y_{2n(k)}, y_{2m(k)-2}\big) + d\big(y_{2m(k)-2}, y_{2m(k)-1}\big) + d\big(y_{2m(k)-1}, y_{2m(k)}\big).$$

By Lemma 4.2 and (4.9), it follows that

$$d\big(y_{2n(k)}, y_{2m(k)}\big) \to \varepsilon \ \text{ as } k \to \infty. \tag{4.10}$$

By the triangle inequality, we have

$$\big|d\big(y_{2n(k)}, y_{2m(k)-1}\big) - d\big(y_{2n(k)}, y_{2m(k)}\big)\big| \leq d\big(y_{2m(k)-1}, y_{2m(k)}\big)$$

and

$$\big|d\big(y_{2n(k)+1}, y_{2m(k)-1}\big) - d\big(y_{2n(k)}, y_{2m(k)}\big)\big| \leq d\big(y_{2m(k)-1}, y_{2m(k)}\big) + d\big(y_{2n(k)}, y_{2n(k)+1}\big).$$

From Lemma 4.2 and eq. (4.10), as $k \to \infty$,

$$d\big(y_{2n(k)}, y_{2m(k)-1}\big) \to \varepsilon \ \text{ and } \ d\big(y_{2n(k)+1}, y_{2m(k)-1}\big) \to \varepsilon. \tag{4.11}$$

Therefore, by (4.5) and (4.6), we have

$$\begin{aligned} d\big(y_{2n(k)}, y_{2m(k)}\big) &\leq d\big(y_{2n(k)}, y_{2n(k)+1}\big) + d\big(y_{2n(k)+1}, y_{2m(k)}\big) \\ &= d\big(y_{2n(k)}, y_{2n(k)+1}\big) + d\big(Ax_{2m(k)}, Bx_{2n(k)+1}\big) \\ &\leq d\big(y_{2n(k)}, y_{2n(k)+1}\big) + \varphi\Big(d\big(Sx_{2m(k)}, Tx_{2n(k)+1}\big), \\ &\qquad d\big(Ax_{2m(k)}, Sx_{2m(k)}\big),\ d\big(Bx_{2n(k)+1}, Tx_{2n(k)+1}\big), \\ &\qquad d\big(Ax_{2m(k)}, Tx_{2n(k)+1}\big),\ d\big(Bx_{2n(k)+1}, Sx_{2m(k)}\big)\Big) \\ &= d\big(y_{2n(k)}, y_{2n(k)+1}\big) + \varphi\Big(d\big(y_{2m(k)-1}, y_{2n(k)}\big),\ d\big(y_{2m(k)}, y_{2m(k)-1}\big), \\ &\qquad d\big(y_{2n(k)+1}, y_{2n(k)}\big),\ d\big(y_{2m(k)}, y_{2n(k)}\big),\ d\big(y_{2n(k)+1}, y_{2m(k)-1}\big)\Big). \end{aligned} \tag{4.12}$$

Since φ is upper semi continuous, as $k \to \infty$ as in (4.11), by Lemma 4.2, eqs (4.10), (4.11) and (4.12) we have

$$\varepsilon \leq \varphi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \gamma(\varepsilon) < \varepsilon,$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in (X, d) and so is $\{y_n\}$ is also a Cauchy sequence in (X, d), that is, $\{y_n\}$ is a perturbed Cauchy sequence in the perturbed metric space (X, D, P). This completes the proof.

Theorem 4.2. Let (A, S) and (B, T) be weakly compatible pairs of self maps of a complete perturbed metric space (X, D, P) satisfying (4.4) and (4.5). One of the subspaces $A(X), B(X), S(X)$ and $T(X)$ is complete subspace of X. Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 4.3, $\{y_n\}$ is a Cauchy sequence in (X, D, P). Since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$ i.e.,

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (4.5),

$$\begin{aligned} D(Au, z) &\le D(Au, Bx_{2n-1}) + D(Bx_{2n-1}, z) \\ &\le \varphi\big(D(Su, Tx_{2n-1}),\ D(Au, Su),\ D(Bx_{2n-1}, Tx_{2n-1}), \\ &\qquad D(Au, Tx_{2n-1}),\ D(Bx_{2n-1}, Su)\big). \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\begin{aligned} D(Au, z) &\le \varphi\big(0,\ D(Au, Su),\ 0,\ D(Au, z),\ D(z, Su)\big) \\ &= \varphi\big(0,\ D(Au, z),\ 0,\ D(Au, z),\ 0\big) \le \beta\, D(Au, z), \end{aligned}$$

where $\beta < 1$. Therefore $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then, again using (4.5),

$$\begin{aligned} D(z, Bv) = D(Au, Bv) &\le \varphi\big(D(Su, Tv),\ D(Au, Su),\ D(Bv, Tv),\ D(Au, Tv),\ D(Bv, Su)\big) \\ &= \varphi\big(0, 0, D(Bv, z), 0, D(Bv, z)\big) \le \varphi(t, t, t, t, t) < t, \end{aligned}$$

where $t = D(z, Bv)$. Therefore $z = Bv = Tv$. Thus $Au = Su = Bv = Tv = z$. Since pair of maps A and S are weakly compatible, then $ASu = SAu$ i.e., $Az = Sz$. Now we show that z is a fixed point of A. If $Az \ne z$, then by (4.5),

$$\begin{aligned} D(Az, z) = D(Az, Bv) &\le \varphi\big(D(Sz, Tv),\ D(Az, Sz),\ D(Bv, Tv),\ D(Az, Tv),\ D(Bv, Sz)\big) \\ &= \varphi\big(D(Az, z), 0, 0, D(Az, z), D(Az, z)\big) \\ &\le \varphi(t, t, t, t, t) < t, \quad \text{where } t = D(Az, z). \end{aligned}$$

Therefore, $Az = z$. Hence $Az = Sz = z$. Similarly, pair of maps B and T are weakly compatible, we have $Bz = Tz = z$, since

$$\begin{aligned} D(z, Bz) = D(Az, Bz) &\le \varphi\big(D(Sz, Tz),\ D(Az, Sz),\ D(Bz, Tz),\ D(Az, Tz),\ D(Bz, Sz)\big) \\ &= \varphi\big(D(z, Tz), 0, 0, D(z, Tz), D(z, Sz)\big) \\ &\le \varphi(t, t, t, t, t) < t, \quad \text{where } t = D(z, Tz) = D(z, Bz). \end{aligned}$$

Thus $z = Az = Bz = Sz = Tz$, and z is a common fixed point of A, B, S and T. Finally, in order to prove the uniqueness of z, suppose that z and w, $z \ne w$, are common fixed points of A, B, S and T. Then by (4.5), we obtain

$$\begin{aligned} D(z, w) = D(Az, Bw) &\le \varphi\big(D(Sz, Tw),\ D(Az, Sz),\ D(Bw, Tw),\ D(Az, Tw),\ D(Bw, Sz)\big) \\ &= \varphi\big(D(z, w), 0, 0, D(z, w), D(z, w)\big) \le \varphi(t, t, t, t, t) < t, \quad \text{where } t = D(z, w). \end{aligned}$$

Therefore, $z = w$. The following corollaries follow immediately from Theorem 4.2.

Corollary 4.2.1 Let (A, S) and (B, T) be weakly compatible pairs of self maps of a complete perturbed metric space (X, D, P) satisfying (4.4), (4.6) and (4.13)

$$D(Ax, By) \le h\, M(x, y),\ \ 0 \le h < 1,\ \ x, y \in X,\ \text{ where}$$

$$M(x,y) = \max\Big\{D(Sx,Ty),\ D(Ax,Sx),\ D(By,Ty),\ [D(Ax,Ty)+D(By,Sx)]/2\Big\}. \tag{4.13}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\varphi : [0,\infty)^5 \to [0,\infty)$ defined by

$$\varphi(x_1,x_2,x_3,x_4,x_5) = h\max\{x_1,x_2,x_3,\tfrac{1}{2}(x_4+x_5)\}.$$

Since $\varphi \in F$, we can apply Theorem 4.2 and deduce the Corollary.

Corollary 4.2.2 Let (A,S) and (B,T) be weakly compatible pairs of self maps of a complete perturbed metric space (X,D,P) satisfying (4.4), (4.6) and (4.14)

$$D(Ax,By) \le h\max\{D(Ax,Sx),\ D(By,Ty),\ \tfrac{1}{2}D(Ax,Ty),\ \tfrac{1}{2}D(By,Sx),\ D(Sx,Ty)\}$$

for all x,y in X, where $0 \le h < 1$.

(4.14)

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\varphi : [0,\infty)^5 \to [0,\infty)$ defined by

$$\varphi(x_1,x_2,x_3,x_4,x_5) = h\max\{x_1,x_2,x_3,\tfrac{1}{2}x_4,\tfrac{1}{2}x_5\}.$$

Since $\varphi \in F$, we can apply Theorem 4.2 to obtain this Corollary.

5. Weakly Compatible Mappings and Variants of $R-$ Weakly Commuting Mappings

In this section, we prove common fixed point theorems for a family of self-mappings in the framework of perturbed metric spaces, using the notions of R-weak commutativity together with mappings of types (A_f), (A_g), and P, in combination with weakly compatible mappings.

Theorem 5.1. Let A, B, S and T be mappings from a perturbed metric space (X,D,P) into itself satisfying the following conditions:

(5.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(5.2) $D(Ax,By) \le \varphi\big(D(Sx,Ty),\ D(Ax,Sx),\ D(By,Ty),\ D(Ax,Ty),\ D(By,Sx)\big)$ for all $x,y \in X$, where $\varphi \in F$.

Let (A,S) and (B,T) be $R-$ weakly commuting pairs of self maps of a complete perturbed metric space (X,D,P) satisfying (5.1) and (5.2). Assume further that $S(X)$ and $T(X)$ are closed subsets of X. Then A, B, S and T have a unique common fixed point in X.

Proof. By Lemma 4.3, $\{y_n\}$ is a Cauchy sequence in (X,D,P). Since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$ i.e.,

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (5.2),

$$\begin{aligned} D(Au,z) &\le D(Au,Bx_{2n-1}) + D(Bx_{2n-1},z) \\ &\le \varphi\big(D(Su,Tx_{2n-1}),\ D(Au,Su),\ D(Bx_{2n-1},Tx_{2n-1}), \\ &\qquad D(Au,Tx_{2n-1}),\ D(Bx_{2n-1},Su)\big). \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\begin{aligned} D(Au,z) &\le \varphi\big(0,\ D(Au,Su),\ 0,\ D(Au,z),\ D(z,Su)\big) \\ &= \varphi\big(0,\ D(Au,z),\ 0,\ D(Au,z),\ 0\big) \le \beta\, D(Au,z), \end{aligned}$$

where $\beta < 1$. Therefore $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then, again using (5.2),

$$\begin{aligned} D(z,Bv) = D(Au,Bv) &\le \varphi\big(D(Su,Tv),\ D(Au,Su),\ D(Bv,Tv),\ D(Au,Tv),\ D(Bv,Su)\big) \\ &= \varphi\big(0,0,D(Bv,z),0,D(Bv,z)\big) \le \varphi(t,t,t,t,t) < t, \end{aligned}$$

where $t = D(z,Bv)$. Therefore $z = Bv = Tv$. Thus $Au = Su = Bv = Tv = z$. Since pair of maps A and S are $R-$ weakly commuting, then $ASu = SAu$ i.e., $Az = Sz$. Now we show that z is a fixed point of A. If $Az \ne z$, then by (5.2),

$$\begin{aligned} D(Az,z) = D(Az,Bv) &\le \varphi\big(D(Sz,Tv),\ D(Az,Sz),\ D(Bv,Tv),\ D(Az,Tv),\ D(Bv,Sz)\big) \\ &= \varphi\big(D(Az,z),0,0,D(Az,z),D(Az,z)\big) \\ &\le \varphi(t,t,t,t,t) < t, \quad \text{where } t = D(Az,z). \end{aligned}$$

Therefore, $Az = z$. Hence $Az = Sz = z$. Similarly, pair of maps B and T are $R-$ weakly commuting, we have $Bz = Tz = z$, since

$$\begin{aligned} D(z,Bz) = D(Az,Bz) &\le \varphi\big(D(Sz,Tz),\ D(Az,Sz),\ D(Bz,Tz),\ D(Az,Tz),\ D(Bz,Sz)\big) \\ &= \varphi\big(D(z,Tz),0,0,D(z,Tz),D(z,Sz)\big) \\ &\le \varphi(t,t,t,t,t) < t, \quad \text{where } t = D(z,Tz) = D(z,Bz). \end{aligned}$$

Thus $z = Az = Bz = Sz = Tz$, and z is a common fixed point of A, B, S and T. Finally, in order to prove the uniqueness of z, suppose that z and w, $z \ne w$, are common fixed points of A, B, S and T. Then by (5.2), we obtain

$$\begin{aligned} D(z,w) = D(Az,Bw) &\le \varphi\big(D(Sz,Tw),\ D(Az,Sz),\ D(Bw,Tw),\ D(Az,Tw),\ D(Bw,Sz)\big) \\ &= \varphi\big(D(z,w),0,0,D(z,w),D(z,w)\big) \le \varphi(t,t,t,t,t) < t, \quad \text{where } t = D(z,w). \end{aligned}$$

Therefore, $z = w$.

Theorem 5.2. Theorem 5.1 remains true if $R-$ weakly commuting property is replaced by any one (retaining the rest of the hypothesis) of the following:

(a) R-weakly commuting property of type (A_g),

(b) R-weakly commuting property of type (A_f),

(c) R-weakly commuting property of type (P).

6. Weakly Compatible Mappings and Property (E.A.)

In this section, we investigate the existence of common fixed points for weakly compatible mappings in the setting of perturbed metric spaces. We consider these mappings under the additional assumption of property (E.A.), which allows us to establish fixed point results without requiring completeness of the space.

Theorem 6.1. [11] Let A, B, S and T be self-maps of a metric space (X,d) satisfying the following conditions:

(6.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

(6.2) $d(Ax,By) \le \phi\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ax,Ty)\}$
for all $x, y \in X$, where $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is in the class $\mathcal{F}$ of all upper semi-continuous mappings, strictly non-decreasing in each coordinate variable.

(6.3) Pairs (A,S) or (B,T) satisfy property (E.A.).

(6.4) Pairs (A,S) and (B,T) are weakly compatible.

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Now we prove fixed point theorems in the setting of perturbed metric space for a pair of weakly compatible maps and property (E.A.).

Theorem 6.2 Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying the following conditions:

(6.5) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

(6.6) $D(Ax, By) \leq \phi\{D(Sx, Ty), D(Ax, Sx), D(By, Ty), D(Sx, By), D(Ax, Ty)\}$ for all $x, y \in X$, where $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is in the class $\mathcal{F}$ of all upper semi-continuous mappings, strictly non-decreasing in each coordinate variable.

(6.7) Pairs (A, S) or (B, T) satisfy property (E.A.).

(6.8) Pairs (A, S) and (B, T) are weakly compatible.

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof: Suppose that (B, T) satisfies the property (E.A.). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} D(Bx_n, z) = \lim_{n\to\infty} D(Tx_n, z) = 0$$

for some $z \in X$.

Since $B(X) \subseteq S(X)$, therefore, there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} D(Bx_n, z) = \lim_{n\to\infty} D(Tx_n, z) = 0.$$

Now we shall show that $\lim_{n\to\infty} D(Ay_n, z) = 0$. We claim that $\lim_{n\to\infty} D(Ay_n, z) = l$.

From (6.6), we have

$$D(Ay_n, Bx_n) \leq \phi\{D(Sy_n, Tx_n), D(Ay_n, Sy_n), D(Bx_n, Tx_n), D(Sy_n, Bx_n), D(Ay_n, Tx_n)\}.$$

Proceeding limit as $n \to \infty$, we have

$$D(l, z) \leq \phi\{D(z, z), D(l, z), D(z, z), D(z, z), D(l, z)\} = \phi\{0, D(l, z), 0, 0, D(l, z)\} \leq \phi(t, t, t, t, t) < t,$$

where $t = D(l, z)$.

Hence, we have $l = z$. Therefore, $\lim\limits_{n\to\infty} D(Ay_n, z) = 0$.

Suppose that $S(X)$ is a closed subset of X. Then $z = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} D(Ay_n, z) = \lim_{n\to\infty} D(Bx_n, z) = \lim_{n\to\infty} D(Tx_n, z) = \lim_{n\to\infty} D(Sy_n, z) = z = Su.$$

From (6.6), we have

$$D(Au, Bx_n) \leq \phi\{D(Su, Tx_n), D(Au, Su), D(Bx_n, Tx_n), D(Su, Bx_n), D(Au, Tx_n)\}.$$

Letting $n \to \infty$, we have $Au = Su = z$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$.

Now, we claim that $z = Tv = Bv$. From (6.6), we have

$$D(Au, Bv) \leq \phi\{D(Su, Tv), D(Au, Su), D(Bv, Tv), D(Su, Bv), D(Au, Tv)\}.$$

Letting $n \to \infty$, we have $z = Bv$. Thus, we have $Au = Su = Tv = Bv = z$.

Since the pair (A,S) is weakly compatible, it follows that

$$ASu = SAu, \quad \text{i.e., } Az = Sz.$$

From (6.6), we have

$$D(Az,Bv) \le \phi\{D(Sz,Tv), D(Az,Sz), D(Bv,Tv), D(Sz,Bv), D(Az,Tv)\},$$

which implies that $Az = Sz = z$.

The weak compatibility of B and T implies that $BTv = TBv$, i.e., $Bz = Tz$.

Now we shall show that z is the common fixed point of B. From (6.6), one obtains $Bz = z$.

Hence $Az = Bz = Sz = Tz = z$, and therefore z is a common fixed point of A, B, S and T.

Example 6.1. Let $X = [0,2]$ equipped with the perturbed metric D. Define A, B, S and T by

$$Ax = Tx = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } 1 \le x \le 2, \end{cases} \qquad Bx = Sx = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 2, & \text{if } 1 \le x \le 2. \end{cases}$$

The mapping $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is defined by

$$\phi(x_1, x_2, x_3, x_4, x_5) = x_1.$$

Then

$$D(Sx,Ty) \le \phi\{D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Ax,Ty), D(By,Sx)\}$$

for all $x, y \in X$.

Consider $x_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0.$$

Hence, pairs (A,S) and (B,T) satisfy property (E.A.). Also, $A(X) = T(X) = \{0,1\}$ and $B(X) = S(X) = \{0,2\}$ are closed subsets of X. Moreover, pairs (A,S) and (B,T) are weakly compatible. Thus, all the conditions of the above theorem are satisfied, and 0 is the unique common fixed point of A, B, S and T.

Corollary 6.2.1. Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying (6.5), (6.7), and (6.8), and the following:

$$D(Ax,By) \le k \max\{D(Sx,Ty), D(Ax,Sx), D(By,Ty), D(Sx,By), D(Ax,Ty)\} \tag{6.9}$$

for all $x, y \in X$, where $k \in (0,1)$.

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof: Take $\phi(x_1, x_2, x_3, x_4, x_5) = \max\{x_1, x_2, x_3, x_4, x_5\}$ in Theorem 6.2.

Example 6.2. Let $X = [0,2]$ equipped with the perturbed metric $D(x,y) = |x-y| + xy$ with perturbing function $P(x,y) = xy$.

Define the self-maps A, B, S and $T : X \to X$ by

$$Ax = \begin{cases} 0, & \text{if } x = 0, \\ 0.25, & \text{if } x > 0, \end{cases} \qquad Bx = \begin{cases} 0, & \text{if } x = 0, \\ 0.45, & \text{if } x > 0, \end{cases}$$

$$Sx = \begin{cases} 0, & \text{if } x = 0, \\ 0.40, & \text{if } 0 < x \le 0.6, \\ x - 0.45, & \text{if } x > 0.6, \end{cases} \qquad Tx = \begin{cases} 0, & \text{if } x = 0, \\ 0.25, & \text{if } 0 < x \le 0.6, \\ x - 0.25, & \text{if } x > 0.6. \end{cases}$$

Then

$$AX = \{0, 0.25\}, \quad BX = \{0, 0.45\}, \quad SX = \{0\} \cup (0.15, 1.55], \quad TX = \{0\} \cup \{0.25\} \cup (0.35, 1.75].$$

Let us consider the sequence $x_n = 0.60 + \frac{1}{n}$. Then

$$\begin{aligned} Ax_n &\to 0.25, & Bx_n &\to 0.45, \\ Sx_n &\to 0.15, & Tx_n &\to 0.35, \\ ASx_n &\to 0.25, & SAx_n &\to 0.40, \\ BTx_n &\to 0.45, & TBx_n &\to 0.25. \end{aligned}$$

Hence, pairs (A, S) and (B, T) are not compatible. If we take $k = 0.6$ and $t = 1$, then A, B, S and T satisfy all the conditions of Corollary 6.2.1, and 0 is the unique common fixed point of A, B, S and T. Moreover, A, B, S and T are discontinuous at the fixed point 0.

Next, we consider a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

$$\begin{cases} \psi \text{ is continuous and nondecreasing on } \mathbb{R}^+, \\ \psi(t) < t \text{ for all } t > 0. \end{cases} \tag{6.10}$$

We note that $\psi(1) = 1$ and $\psi(t) < t$ for all $t > 0$. Also, $\psi(D(x, y)) < D(x, y)$ holds for all $x, y \in X$.
Theorem 6.3. Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying (6.5), (6.7), (6.8) and the following condition:

$$D(Ax, By) \leq \psi\big(\max\big\{D(Sx, Ty), D(Ax, Sx), D(By, Ty), D(Sx, By), D(Ax, Ty)\big\}\big) \tag{6.11}$$

for all $x, y \in X$. If the range of one of A, B, S or T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.
Proof: Suppose that (B, T) satisfies the property (E.A.). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n = z$. Hence, $\lim_{n\to\infty} Sy_n = z$.

We shall show that $\lim_{n\to\infty} Ay_n = z$.

From (6.11), we have

$$D(Ay_n, Bx_n) \leq \psi\big(\max\big\{D(Sy_n, Tx_n),\ D(Ay_n, Sy_n),\ D(Bx_n, Tx_n),\ D(Sy_n, Bx_n),\ D(Ay_n, Tx_n)\big\}\big).$$

Proceeding to the limit as $n \to \infty$, one obtains

$$\lim_{n\to\infty} Ay_n = z.$$

Suppose that $S(X)$ is a closed subspace of X. Then $z = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z = Su.$$

Now, we shall show that $Au = Su$. From (6.11), we have

$$D(Au, Bx_n) \leq \psi\big(\max\big\{D(Su, Tx_n),\ D(Au, Su),\ D(Bx_n, Tx_n),\ D(Su, Bx_n),\ D(Au, Tx_n)\big\}\big).$$

Letting $n \to \infty$, we get

$$D(Au, z) \leq \psi\big(\max\big\{D(z, z),\ D(Au, z),\ D(z, z),\ D(z, z),\ D(Au, z)\big\}\big).$$

Using (6.10), we have $Au = Su = z$.
Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that

$$z = Au = Tv.$$

Now, we claim that $z = Tv = Bv$. From (6.11), we have

$$D(Au, Bv) \leq \psi\big(\max\{D(Su,Tv),\, D(Au,Su),\, D(Bv,Tv),\, D(Su,Bv),\, D(Au,Tv)\}\big)\,.$$

Using (6.10), we have $z = Bv$. Thus, we have

$$Au = Su = Tv = Bv = z.$$

Since the pair (A, S) is weakly compatible, it implies

$$ASu = SAu, \quad \text{i.e., } Az = Sz.$$

From (6.11), we have

$$D(Az, Bz) \leq \psi\big(\max\{D(Sz,Tz),\, D(Az,Sz),\, D(Bz,Tz),\, D(Sz,Bz),\, D(Az,Tz)\}\big)\,.$$

$$D(Az, Bv) \leq \psi\big(\max\{D(Sz,Tv),\, D(Az,Sz),\, D(Bv,Tv),\, D(Sz,Bv),\, D(Az,Tv)\}\big)\,.$$

Using (6.10), we have $Az = Sz = z$.
The weak compatibility of B and T implies that

$$BTv = TBv, \quad \text{i.e., } Bz = Tz.$$

Now, we shall show that z is the common fixed point of A, B, S and T.
Suppose that $Bz \neq z$. Then, using (6.11), one obtains $Bz = z$.
Hence,

$$Az = Bz = Sz = Tz = z,$$

and therefore z is a common fixed point of A, B, S and T.
Uniqueness follows easily.
Theorem 6.4. Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying (6.5), (6.8) and the following conditions:

(6.11) pairs (A, S) and (B, T) satisfy a common property (E.A.).

If the range of S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.
Proof: Suppose that (A, S) and (B, T) satisfy a common (E.A.) property. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$.
Since $S(X)$ and $T(X)$ are closed subsets of X, we obtain

$$z = Su = Tv \quad \text{for some } u, v \in X.$$

From (6.11), we have

$$D(Au, By_n) \leq \psi\big(\max\{D(Su,Ty_n),\, D(Au,Su),\, D(By_n,Ty_n),\, D(Su,By_n),\, D(Au,Ty_n)\}\big)\,.$$

Letting $n \to \infty$ and using (6.10), we have

$$z = Au = Su = Tv.$$

The rest of the proof follows from Theorem 6.3.
A common fixed point theorem for weakly compatible mappings with common limit range property is now proved in perturbed metric spaces.
Theorem 6.5. Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying (6.5), (6.6) and (6.8)

(6.12) Pairs (A, S) and (B, T) satisfy CLR_S and CLR_T, respectively (common-limit-in-range property),

Then A, B, S and T have a unique common fixed point in X.

Proof. Since the pair (B, T) satisfies the property (CLR), there exists a sequence $\{x_n\}$ in X and a point $v \in X$ such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tv = z.$$

As $B(X) \subseteq S(X)$, for each n there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = z.$$

Now, we shall show that $\lim_{n\to\infty} D(Ay_n, z) = 0$. Let $l = \lim_{n\to\infty} D(Ay_n, z)$.

From (6.6), we have

$$D(Ay_n, Bx_n) \leq \phi\{D(Sy_n, Tx_n), D(Ay_n, Sy_n), D(Bx_n, Tx_n), D(Sy_n, Bx_n), D(Ay_n, Tx_n)\}.$$

Taking limit as $n \to \infty$, and using the monotonicity and upper semi-continuity of ϕ, we obtain

$$l \leq \phi\{l, l, 0, 0, l\}.$$

Let $t = l$. Then $\phi(t, t, t, t, t) < t$ for all $t > 0$, which implies $t = 0$. Hence $l = 0$ and therefore $\lim_{n\to\infty} D(Ay_n, z) = 0$.

Since (A, S) also satisfies the property (CLR), there exists $u \in X$ such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = Su = z.$$

Then, using (6.6), we have

$$D(Au, Bx_n) \leq \phi\{D(Su, Tx_n), D(Au, Su), D(Bx_n, Tx_n), D(Su, Bx_n), D(Au, Tx_n)\}.$$

Letting $n \to \infty$, we obtain $Au = Su = z$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$. Again, from (6.6), we have

$$D(Au, Bv) \leq \phi\{D(Su, Tv), D(Au, Su), D(Bv, Tv), D(Su, Bv), D(Au, Tv)\}.$$

Letting $n \to \infty$, we have $z = Bv$. Thus, we have $Au = Su = Tv = Bv = z$.

Since the pair (A, S) is weakly compatible, it follows that

$$ASu = SAu, \quad \text{i.e., } Az = Sz.$$

From (6.6), we have

$$D(Az, Sz) \leq \phi\{D(Sz, Tz), D(Az, Sz), D(Bz, Tz), D(Sz, Bz), D(Az, Tz)\}.$$

$$D(Az, Bv) \leq \phi\{D(Sz, Tv), D(Az, Sz), D(Bv, Tv), D(Sz, Bv), D(Az, Tv)\}.$$

This implies that $Az = Sz = z$.

The weak compatibility of B and T implies that $BTv = TBv$, i.e., $Bz = Tz$.

Now, we shall show that z is the common fixed point of A, B, S and T. From (6.6), one obtains

$$D(z, Bz) = D(Az, Bz) \leq \phi\{D(Sz, Tz), D(Az, Sz), D(Bz, Tz), D(Az, Tz), D(Bz, Sz)\} \leq \phi(t, t, t, t, t) < t,$$

where $t = D(z, Bz)$. Hence $Bz = z$. Therefore $Az = Bz = Sz = Tz = z$, and z is a common fixed point of A, B, S and T.

Finally, to prove the uniqueness of z, suppose that z and w $(z \neq w)$ are two common fixed points of A, B, S and T. Then, by (6.6),

$$D(z,w) = D(Az, Bw) \leq \phi\{D(Sz, Tw), D(Az, Sz), D(Bw, Tw), D(Az, Tw), D(Bw, Sz)\} \leq \phi(t,t,t,t,t) < t,$$

where $t = D(z,w)$. This is a contradiction. Hence, the common fixed point is unique.

Theorem 6.6. Let A, B, S and T be self-maps of a perturbed metric space (X, D, P) satisfying (6.5), (6.6) and (6.8)

(6.13) The pair (B, T) satisfies the property (CLR).

If the range of S is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Since (B, T) satisfies the property (CLR), there exists a sequence $\{x_n\}$ in X and a point $v \in X$ such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tv = z.$$

As $B(X) \subseteq S(X)$, for each n there exists $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = z.$$

Since $S(X)$ is closed, there exists $u \in X$ such that $Su = z$.
Now, we shall show that $\lim_{n\to\infty} D(Ay_n, z) = 0$. Let $l = \lim_{n\to\infty} D(Ay_n, z)$.

From (6.6), we have

$$D(Ay_n, Bx_n) \leq \phi\{D(Sy_n, Tx_n), D(Ay_n, Sy_n), D(Bx_n, Tx_n), D(Sy_n, Bx_n), D(Ay_n, Tx_n)\}.$$

Taking limit as $n \to \infty$, and using the monotonicity and upper semi-continuity of ϕ, we obtain

$$l \leq \phi\{l, l, 0, 0, l\}.$$

Let $t = l$. Then $\phi(t,t,t,t,t) < t$ for all $t > 0$, which implies $t = 0$. Hence $l = 0$ and therefore $\lim_{n\to\infty} D(Ay_n, z) = 0$.

Using (6.6) with Bx_n and letting $n \to \infty$, we obtain $Au = Su = z$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$. Again, from (6.6), we have

$$D(Au, Bv) \leq \phi\{D(Su, Tv), D(Au, Su), D(Bv, Tv), D(Su, Bv), D(Au, Tv)\}.$$

Letting $n \to \infty$, we have $z = Bv$. Thus, we have $Au = Su = Tv = Bv = z$.

Since the pair (A, S) is weakly compatible, it follows that

$$ASu = SAu, \quad \text{i.e., } Az = Sz.$$

From (6.6), we have

$$D(Az, Sz) \leq \phi\{D(Sz, Tz), D(Az, Sz), D(Bz, Tz), D(Sz, Bz), D(Az, Tz)\}.$$

$$D(Az, Bv) \leq \phi\{D(Sz, Tv), D(Az, Sz), D(Bv, Tv), D(Sz, Bv), D(Az, Tv)\}.$$

This implies that $Az = Sz = z$.

The weak compatibility of B and T implies that $BTv = TBv$, i.e., $Bz = Tz$.

Now, we shall show that z is the common fixed point of A, B, S and T. From (6.6), one obtains

$$D(z, Bz) = D(Az, Bz) \leq \phi\{D(Sz, Tz), D(Az, Sz), D(Bz, Tz), D(Az, Tz), D(Bz, Sz)\} \leq \phi(t,t,t,t,t) < t,$$

where $t = D(z, Bz)$. Hence $Bz = z$. Therefore $Az = Bz = Sz = Tz = z$, and z is a common fixed point of A, B, S and T.

Finally, to prove the uniqueness of z, suppose that z and w $(z \neq w)$ are two common fixed points of A, B, S and T. Then, by (6.6),

$$D(z,w) = D(Az, Bw) \leq \phi\{D(Sz, Tw), D(Az, Sz), D(Bw, Tw), D(Az, Tw), D(Bw, Sz)\} \leq \phi(t,t,t,t,t) < t,$$

where $t = D(z,w)$. This is a contradiction. Hence, the common fixed point is unique.

7. Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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