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Perturbation Analysis of Fractional mKdV and Generalized KdV-Burgers Equations with Cubic Nonlinearity

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ABSTRACT: This work deals with an analysis of two singularly perturbed partial differential equations with cubic nonlinearities using the conformable derivative approach. We investigate a modified Korteweg-de Vries type equation and a generalized KdV-Burgers equation, both incorporating conformable fractional derivatives and singular perturbation parameters. The conformable derivatives enable a treatment of fractional-order effects in these two nonlinear problems. Using asymptotic expansion methods, we derive solutions up to first-order corrections and examine the differences between perturbed and unperturbed cases. Our analysis shows how the conformable order and perturbation parameters influence wave propagation characteristics, and solution behavior. The results demonstrate qualitative differences from classical integer-order models, particularly in soliton dynamics and pattern formation.

Keywords: Singular perturbations, Khalil derivatives, mKdV equation, mKdV-Burgers equation, cubic nonlinearity, asymptotic methods, soliton dynamics.

Contents

1. Introduction

The study of nonlinear wave phenomena has been discussed by differential equations that capture essential physical mechanisms [7,20,24,25]. Among these, the modified Korteweg-de Vries (mKdV) equation $aV_t + bV^2V_x + cV_{xxx} = 0$ and its generalizations stand as crucial models for understanding wave propagation in dispersive media with cubic nonlinearities including plasma physics, nonlinear optics, and fluid dynamics, see [6,10,13,21,18,31]. Concurrently, the modified KdV-Burgers equation, with dispersive and dissipative effects through the combination

$$V_t + a_2V^2V_x + a_3V_{xx} + a_4V_{xxx} = 0,$$

has provided applications into wave dynamics in viscous media, atmospheric sciences, and condensed matter systems, see [5,19,32].

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Recent advances in fractional calculus have opened new ways for extending these classical models to describe phenomena in complex and heterogeneous media [3,8,14,30]. The conformable derivative, introduced by Khalil et al. [12], provides a local definition that satisfies classical rules such as the product rule, quotient rule, and chain rule, making it suitable for analytical studies in nonlinear problems. This approach has given significant research interest, and recent work by Abdeljawad [1], and others [2,4,15,17,26] explored its applications to various dynamical systems and physical models.

Singular perturbation theory provides important tools for studying problems where infinitesimal parameters multiply the highest-order derivatives [22]. Such problems show multi-scale behavior. In the case of partial differential equations, singular perturbations arise in problems involving boundary layers, phase transitions, and regularized nonlinear waves [9,11,23,28,29].

The intersection of fractional calculus and singular perturbation theory remains relatively unexplored territory, particularly for equations with cubic nonlinearities. Recent studies have examined fractional versions of KdV-type equations using various derivative definitions, but the specific combination of conformable derivatives with singular perturbations in mKdV and generalized KdV-Burgers frameworks represents a novel direction of investigation. This gap motivates our present work, which aims to develop a comprehensive theoretical framework for such hybrid problems.

In this paper, we investigate two singularly perturbed fractional differential equations incorporating Khalil conformable derivatives. We consider the following perturbed problems:

$$\varepsilon D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha V + \delta V_t + \gamma V^2 V_x + \beta V_{xxx} = 0, \quad (0 < \alpha \leq 1) \tag{1.1}$$

and

$$\varepsilon D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha V + V_t + dV^2 V_x + \beta V_{xx} + \gamma V_{xxx} + \mu V = 0, \quad (0 < \alpha \leq 1), \tag{1.2}$$

where, the coefficients of the terms are non zero real constants, $0 < \alpha \leq 1$ and $0 < \varepsilon \ll 1$.

In the above problems, the quantity $V(x,t)$ represents the unknown function that describes the wave amplitude or potential, x and t denote the spatial and temporal coordinates, respectively. The operator D_x^α signifies the Khalil conformable derivative of order α; $0 < \alpha \leq 1$ with respect to the spatial variable x. The parameter ε is a small positive parameter that multiplies the highest-order fractional derivative term, giving the problems in singularly perturbed forms. The coefficients $\delta, \gamma, \beta, d, \mu$ are non-zero real constants that balance the influence of various effects, including dispersion (β, γ), nonlinearity (γ, d), and dissipation (or damping) (δ, μ)-terms.

The unperturbed problems are obtained when $\varepsilon = 0$ by:

$$\delta V_t + \gamma V^2 V_x + \beta V_{xxx} = 0, \tag{1.3}$$

and

$$V_t + dV^2 V_x + \beta V_{xx} + \gamma V_{xxx} + \mu V = 0. \tag{1.4}$$

The unperturbed cases yield two fundamental equations that form a part of our analysis. Equation (1.3) represents a modified Korteweg-de Vries (mKdV) equation with cubic nonlinearity, governing dispersive wave propagation without higher-order regularization. Equation (1.4) constitutes a generalized KdV-Burgers type equation that incorporates the effects of nonlinearity, diffusion, dispersion, and reaction. These unperturbed problems form the basis for studying how singular perturbations given through higher-order conformable derivative terms ($\varepsilon > 0$) can affect regularization mechanisms and stability properties.

Our analysis focuses on the relation between fractional order effects, singular perturbations, and nonlinear dynamics, with emphasis on asymptotic behavior. The tanh method [16] is used in this analysis; it is a powerful technique for finding traveling wave solutions to nonlinear PDEs. It exploits the fact that many soliton solutions can be expressed as polynomials in the hyperbolic tangent function. The method works by using a variable $z = \tanh(\xi)$, which transforms the PDE into a polynomial equation in z.

The paper is organized as follows: Section 2 reviews essential mathematical preliminaries on conformable derivatives and transformation of traveling wave solutions. Section 3 and 4 present a detailed analysis of the unperturbed equations (1.3) and (1.4), establishing fundamental solution properties and conservation laws. Section 5 develops the asymptotic framework for the singular perturbation problems (1.1) and (1.2), deriving first-order corrections and examining boundary layer effects [22]. Section 6 discusses numerical results and computational aspects. Finally, Section 7 summarizes our main results and suggests directions for future research.

2. Traveling-Wave Reduction Using Khalil Derivatives

For a differentiable function $g(x)$ with $x > 0$, the Khalil conformable derivative of order α is:

$$D_x^\alpha g(x) = x^{1-\alpha}\frac{dg}{dx}(x). \tag{2.1}$$

Now, using the transformation,

$$\xi = \frac{x^\alpha}{\alpha} - ct, \quad V(x,t) = U(\xi), \tag{2.2}$$

we obtain the following new classical derivatives:

$$D_x^\alpha V = U'(\xi), \quad V_t = -cU'(\xi), \quad V_x = U'(\xi), \quad V_{xx} = U''(\xi), \quad V_{xxx} = U^{(3)}(\xi). \tag{2.3}$$

3. Unperturbed Case of Equat.(1.1)

We proceed to analyze the first stage of the perturbed problem (1.1). So, applying the traveling wave transformation to (1.3) which represents the unperturbed case of Equat.(1.1), we obtain:

$$\begin{aligned}
\delta V_t &= -\delta c U'(\xi) \\
\gamma V^2 V_x &= \gamma U^2 U' \\
\beta V_{xxx} &= \beta U^{(3)}(\xi).
\end{aligned}$$

The ODE becomes

$$-\delta c U' + \gamma U^2 U' + \beta U^{(3)} = 0. \tag{3.1}$$

We balance the highest order derivative term $U^{(3)}$ with the nonlinear term U^2U' (see [16]): $U^{(3)}$ has degree $N+3$ and U^2U' has degree $2N + (N+1) = 3N+1$, $N = 1$.

Thus:

$$U(\xi) = a_0 + a_1 Y, \quad Y = \tanh(\lambda\xi). \tag{3.2}$$

Now, we compute

$$U'(\xi) = \frac{dU}{dY}\frac{dY}{d\xi} = a_1\lambda(1 - Y^2)$$

$$\begin{aligned}
U^{(3)}(\xi) = \frac{d^3U}{d\xi^3} &= \lambda^3 \frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{dU}{dY}\right]\right] \\
&= \lambda^3 \frac{d}{dY}\left[(1-Y^2)\frac{d}{dY}\left[(1-Y^2)a_1\right]\right] \\
&= \lambda^3 \frac{d}{dY}\left[(1-Y^2)(-2a_1Y)\right] \\
&= \lambda^3\left[-2Y(-2a_1Y) + (1-Y^2)(-2a_1)\right] \\
&= \lambda^3\left[4a_1Y^2 - 2a_1 + 2a_1Y^2\right] \\
&= \lambda^3\left[-2a_1 + 6a_1Y^2\right],
\end{aligned}$$

then,

$$\begin{aligned}
U^2U' &= (a_0 + a_1Y)^2 \cdot a_1\lambda(1 - Y^2) \\
&= a_1\lambda(1 - Y^2)(a_0^2 + 2a_0a_1Y + a_1^2Y^2).
\end{aligned}$$

Substitute into equation (3.1), we obtain

$$
\begin{aligned}
-\delta c U' &= -\delta c a_1 \lambda (1 - Y^2) \\
\gamma U^2 U' &= \gamma a_1 \lambda (1 - Y^2)(a_0^2 + 2a_0 a_1 Y + a_1^2 Y^2) \\
\beta U^{(3)} &= \beta \lambda^3 (-2a_1 + 6a_1 Y^2).
\end{aligned}
$$

The equation becomes

$$(1 - Y^2)\left[-\delta c a_1 \lambda + \gamma a_1 \lambda (a_0^2 + 2a_0 a_1 Y + a_1^2 Y^2)\right] + \beta \lambda^3 (-2a_1 + 6a_1 Y^2) = 0. \tag{3.3}$$

Therefore,

$$
\begin{aligned}
Y^0 :&\quad -\delta c a_1 \lambda + \gamma a_0^2 a_1 \lambda - 2\beta a_1 \lambda^3 = 0 \\
Y^1 :&\quad 2\gamma a_0 a_1^2 \lambda = 0 \\
Y^2 :&\quad \gamma a_1^3 \lambda + 6\beta a_1 \lambda^3 = 0.
\end{aligned}
$$

From coefficient of Y^1, we have $2\gamma a_0 a_1^2 \lambda = 0$.
But, since $\gamma \neq 0$, $\lambda \neq 0$, and $a_1 \neq 0$, so we get:

$$a_0 = 0. \tag{3.4}$$

From coefficient of Y^2, we have $\gamma a_1^3 \lambda + 6\beta a_1 \lambda^3 = 0$.
But, $a_1 \neq 0$, $\lambda \neq 0$, hence we get

$$\gamma a_1^2 + 6\beta \lambda^2 = 0 \Rightarrow a_1^2 = -\frac{6\beta \lambda^2}{\gamma}. \tag{3.5}$$

From coefficient of Y^0, with $a_0 = 0$, we get $-\delta c a_1 \lambda - 2\beta a_1 \lambda^3 = 0$.
Consequently,

$$\delta c + 2\beta \lambda^2 = 0 \Rightarrow c = -\frac{2\beta \lambda^2}{\delta}. \tag{3.6}$$

Hence, the Solution for (1.3) is given by:

$$V(x,t) = \pm \sqrt{-\frac{6\beta \lambda^2}{\gamma}} \tanh \left[\lambda \left(\frac{x^\alpha}{\alpha} + \frac{2\beta \lambda^2}{\delta} t \right) \right], \tag{3.7}$$

where, λ is an arbitrary nonzero constant, and $\frac{\beta}{\gamma} < 0$.

4. Unperturbed Case of Equat. (1.2)

We now proceed to discuss the first step of the study of (1.2). We begin by applying the traveling wave transformation to (1.4), which represents the unperturbed case of Equat. (1.2). We obtain:

$$
\begin{aligned}
V_t &= -cU'(\xi) \\
dV^2 V_x &= dU^2 U' \\
\beta V_{xx} &= \beta U''(\xi) \\
\gamma V_{xxx} &= \gamma U^{(3)}(\xi) \\
\mu V &= \mu U(\xi).
\end{aligned}
$$

The ODE becomes:

$$-cU' + dU^2 U' + \beta U'' + \gamma U^{(3)} + \mu U = 0. \tag{4.1}$$

We balance the highest order derivative term $U^{(3)}$ with the nonlinear term U^2U' to get $N = 1$.

Thus, we have:

$$U(\xi) = a_0 + a_1 Y, \quad Y = \tanh(\lambda\xi). \tag{4.2}$$

Therefore,

$$\begin{aligned} U''(\xi) &= \lambda^2 \frac{d}{dY}\left[(1-Y^2)\frac{dU}{dY}\right] \\ &= \lambda^2 \frac{d}{dY}\left[(1-Y^2)a_1\right] \\ &= \lambda^2(-2a_1 Y). \end{aligned}$$

(The term $U^{(3)}(\xi)$ is already computed.)
Substitute into the ODE (4.1), we can write

$$\begin{aligned} &-cU' = -ca_1\lambda(1-Y^2) \\ &dU^2U' = da_1\lambda(1-Y^2)(a_0^2 + 2a_0a_1Y + a_1^2Y^2) \\ &\beta U'' = \beta\lambda^2(-2a_1Y) \\ &\gamma U^{(3)} = \gamma\lambda^3(-2a_1 + 6a_1Y^2) \\ &\mu U = \mu(a_0 + a_1Y). \end{aligned}$$

Collecting coefficients, it yields that:

$$\begin{aligned} Y^0 :& \quad \mu a_0 - ca_1\lambda + da_0^2a_1\lambda - 2\gamma a_1\lambda^3 = 0 \\ Y^1 :& \quad \mu a_1 + 2da_0a_1^2\lambda - 2\beta a_1\lambda^2 = 0 \\ Y^2 :& \quad da_1^3\lambda + 6\gamma a_1\lambda^3 = 0. \end{aligned}$$

From coefficient of Y^2, we have $da_1^3\lambda + 6\gamma a_1\lambda^3 = 0$.
Hence,

$$da_1^2 + 6\gamma\lambda^2 = 0 \Rightarrow a_1^2 = -\frac{6\gamma\lambda^2}{d}. \tag{4.3}$$

From coefficient of Y^1, we have $\mu a_1 + 2da_0a_1^2\lambda - 2\beta a_1\lambda^2 = 0$,
which gives

$$\mu + 2da_0a_1\lambda - 2\beta\lambda^2 = 0 \Rightarrow a_0 = \frac{2\beta\lambda^2 - \mu}{2da_1\lambda}. \tag{4.4}$$

From coefficient of Y^0, we have $\mu a_0 - ca_1\lambda + da_0^2a_1\lambda - 2\gamma a_1\lambda^3 = 0$.
Substitute a_0 from (4.4) and a_1^2 from (4.3) to find c.

Hence, the solution of (1.4) is given by

$$V(x,t) = a_0 + a_1 \tanh\left[\lambda\left(\frac{x^\alpha}{\alpha} - ct\right)\right], \tag{4.5}$$

where,

$$\begin{aligned} a_1 &= \pm\sqrt{-\frac{6\gamma\lambda^2}{d}} \\ a_0 &= \frac{2\beta\lambda^2 - \mu}{2da_1\lambda}. \end{aligned}$$

5. Analysis of Perturbed Problems

To study (1.1) and (1.2), we begin by considering the traveling wave transformation:

$$\xi = \frac{x^\alpha}{\alpha} - ct, \quad V(x,t) = U(\xi). \tag{5.1}$$

Then, we have

$$D_x^\alpha V = U'(\xi), \quad D_x^\alpha D_x^\alpha V = U''(\xi), \quad D_x^\alpha D_x^\alpha D_x^\alpha V = U^{(3)}(\xi), \quad D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha V = U^{(4)}(\xi). \tag{5.2}$$

Based on the derivatives obtained above, we are in a position to analyze the following ODEs:

$$\varepsilon U^{(4)} - \delta c U' + \gamma U^2 U' + \beta U^{(3)} = 0 \tag{5.3}$$

and

$$\varepsilon U^{(4)} - cU' + dU^2 U' + \beta U'' + \gamma U^{(3)} + \mu U = 0. \tag{5.4}$$

To do this analysis, we need to employ the asymptotic expansion:

$$U(\xi) = U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) + O(\varepsilon^3) \tag{5.5}$$

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + O(\varepsilon^3). \tag{5.6}$$

5.1. Analysis of Singularly Perturbed mKdV

We proceed into four steps:
1-Zero-Order Solution (ε^0):
Substituting the expansion into (5.3) and collecting terms of order ε^0, we have

$$-\delta c_0 U_0' + \gamma U_0^2 U_0' + \beta U_0^{(3)} = 0. \tag{5.7}$$

Taking $U_0(\xi) = a_0 + a_1 \tanh(\lambda\xi)$ and balancing terms, we obtain

$$a_0 = 0 \tag{5.8}$$

$$a_1^2 = -\frac{6\beta\lambda^2}{\gamma} \tag{5.9}$$

$$c_0 = -\frac{2\beta\lambda^2}{\delta}. \tag{5.10}$$

Thus, the zero-order solution is

$$U_0(\xi) = a_1 \tanh(\lambda\xi), \quad a_1 = \pm\sqrt{-\frac{6\beta\lambda^2}{\gamma}}, \quad c_0 = -\frac{2\beta\lambda^2}{\delta}. \tag{5.11}$$

2-First-Order Correction (ε^1):
Collecting terms of order ε^1, we obtain

$$U_0^{(4)} - \delta c_0 U_1' - \delta c_1 U_0' + \gamma(2U_0 U_1 U_0' + U_0^2 U_1') + \beta U_1^{(3)} = 0, \tag{5.12}$$

that is

$$\mathcal{L}_1[U_1] = F_1(\xi), \tag{5.13}$$

where,

$$\mathcal{L}_1[U_1] = -\delta c_0 U_1' + \gamma(2U_0 U_0' U_1 + U_0^2 U_1') + \beta U_1^{(3)} \tag{5.14}$$

$$F_1(\xi) = -U_0^{(4)} + \delta c_1 U_0'. \tag{5.15}$$

Computation of Derivatives:

For $U_0(\xi) = a_1 \tanh(\lambda\xi)$, we get

$$U_0'(\xi) = a_1\lambda(1 - \tanh^2(\lambda\xi)) \tag{5.16}$$
$$U_0''(\xi) = -2a_1\lambda^2 \tanh(\lambda\xi)(1 - \tanh^2(\lambda\xi)) \tag{5.17}$$
$$U_0^{(3)}(\xi) = -2a_1\lambda^3(1 - \tanh^2(\lambda\xi))(1 - 3\tanh^2(\lambda\xi)) \tag{5.18}$$
$$U_0^{(4)}(\xi) = -2a_1\lambda^4 \tanh(\lambda\xi)(1 - \tanh^2(\lambda\xi))(5 - 3\tanh^2(\lambda\xi)). \tag{5.19}$$

Thus, the forcing term becomes

$$F_1(\xi) = 2a_1\lambda^4 \tanh(\lambda\xi)(1 - \tanh^2(\lambda\xi))(5 - 3\tanh^2(\lambda\xi)) + \delta c_1 a_1\lambda(1 - \tanh^2(\lambda\xi)). \tag{5.20}$$

Now, since

$$\mathcal{L}_1^*[\psi] = \delta c_0\psi' - 2\gamma U_0 U_0'\psi + \gamma U_0^2\psi' - \beta\psi^{(3)}, \tag{5.21}$$

then, we verify that $\psi(\xi) = U_0'(\xi)$ satisfies $\mathcal{L}_1^*[\psi] = 0$. The solvability condition [27] is:

$$\int_{-\infty}^{+\infty} F_1(\xi)U_0'(\xi)d\xi = 0. \tag{5.22}$$

This condition guaranties that the perturbation does not enter into resonance with the natural modes of the system; it ensures bounded solutions over time.

Substituting $F_1(\xi)$ and $U_0'(\xi)$ gives

$$\int_{-\infty}^{+\infty} \left[2a_1\lambda^4 \tanh(1 - \tanh^2)(5 - 3\tanh^2) + \delta c_1 a_1\lambda(1 - \tanh^2)\right] \cdot \left[a_1\lambda(1 - \tanh^2)\right] d\xi = 0. \tag{5.23}$$

Using $z = \tanh(\lambda\xi)$, $dz = \lambda(1 - z^2)d\xi$, and noting that the integration is taken over $[-1, 1]$, we obtain the following result:

$$\int_{-\infty}^{+\infty} \tanh(1 - \tanh^2)^2(5 - 3\tanh^2)d\xi = 0 \quad \text{(odd function)} \tag{5.24}$$
$$\int_{-\infty}^{+\infty} (1 - \tanh^2)^2 d\xi = \int_{-1}^{1} (1 - z^2)\frac{dz}{\lambda} = \frac{1}{\lambda}\left[z - \frac{z^3}{3}\right]_{-1}^{1} = \frac{1}{\lambda}\left(\frac{4}{3}\right). \tag{5.25}$$

Therefore,

$$\delta c_1 a_1^2\lambda^2 \cdot \frac{4}{3\lambda} = 0 \quad \Rightarrow \quad c_1 = 0. \tag{5.26}$$

3-First-Order Solution:
With $c_1 = 0$, we have

$$\mathcal{L}_1[U_1] = 2a_1\lambda^4 \tanh(\lambda\xi)(1 - \tanh^2(\lambda\xi))(5 - 3\tanh^2(\lambda\xi)). \tag{5.27}$$

We seek a solution of the form

$$U_1(\xi) = A\tanh^3(\lambda\xi) + B\tanh(\lambda\xi). \tag{5.28}$$

We have

$$U_1'(\xi) = \lambda(1 - \tanh^2)(3A\tanh^2 + B) \tag{5.29}$$
$$U_1^{(3)}(\xi) = -2\lambda^3(1 - \tanh^2)[(15A + B) - (18A + 3B)\tanh^2 + 9A\tanh^4], \tag{5.30}$$

which implies

$$A = -\frac{a_1 \lambda}{10\beta} \tag{5.31}$$

$$B = \frac{3a_1 \lambda}{10\beta}. \tag{5.32}$$

4-Final Solution:

$$U(\xi) = a_1 \tanh(\lambda\xi) + \varepsilon \left(-\frac{a_1\lambda}{10\beta} \tanh^3(\lambda\xi) + \frac{3a_1\lambda}{10\beta} \tanh(\lambda\xi) \right) + O(\varepsilon^2), \tag{5.33}$$

$$c = -\frac{2\beta\lambda^2}{\delta} + O(\varepsilon^2), \tag{5.34}$$

where $a_1 = \pm\sqrt{-\frac{6\beta\lambda^2}{\gamma}}$.

5.2. Analysis of Singularly Perturbed mKdV-Buregrs

Step1: Zero-Order Solution (ε^0):

At order ε^0, we have

$$-c_0 U_0' + dU_0^2 U_0' + \beta U_0'' + \gamma U_0^{(3)} + \mu U_0 = 0. \tag{5.35}$$

Taking $U_0(\xi) = a_0 + a_1 \tanh(\lambda\xi)$ and substituting, we get:

$$Y^0: \quad \mu a_0 - c_0 a_1 \lambda + d a_0^2 a_1 \lambda - 2\gamma a_1 \lambda^3 = 0 \tag{5.36}$$

$$Y^1: \quad \mu a_1 + 2d a_0 a_1^2 \lambda - 2\beta a_1 \lambda^2 = 0 \tag{5.37}$$

$$Y^2: \quad d a_1^3 \lambda + 6\gamma a_1 \lambda^3 = 0 \tag{5.38}$$

$$Y^3: \quad -2d a_0 a_1^2 \lambda + 2\beta a_1 \lambda^2 = 0. \tag{5.39}$$

From Y^2, we have $d a_1^3 \lambda + 6\gamma a_1 \lambda^3 = 0$, hence, $a_1^2 = -\frac{6\gamma\lambda^2}{d}$.

Using Y^3, we get $-2d a_0 a_1^2 \lambda + 2\beta a_1 \lambda^2 = 0$, then, $a_0 = \frac{\beta\lambda}{d a_1}$.
For the case $\mu \neq 0$, we use Y^1 to obtain $\mu a_1 + 2d a_0 a_1^2 \lambda - 2\beta a_1 \lambda^2 = 0$, and then

$$a_0 = \frac{2\beta\lambda^2 - \mu}{2d a_1 \lambda}. \tag{5.40}$$

From Y^0, we determine c_0.
Step 2: First-Order Correction (ε^1):
At order ε^1, we have

$$U_0^{(4)} - c_0 U_1' - c_1 U_0' + d(2U_0 U_1 U_0' + U_0^2 U_1')' + \beta U_1'' + \gamma U_1^{(3)} + \mu U_1 = 0. \tag{5.41}$$

So,

$$\mathcal{L}_2[U_1] = F_2(\xi), \tag{5.42}$$

where

$$\mathcal{L}_2[U_1] = -c_0 U_1' + \beta U_1'' + \gamma U_1^{(3)} + \mu U_1 + d(2U_0 U_1 U_0' + U_0^2 U_1')' \tag{5.43}$$

$$= \gamma U_1^{(3)} + \beta U_1'' - c_0 U_1' + \mu U_1 + d\left[U_0^2 U_1'' + 4U_0 U_0' U_1' + 2U_0 U_1 U_0'' + 2(U_0')^2 U_1\right]. \tag{5.44}$$

The forcing term is

$$F_2(\xi) = -U_0^{(4)} + c_1 U_0'. \tag{5.45}$$

Computation of Derivatives gives us:

For $U_0(\xi) = a_0 + a_1 \tanh(\lambda\xi)$, let $y = \tanh(\lambda\xi)$, $\frac{dy}{d\xi} = \lambda(1-y^2)$:

$$U_0'(\xi) = a_1\lambda(1-y^2) \tag{5.46}$$
$$U_0''(\xi) = -2a_1\lambda^2 y(1-y^2) \tag{5.47}$$
$$U_0^{(3)}(\xi) = -2a_1\lambda^3(1-y^2)(1-3y^2) \tag{5.48}$$
$$U_0^{(4)}(\xi) = 8a_1\lambda^4 y(1-y^2)(2-3y^2) \tag{5.49}$$

Thus, the forcing term becomes:

$$F_2(\xi) = -8a_1\lambda^4 y(1-y^2)(2-3y^2) + c_1 a_1\lambda(1-y^2). \tag{5.50}$$

Now, since the operator $\mathcal{L}_2^*$ is defined by $\int \psi\mathcal{L}_2[U_1]d\xi = \int U_1\mathcal{L}_2^*[\psi]d\xi$, we verify that $\psi(\xi) = U_0'(\xi)$ satisfies $\mathcal{L}_2^*[\psi] = 0$. The solvability condition is:

$$\int_{-\infty}^{+\infty} F_2(\xi)U_0'(\xi)d\xi = 0. \tag{5.51}$$

Substituting $F_2(\xi)$ and $U_0'(\xi)$ yields that

$$\int_{-\infty}^{+\infty} \left[-8a_1^2\lambda^5 y(1-y^2)^2(2-3y^2) + c_1 a_1^2\lambda^2(1-y^2)^2\right] d\xi = 0. \tag{5.52}$$

Using the fact that $z = \tanh(\lambda\xi)$, $dz = \lambda(1-z^2)d\xi$, $d\xi = \frac{dz}{\lambda(1-z^2)}$, and noting the integration interval $[-1,1]$, we can write

$$\int_{-\infty}^{+\infty} y(1-y^2)^2(2-3y^2)d\xi = \frac{1}{\lambda}\int_{-1}^{1} z(1-z^2)(2-3z^2)dz = 0 \tag{5.53}$$
$$\int_{-\infty}^{+\infty} (1-y^2)^2 d\xi = \frac{1}{\lambda}\int_{-1}^{1}(1-z^2)dz = \frac{1}{\lambda}\left[z - \frac{z^3}{3}\right]_{-1}^{1} = \frac{4}{3\lambda}. \tag{5.54}$$

Therefore,

$$0 + c_1 a_1^2\lambda^2 \cdot \frac{4}{3\lambda} = 0 \quad \Rightarrow \quad c_1 = 0. \tag{5.55}$$

Step 3: First-Order Solution:
With $c_1 = 0$, the equation for U_1 is given by:

$$\mathcal{L}_2[U_1] = -8a_1\lambda^4 \tanh(\lambda\xi)(1-\tanh^2(\lambda\xi))(2-3\tanh^2(\lambda\xi)). \tag{5.56}$$

We seek a particular solution as:

$$U_1(\xi) = A\tanh^3(\lambda\xi) + B\tanh(\lambda\xi). \tag{5.57}$$

After substituting into $\mathcal{L}_2[U_1]$, we can write:

$$A = \frac{2a_1\lambda}{5\beta}, B = -\frac{3a_1\lambda}{5\beta}. \tag{5.58}$$

The velocity c_0 is determined by Y^0 as follows:

$$c_0 = \frac{\mu a_0 + d a_0^2 a_1\lambda - 2\gamma a_1\lambda^3}{a_1\lambda}. \tag{5.59}$$

Substituting $a_0 = \frac{2\beta\lambda^2-\mu}{2da_1\lambda}$, we get

$$c_0 = \frac{\mu(2\beta\lambda^2-\mu)}{2da_1^2\lambda^2} + \frac{(2\beta\lambda^2-\mu)^2}{4da_1^2\lambda^2} - \frac{2\gamma\lambda^2}{a_1}. \tag{5.60}$$

Step 4: Final solutions of (1.2):
Finally, we have

$$U(\xi) = \frac{2\beta\lambda^2-\mu}{2da_1\lambda} + a_1\tanh(\lambda\xi) + \varepsilon\left(\frac{2a_1\lambda}{5\beta}\tanh^3(\lambda\xi) - \frac{3a_1\lambda}{5\beta}\tanh(\lambda\xi)\right) + O(\varepsilon^2), \tag{5.61}$$

$$c = \frac{\mu(2\beta\lambda^2-\mu)}{2da_1^2\lambda^2} + \frac{(2\beta\lambda^2-\mu)^2}{4da_1^2\lambda^2} - \frac{2\gamma\lambda^2}{a_1} + O(\varepsilon^2), \tag{5.62}$$

where, $a_1 = \pm\sqrt{-\frac{6\gamma\lambda^2}{d}}$.

6. Nuemrical Simulation

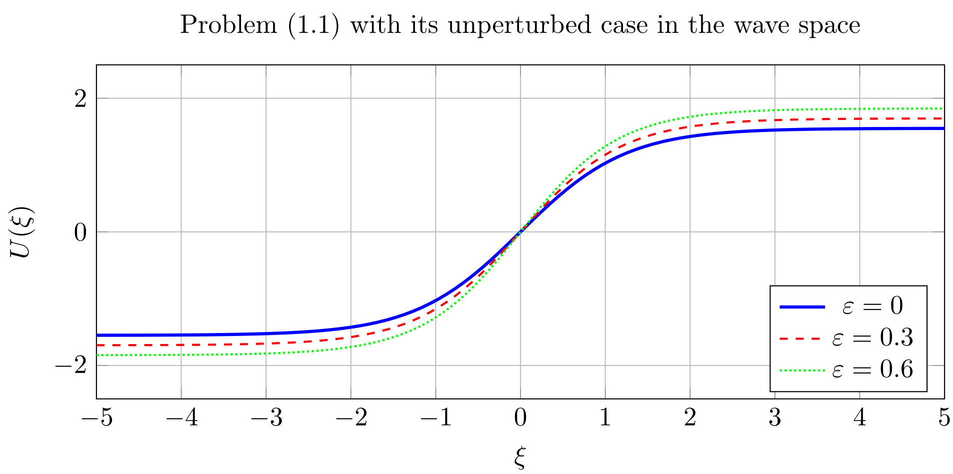

Figure 1: Solutions for (1.1)-(1.3) in the wave space, with $\beta = 0.5$, $\gamma = -0.8$, $\delta = 1.0$, $\lambda = 0.8$, $a_1 \approx 1.549$.

6.1. Analysis of Figure 1

The perturbed solutions in Figure 1 show sensitivity to ε, with modifications occurring still at moderate values of $\varepsilon = 0.3$. The antisymmetric character of the solutions is preserved across all perturbation levels, it maintains the odd symmetry property. As ε increases from 0 to 0.6, the solutions develop excess peaks near the transition region.

6.2. Analysis of Figure 2

Figure 2 shows a different perturbation response which is characterized by amplitude reduction as ε increases. The shifted asymmetric transition solutions maintain their character of connecting distinct asymptotic states, but the transition region becomes broader and more gradual with higher ε values. The perturbations in Figure 2 produce a damping effect that reduces the maximum slope and decreases the wave amplitude, evident when $\varepsilon = 0.6$.

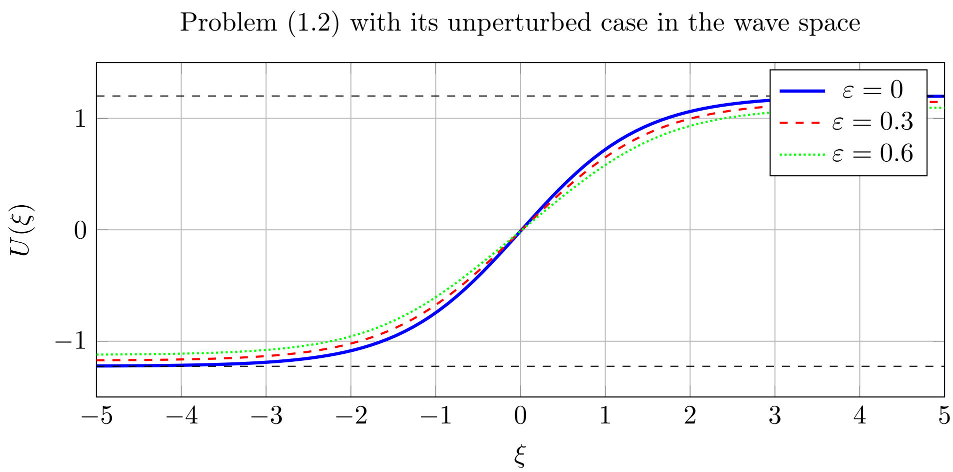

Figure 2: Solutions of (1.2)-(1.4) in the wave space, with $\beta = 1.0$, $\gamma = -0.5$, $d = 1.0$, $\mu = 1.0$, $\lambda = 0.7$, $a_0 \approx -0.012$, $a_1 \approx 1.212$.

7. Conclusion

This study has analyzed singularly perturbed fractional differential equations using Khalil conformable derivatives. We have proved that term containing $D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha V$, introduced through the infinitesimal parameter ε, acts as a significant singular perturbation. We have shown that its impact is highly dependent on the structure of the unperturbed equation. For the modified Korteweg-de Vries (mKdV) type equation, the perturbation generates excess peaks, but for the generalized KdV-Burgers equation, it induces a damping effect, enlarging the transition region and reducing the wave amplitude.
We have proved that the Khalil derivative offers an important tool for introducing higher-order regularization effects, which can alter stability and asymptotic behavior. These results open new avenues for future work, particularly in applying this asymptotic framework to more complex systems, and exploring different types of nonlinearities.

References

1. T. Abdeljawad, On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, May 2015.
2. M. Aljazzazi, F. Seddiki, I. M. Batiha, A. Bouchenak, and R. I. Hajaj, Results on the existence of solutions for conformable evolution differential equations with non-local conditions, *Gulf Journal of Mathematics*, vol. 20, no. 2, pp. 26–38, 2025.
3. H. A. Alyousef, R. Shah, N. A. Shah, J. D. Chung, S. M. E. Ismaeel, and S. A. El-Tantawy, The fractional analysis of a nonlinear mKdV equation with Caputo operator, *Fractal and Fractional*, vol. 7, no. 2, Art. 161, pp. 1–20, 2023.
4. A. Bouchenak, I. M. Batiha, R. Hatamleh, M. Aljazzazi, I. H. Jebril, and M. Al-Horani, Study and Analysis of the Second Order Constant Coefficients and Cauchy–Euler Equations via Modified Conformable Operator, *International Journal of Robotics and Control Systems*, vol. 5, no. 2, pp. 794–812, 2025. ISSN 2775-2658.
5. L. F. Calazans de Brito and A. M. Kamchatnov, Undular bore theory for the modified Korteweg–de Vries–Burgers equation, *Physical Review E*, vol. 109, p. 015102, Jan. 2024.
6. F. Calogero and A. Degasperis, A Modified Modified Korteweg–de Vries Equation, *Inverse Problems*, vol. 1, no. 1, pp. 57–66, 1985.
7. B. Fornberg and G. B. Whitham, A Numerical and Theoretical Study of Certain Nonlinear Wave Phenomena, *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 289, no. 1361, pp. 373–404, May 1978.
8. A. Jamal, A. Ullah, S. Ahmad, S. Sarwar, and A. Shokri, A survey of (2+1)-dimensional KdV–mKdV equation using nonlocal Caputo fractal–fractional operator, *Results in Physics*, vol. 46, Art. 106294, pp. 1–12, Mar. 2023.
9. I. Jebril and Z. Dahmani, New Results on Singularly Perturbed Fractional KdV and KdV–Burgers Equations, *Journal of Prime Research in Mathematics*, vol. 21, no. 2, pp. 25–36, 2025.

10. J. M. Heris and M. Bagheri, Exact Solutions for the Modified KdV and the Generalized KdV Equations via Exp-Function Method, *Journal of Mathematical Extension*, vol. 4, no. 2, pp. 75–95, 2010.

11. M. K. Kadalbajoo and K. C. Patidar, Singularly perturbed problems in partial differential equations: A survey, *Applied Mathematics and Computation*, vol. 134, nos. 2–3, pp. 371–429, Jan. 2003.

12. R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, vol. 264, pp. 65–70, 2014.

13. R. Killip, Z. Ouyang, M. Visan, and L. Wu, The Modified Korteweg–de Vries Limit of the Ablowitz–Ladik System, *Discrete and Continuous Dynamical Systems*, vol. 45, no. 3, pp. 821–846, Mar. 2025.

14. R. Kumar, R. Dharra, and S. Kumar, Comparative qualitative analysis and numerical solution of conformable fractional derivative generalized KdV–mKdV equation, *International Journal of System Assurance Engineering and Management*, vol. 14, pp. 1247–1254, 2023.

15. A. Lamamri, I. Jebril, Z. Dahmani, A. Anber, M. Rakah, and S. Alkhazaleh, Fractional calculus in beam deflection: Analyzing nonlinear systems with Caputo and conformable derivatives, *AIMS Mathematics*, vol. 9, no. 8, pp. 21609–21627, 2024.

16. W. Malfliet, The tanh method: A tool for solving certain classes of nonlinear evolution and wave equations, *Journal of Computational and Applied Mathematics*, vols. 164–165, pp. 529–541, Mar. 2004.

17. F. Martínez, I. Martínez, M. K. A. Kaabar, and S. Paredes, Solving systems of conformable linear differential equations via the conformable exponential matrix, *Ain Shams Engineering Journal*, vol. 12, pp. 4075–4080, 2021.

18. W. W. Mohammed and F. M. Al-Askar, New Stochastic Solitary Solutions for the Modified Korteweg–de Vries Equation with Stochastic Term/Random Variable Coefficients, *AIMS Mathematics*, vol. 9, no. 8, pp. 20467–20481, Jun. 2024.

19. S. K. Mohanty, S. Kumar, A. N. Dev, M. K. Deka, D. V. Churikov, and O. V. Kravchenko, An efficient technique of $\exp(-\phi(\xi))$-expansion method for modified KdV and Burgers equations with variable coefficients, *Results in Physics*, vol. 37, p. 105504, Jun. 2022.

20. S. Mukhtar, W. Alhejaili, M. Alqudah, A. M. Mahnashi, R. Shah, and S. A. El-Tantawy, Kink Wave Phenomena in the Nonlinear Partial Differential Equation Representing the Transmission Line Model of Microtubules for Nanoionic Currents, *Axioms*, vol. 13, no. 10, p. 686, 2024.

21. D. Ntiamoah, W. Ofori-Atta, and L. Akinyemi, The Higher-Order Modified Korteweg–de Vries Equation: Its Soliton, Breather and Approximate Solutions, *Journal of Ocean Engineering and Science*, vol. 9, no. 6, pp. 554–565, Dec. 2024.

22. R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, 1st ed., New York, NY, USA: Springer-Verlag, 1991.

23. N. Ramanujam and U. N. Srivastava, Singular perturbation problems for systems of partial differential equations of elliptic type, *Journal of Mathematical Analysis and Applications*, vol. 71, no. 1, pp. 18–35, Sep. 1979.

24. S. Sen, J. Hong, J. Bang, E. Avalos, and R. Doney, Solitary Waves in the Granular Chain, *Physics Reports*, vol. 462, no. 2, pp. 21–66, Jun. 2008.

25. M. Tanaka, *Physics of Nonlinear Waves*, 1st ed., Cham, Switzerland: Springer Nature, 2020.

26. J. D. Villacís Lascano and M. J. Vivas-Cortez, Khalil conformable fractional derivative and its applications to population growth and body cooling models, *Selecciones Matemáticas*, vol. 9, no. 1, pp. 44–52, 2022.

27. V. Vougalter and V. Volpert, Solvability conditions for some non-Fredholm operators, *Proc. Edinburgh Math. Soc.*, vol. 54, pp. 249–271, 2011.

28. G. M. Wondimu, M. M. Woldaregay, T. G. Dinka, and G. F. Duressa, Numerical treatment of singularly perturbed parabolic partial differential equations with nonlocal boundary condition, *Frontiers in Applied Mathematics and Statistics*, vol. 8, Art. no. 1005330, 2022.

29. W.-A. Yong, Singular perturbations of first-order hyperbolic systems with stiff source terms, *Journal of Differential Equations*, vol. 155, no. 1, pp. 89–132, Jun. 1999.

30. A. Zafar and A. R. Seadawy, The conformable space-time fractional mKdV equations and their exact solutions, *Journal of King Saud University - Science*, vol. 31, no. 4, pp. 1478–1484, Oct. 2019.

31. D.-J. Zhang, S.-L. Zhao, Y.-Y. Sun, and J. Zhou, Solutions to the Modified Korteweg–de Vries Equation, *Reviews in Mathematical Physics*, vol. 26, no. 7, pp. 1430006:1–1430006:42, 2014.

32. W. Zhang, Y. Sun, Z. Li, S. Pei, and X. Li, Bounded traveling wave solutions for MKdV-Burgers equation with the negative dispersive coefficient, *Discrete and Continuous Dynamical Systems - Series B*, vol. 21, no. 8, pp. 2883–2903, Oct. 2016.

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