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Invariant Muirhead Mean Inequality and Some Schur Convexity Properties

Sowmyashree M. K., Narasimhan G., Sudhamani M. J., K. M. Nagaraja and Sampath Kumar R.

ABSTRACT: The aim of this paper is to define the invariant Muirhead mean, denoted by ${}^iM_h(a,b)$ and to discuss the properties of Schur, Schur geometric and Schur harmonic convexities of this mean. Also, an inequality involving the invariant Muirhead mean is constructed by applying the Newton Raphson formula. These inspections contribute to the theoretical development of generalized symmetric means in Majorization theory and Functional inequalities.

Keywords: Invariant, Muirhead mean, Schur convexities, inequality.

Contents

1. Introduction

The Greek Mathematicians introduced the concept of Mathematical means based on proportions in the 4^{th} century A.D in the Pythagorean School. In literature, it is evident that Mathematical means have a lot of applications in geometry and music. Later on, several authors contributed and developed a good number of results that are applicable to various branches of science & technology. In recently, Lokesha et al., obtained the solution for an open problem raised by Rooin involving means [11] and an investigation on the homogeneous functions; as a result, some inequalities involving means are established [9,10,11]. Studies on Greek means, the approach of new means, and their generalizations leads to several inequality results were found in [13,14,15]. The concept and detailed study on invariant and complementary means found in [26]. In [29] Zhen Hang yang et al. proposed the power exponential mean and invariant power exponential mean are respectively given by;

$$A(a,b) = \frac{a+b}{2} \tag{1.1}$$

$$G(a,b) = \sqrt{ab} \tag{1.2}$$

and

$$H(a,b) = \frac{2ab}{a+b} \tag{1.3}$$

About Muirhead means: Bullen [2] provided brief discussions in his book "Handbook of Means and Their Inequalities" for n arguments as well as for two arguments defined as follows :

$$A_{2,\alpha,\beta}(a,b) = \left(\frac{a^\alpha b^\beta + a^\beta b^\alpha}{2}\right)^{\frac{1}{\alpha+\beta}} \quad \text{such that} \quad \alpha+\beta = 1. \tag{1.4}$$

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The theory of means forms a foundational element in mathematical analysis and has seen extensive exploration across various domains. Bullen [2] , in his authoritative text Handbook of Means and Their Inequalities, presents a detailed discussion on Muirhead means, emphasizing their structural richness and inequality properties [5,6,28]. In a complementary direction, Gheorghe Toader and Silvia Toader [26], in their monograph Greek Means and the Arithmetic-Geometric Mean, introduce and elaborate on the concept of invariant means, offering insights into their mathematical behavior and applications.

Furthering this line of inquiry, Miao-Kun Wang et al. [13] conducted a comparative study between the generalized Muirhead mean and the identric mean, highlighting subtle distinctions and interrelations between these sophisticated mean functions. On the applied side, Brinda et al. [1] demonstrated the utility of Muirhead mean filtering techniques in the field of pattern recognition, specifically in identifying hand gestures, thereby showcasing the interdisciplinary relevance of these mathematical constructs.

In addition, the Schur-convexity properties of the Invariant Contra-Harmonic Mean have been investigated by Nagaraja et al. [24] contributing to the deeper understanding of generalized mean behavior under majorization.

Motivated by these diverse and interdisciplinary connections, as well as the elegant theoretical framework of the Muirhead mean, the present paper seeks to build upon this foundation and explore further developments in the field.

2. Preliminaries

In this section, the fundamental definitions, notations, and lemmas that form the basis of the discussions and results in the subsequent sections are presented. These preliminaries are essential for understanding the properties and comparisons of the means considered in this paper, which are discussed in ([3,4,7,8], [16]- [23], [25,27]) such as various properties of Schur convexity related results.

Definition 2.1 [2] For any two positive real numbers a and b, the *Muirhead mean* is defined as

$$M_h(a,b) = \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}$$

Definition 2.2 [26] Let $M(a,b)$ and $N(a,b)$ be two means such that $M(a,b)\cdot N(a,b) = G^2(a,b)$, where $G(a,b) = \sqrt{ab}$ is the geometric mean. Then $N(a,b)$ is said to be the *invariant mean* corresponding to $M(a,b)$.

Definition 2.3 [12] Let $E \subseteq R^n (n \geq 2)$ be a set with nonempty interior, a real valued function $f : E \to \mathbb{R}$ is said to be Schur convex on E if $f(x) \leq f(y)$ for each pair of n-tuples $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in E with $\mathbf{x} \prec \mathbf{y}$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, k = 1, 2, 3, ...n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

Where $x_{[i]}$ denotes the i^{th} largest component in x. f is called Schur concave if $-f$ is Schur convex.

Lemma 2.1 *[12] Let $\Omega \in \mathbb{R}^{\ltimes}$ be symmetric with non empty interior geometrically convex set and let $\xi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \xi}{\partial x_1} - \frac{\partial \xi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{2.1}$$

$$(ln x_1 - ln x_2)\left(x_1\frac{\partial \xi}{\partial x_1} - x_2\frac{\partial \xi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{2.2}$$

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \xi}{\partial x_1} - x_2^2 \frac{\partial \xi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{2.3}$$

holds for any $x = (x_1, x_2, ..., x_n)$ in Ω^0, then ϕ is a Schur convex (concave), Schur-geometrically convex (concave) and Schur-harmonically convex(concave) function respectively.

3. Invariant Muirhead Mean and its Schur Convexity

In this section, we define the Invariant Muirhead Mean and establish that it lies between the minimum and maximum of the input values. We further investigate its behavior under the framework of Schur convexity. Specifically, we prove that the Invariant Muirhead Mean satisfies three important types of convexity: Schur convexity, Schur geometric convexity and Schur harmonic convexity. Each of these properties is demonstrated through a dedicated theorem, highlighting the symmetric and ordering-preserving nature of this mean under majorization.

Definition 3.1 For any two positive real numbers a and b, the *invariant Muirhead mean* is defined as;

$$^{i}M_h(a,b) = ab\left(\frac{2}{a^b b^a + a^a b^b}\right)^{\frac{1}{a+b}}$$

which satisfies the identity $M_h(a,b) \cdot {}^{i}M_h(a,b) = ab = G^2(a,b)$.

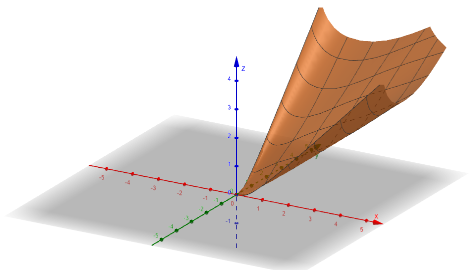

Figure 1: The graph of Muirhead mean

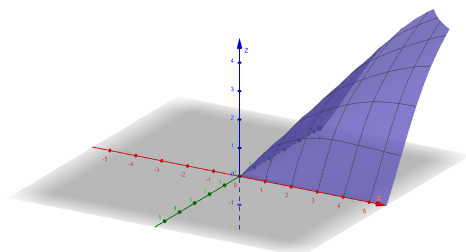

Figure 2: The graph of invariant Muirhead mean

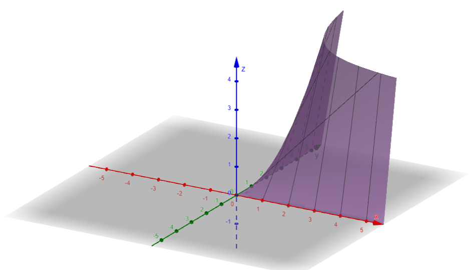

Figure 3: The graph of the geometric mean

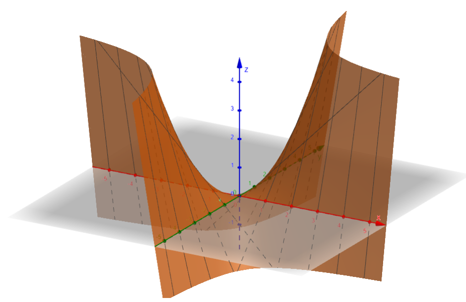

Figure 4: The graph of product of Muirhead and its invariant mean

Remark: Figure 1 and 2 are the geometrical representations of Muirhead mean and invariant Muirhead mean respectively. Figure 3 shows the first octant portion of figure 4. Hence,

$$M_h(a,b).^{i}M_h(a,b) = ab$$

It is necessary to establish that $^{i}M_h(a,b)$ lies strictly between a and b, thereby confirming that it is a valid mean.

Theorem 3.1 *For all $a, b > 0$ with $a \neq b$, the invariant Muirhead mean $M_h(a, b)$ satisfies $a < {}^iM_h(a, b) < b$.*

Proof: It is clear that Muirhead mean satisfies the inequality

$$a < M_h(a, b) < b$$

On reciprocating,

$$\frac{1}{b} < \frac{1}{M_h(a, b)} < \frac{1}{a}$$

On multiplying ab,

$$\frac{ab}{b} < \frac{ab}{M_h(a, b)} < \frac{ab}{a}$$

Therefore,

$$a < {}^iM_h(a, b) < b$$

Hence, the function ${}^iM_h(a, b)$ is a valid mean, which is symmetric. □

Now, the objective is to analyze certain Schur convexity properties of the invariant Muirhead Mean.

Theorem 3.2 *The invariant Muirhead mean ${}^iM_h(a, b)$ is Schur convex for all $a, b > 0$.*

Proof: By the definition of invariant Muirhead mean,

$${}^iM_h(a, b) = ab \left(\frac{2}{a^b b^a + a^a b^b} \right)^{\frac{1}{a+b}}$$

By taking logarithms,

$$\log {}^iM_h = \log a + \log b + \frac{1}{a+b} \left(\log 2 - \log(a^a b^b + a^b b^a) \right)$$

On differentiating (1) partially with respect to a,

$$\frac{\partial(\log {}^iM_h)}{\partial a} = \frac{1}{a} + \frac{\partial}{\partial a} \left[\frac{1}{a+b} \left(\log 2 - \log(a^a b^b + a^b b^a) \right) \right]$$

$$= \frac{1}{a} + \frac{log(a^b b^a + a^a b^b)}{(a+b)^2} - \frac{b^{a+1} a^{b-1} + b^a a^b logb + a^a b^b (loga + 1)}{(a+b)(a^b b^a + a^a b^b)} - \frac{\log 2}{(a+b)^2}$$

On differentiating (1) partially with respect to b,

$$\frac{\partial(\log {}^iM_h)}{\partial b} = \frac{1}{b} + \frac{\partial}{\partial b} \left[\frac{1}{a+b} \left(\log 2 - \log(a^a b^b + a^b b^a) \right) \right]$$

$$= \frac{1}{b} + \frac{log(a^b b^a + a^a b^b)}{(a+b)^2} - \frac{a^{b+1} b^{a-1} + b^a a^b loga + a^a b^b (logb + 1)}{(a+b)(a^b b^a + a^a b^b)} - \frac{\log 2}{(a+b)^2}$$

By subtracting,

$$\frac{\partial({}^iM_h)}{\partial a} - \frac{\partial({}^iM_h)}{\partial b}$$

$$= {}^iM_h \left[\frac{b-a}{ab} + \frac{1}{(a+b)(a^a b^b + a^b b^a)} \left(a^a b^b \log \frac{b}{a} + a^b b^a \log \frac{a}{b} + a^b b^a \frac{a^2 - b^2}{ab} \right) \right]$$

$$= {}^iM_h \left[\frac{(t^2-1)(t+t^t) + (t-t^3) + (t^t - t) t \log t}{t(t+1)(t+t^t)} \right], for \quad a = 1 \quad and \quad b = t$$

$$= {}^iM_h\left[\frac{(t^2-1)(t^t)+(t^t-t)t\log t}{t(t+1)(t+t^t)}\right], which \quad is \quad always \quad positive.$$

Since $0<a<b$,

$$(a-b)\left(\frac{\partial({}^iM_h)}{\partial a}-\frac{\partial({}^iM_h)}{\partial b}\right)<0$$

By Lemma 2.4, theorem is proved. □

Theorem 3.3 *Let $(a,b)\in(0,1)$ with $a<b$ and $t=\frac{b}{a}>1$. Then the invariant Muirhead mean ${}^iM_h(a,b)$ is Schur-geometrically convex.*

Proof: By the definition of invariant Muirhead mean,

$${}^iM_h(a,b)=ab\left(\frac{2}{a^bb^a+a^ab^b}\right)^{\frac{1}{a+b}}$$

By taking logarithms,

$$\log {}^iM_h=\log a+\log b+\frac{1}{a+b}\left(\log 2-\log(a^ab^b+a^bb^a)\right).$$

On multiplying gradient a and differentiating partially with respect to a ,

$$a\frac{\partial({}^iM_h)}{\partial a}=({}^iM_h)a\left\{\frac{1}{a}+\frac{a^ab^b(1+\log a)+a^{b-1}b^{a+1}}{(a+b)^2(a^ab^b+a^bb^a)}\right\} \tag{3.1}$$

On multiplying gradient b and differentiating partially with respect to b

$$b\frac{\partial({}^iM_h)}{\partial b}=({}^iM_h)b\left\{\frac{1}{b}+\frac{a^ab^b(1+\log b)+a^{b+1}b^{a-1}}{(a+b)^2(a^ab^b+a^bb^a)}\right\} \tag{3.2}$$

By subtracting (3.2) from (3.1),

$$a\frac{\partial({}^iM_h)}{\partial a}-b\frac{\partial({}^iM_h)}{\partial b}=\frac{{}^iM_h}{(a+b)^2(a^ab^b+a^bb^a)}\left\{a^{a+1}b^b(1+\log a)-a^ab^{b+1}(1+\log b)\right\}$$

This is enough to check the sign of

$$g(a,b)=a^{a+1}b^b(1+\log a)-a^ab^{b+1}(1+\log b)$$

To check the sign of $g(a,b)$, *let* $a=1$ *and* $b=t>1$.

$$g(1,t)=t^t-t^{t+1}(1+\log t)=t^t\left(1-t(1+\log t)\right).$$

Clearly, for $t>1$, the term $1-t(1+\log t)<0$, so $g(1,t)<0$. Since $\log a<\log b$ and $g(a,b)<0$,

$$(\log a-\log b)\left(a\frac{\partial({}^iM_h)}{\partial a}-b\frac{\partial({}^iM_h)}{\partial b}\right)>0$$

By Lemma 2.4, theorem is proved. □

Theorem 3.4 *Let $a,b\in(0,1)$ with $a<b$ and $t=\frac{b}{a}>1$. Then the invariant Muirhead mean ${}^iM_h(a,b)$ is Schur-harmonically convex.*

Proof: By the definition of invariant Muirhead mean,

$$^{i}M_h(a,b) = ab\left(\frac{2}{a^b b^a + a^a b^b}\right)^{\frac{1}{a+b}}$$

By taking logarithms,

$$\log(^{i}M_h) = \log a + \log b + \frac{1}{a+b}\left(\log 2 - \log(a^a b^b + a^b b^a)\right)$$

On multiplying the gradient a^2 and differentiating partially with respect to a,

$$a^2\frac{\partial(^{i}M_h)}{\partial a} = (^{i}M_h)a^2\left\{\frac{1}{a} + \frac{a^a b^b(1+\log a) + a^{b-1}b^{a+1}}{(a+b)^2(a^a b^b + a^b b^a)}\right\} \tag{3.3}$$

On multiplying the gradient b^2 and differentiating partially with respect to b,

$$b^2\frac{\partial(^{i}M_h)}{\partial b} = (^{i}M_h)b^2\left\{\frac{1}{b} + \frac{a^a b^b(1+\log b) + a^{b+1}b^{a-1}}{(a+b)^2(a^a b^b + a^b b^a)}\right\} \tag{3.4}$$

By subtracting (3.4) from (3.3),

$$a^2\frac{\partial(^{i}M_h)}{\partial a} - b^2\frac{\partial(^{i}M_h)}{\partial b}$$

$$= {}^{i}M_h(a-b) + \frac{^{i}M_h}{(a+b)^2(a^a b^b + a^b b^a)}\left\{a^a b^b(a^2 - b^2 + a^2\log a - b^2\log b)\right\}$$

This is enough to check the sign of $h(a,b) = a^a b^b\left(a^2 - b^2 + a^2\log a - b^2\log b\right)$. To evaluate the sign of $h(a,b)$, take $a = 1$, $b = t > 1$.

$$h(1,t) = t^t\left(1 - t^2(1+\log t)\right).$$

Since $1 - t^2(1+\log t) < 0$ for $t > 1$, we get $h(1,t) < 0$ Therefore,

$$(a-b)\left(a^2\frac{\partial(^{i}M_h)}{\partial a} - b^2\frac{\partial(^{i}M_h)}{\partial b}\right) > 0$$

By lemma 2.4, theorem is proved. □

Theorem 3.5 *Let $a,b \in (0,1)$ with $a < b$ and $t = \frac{b}{a} > 1$. Then the invariant Muirhead mean $^{i}M_h(a,b)$ is Schur-harmonically convex.*

Proof: By the definition of invariant Muirhead mean,

$$^{i}M_h(a,b) = ab\left(\frac{2}{a^b b^a + a^a b^b}\right)^{\frac{1}{a+b}}$$

By taking logarithms,

$$\log(^{i}M_h) = \log a + \log b + \frac{1}{a+b}\left(\log 2 - \log(a^a b^b + a^b b^a)\right)$$

On multiplying the gradient a^2 and differentiating partially with respect to a,

$$a^2\frac{\partial(^{i}M_h)}{\partial a} = (^{i}M_h)a^2\left\{\frac{1}{a} + \frac{a^a b^b(1+\log a) + a^{b-1}b^{a+1}}{(a+b)^2(a^a b^b + a^b b^a)}\right\} \tag{3.5}$$

On multiplying the gradient b^2 and differentiating partially with respect to b,

$$b^2\frac{\partial(^{i}M_h)}{\partial b} = (^{i}M_h)b^2\left\{\frac{1}{b} + \frac{a^a b^b(1+\log b) + a^{b+1}b^{a-1}}{(a+b)^2(a^a b^b + a^b b^a)}\right\} \tag{3.6}$$

By subtracting (3.4) from (3.3),

$$a^2\frac{\partial({}^iM_h)}{\partial a}-b^2\frac{\partial({}^iM_h)}{\partial b}$$
$$={}^iM_h(a-b)+\frac{{}^iM_h}{(a+b)^2(a^ab^b+a^bb^a)}\left\{a^ab^b(a^2-b^2+a^2\log a-b^2\log b)\right\}$$

This is enough to check the sign of

$$h(a,b)=a^ab^b\left(a^2-b^2+a^2\log a-b^2\log b\right)$$

To evaluate the sign of $h(a,b)$, take $a=1$, $b=t>1$.

$$h(1,t)=t^t\left(1-t^2(1+\log t)\right)$$

Since $1-t^2(1+\log t)<0$ for $t>1$, we get $h(1,t)<0$ Therefore,

$$(a-b)\left(a^2\frac{\partial({}^iM_h)}{\partial a}-b^2\frac{\partial({}^iM_h)}{\partial b}\right)>0$$

By lemma 2.4, theorem is proved. □

Theorem 3.6 *Let* $0<a<b$, *and* $t=\frac{b}{a}>1$. *Then*

1. $A(1,t)>M_h(1,t)$ *if* $t\in(1,3.2430)$, *otherwise the inequality reverses.*
2. $H(1,t)<{}^iM_h(1,t)$ *if* $t\in(1,3.2430)$, *otherwise the inequality reverses.*
3. ${}^iM_h(a,b)<G(a,b)<M_h(a,b)$

Proof: Case (1)
Let

$$f_1(t)=A(1,t)-M_h(1,t)=\frac{1+t}{2}-\left(\frac{t^t+t}{2}\right)^{\frac{1}{1+t}}$$

By Newton Raphson formula,

$$t_{n+1}=t_n+\frac{f(t_n)}{f'(t_n)}$$

Where,

$$f_1(t)=\frac{1+t}{2}-\left(\frac{t^t+t}{2}\right)^{\frac{1}{1+t}}$$

and

$$f_1^{'}(t)=-2^{\frac{-1}{t+1}}(t^t+t)^{\frac{1}{t+1}}\left[\frac{t^t(\log t+1)+1}{(t+1)(t^t+t)}-\frac{\log(t^t+t)}{(t+1)^2}\right]-\frac{2^{\frac{-1}{t+1}}(t^t+t)^{\frac{1}{t+1}}\log 2}{(t+1)^2}+\frac{1}{2}$$

By putting $n=0$ and $t_0=3$ initial approximation value can be obtained. After three iterations, the solution $t_3=3.2340$ can be obtained by using the above mentioned Newton's Raphson formula. By the figure 5, $f(t)$ is positive if $t\in(1,3.2430)$ and negative otherwise.
Hence case (1) is proved.

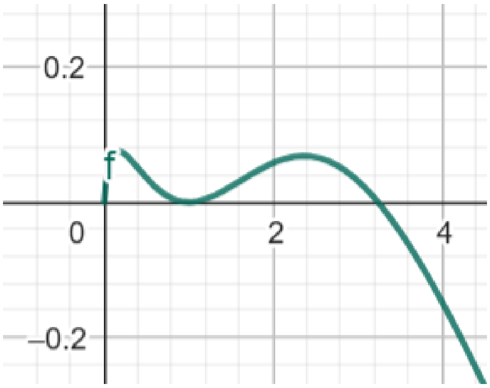

Figure 5: The graph of $f_1(t) = A(1,t) - M_h(1,t)$

Case (2) Let

$$f_2(t) = H(1,t) - {}^{i}M_h(1,t) = \frac{2t}{t+1} - t\left(\frac{2}{t^t+t}\right)^{\frac{1}{1+t}}$$

By Newton Raphson method,

$$t_{n+1} = t_n + \frac{f(t_n)}{f'(t_n)}$$

Where,

$$f_2(t) = \frac{2t}{1+t} - t\left(\frac{2}{t^t+t}\right)^{\frac{1}{1+t}} \quad \text{and} \quad f_2'(t) = -\sqrt[t+1]{2} \times \sqrt[t+1]{\frac{1}{t^t+t}}$$

$$-\sqrt[t+1]{2}t\sqrt[t+1]{\frac{1}{t^t+t}}\left[-\frac{t^t(\log t+1)+1}{(t+1)(t^t+t)} - \frac{\log(\frac{1}{t^t+t})}{(t+1)^2} + \frac{\sqrt[t+1]{2}t\sqrt[t+1]{\frac{1}{t^t+t}}}{(t+1)^2}\right]$$

$$+\frac{\sqrt[t+1]{2}t\sqrt[t+1]{\frac{1}{t^t+t}}\log 2}{(t+1)^2} + \frac{2}{t+1} - \frac{2t}{(t+1)^2}$$

By putting $n = 0$ and $t_0 = 3$ initial approximation value, one can be obtained. After three iterations, the solution $t_3 = 3.2340$ can be obtained by using the above mentioned Newton-Raphson formula. By the figure 6, $f(t)$ is negative if $t \in (1, 3.2430)$ and positive otherwise. Hence, case (2) is proved.

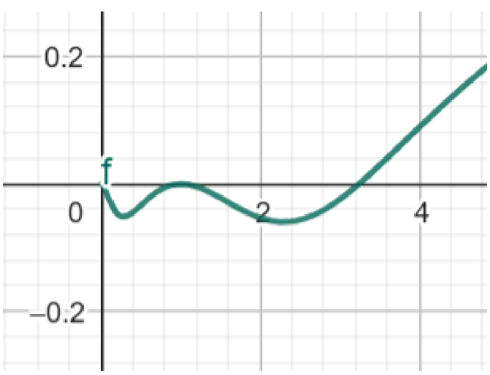

Figure 6: The graph of $f_2'(t) = H(1,t) - {}^{i}M_h(1,t)$

Case (3) To prove,

$$^{i}M_h(a,b) < G(a,b) < M_h(a,b)$$

Consider,

$$[M_h(a,b)]^{a+b} - [G^2(a,b)]^{a+b} = \left(\frac{a^a b^b + a^b b^a}{2}\right) - (ab)^{a+b}$$
$$= \frac{(a^a b^b - a^b b^a)^2}{2} > 0$$

Therefore, $G(a,b) < M_h(a,b)$
Also,

$$[G^2(a,b)]^{a+b} - [^{i}M_h(a,b)]^{a+b} = (ab)^{a+b} - (ab)^{a+b}\left(\frac{2}{a^a b^b + a^b b^a}\right)$$
$$= \frac{(ab)^{a+b}}{a^a b^b + a^b b^a}(a^b b^a + a^a b^b - 2) > 0$$

Therefore, $^{i}M_h(a,b) < G(a,b)$
Hence case (3) is proved. Hence the theorem. □

Example 4.1: Comparison Table

Let us take a few sample values of a and b and compute the respective means:

a	b	$A(a,b)$	$G(a,b)$	$H(a,b)$	$M_h(a,b)$	$^{i}M_h(a,b)$
1.0	2.0	1.5	1.4142	1.3333	1.4644	1.3646
0.5	2.5	1.5	1.1180	0.8333	1.2722	1.1800
1.0	3.0	2.0	1.7321	1.5000	1.9333	1.5549
0.8	1.2	1.0	0.9798	0.9600	0.9954	0.9848

We observe that in each case: $a <^{i} M_h(a,b) < b$, and that $^{i}M_h(a,b)$ lies between $G(a,b)$ and $M_h(a,b)$, consistent with its construction as an invariant mean.

Example 4.2: Graphical Comparison
Let us fix $a = 1$ and plot the behavior of various means as $b \in [1,5]$.
Figure 7 below illustrates:

1. The monotonic growth of all means with b.
2. $H(a,b) < G(a,b) < {^{i}M_h(a,b)} < M_h(a,b) < A(a,b)$
3. $^{i}M_h(a,b)$ stays smoothly within bounds and reflects convex behavior.

Remark: The above figure was generated using Python by evaluating the respective mean functions for $b \in [1,5]$ with $a = 1$.

Observations:

1. The invariant mean $^{i}M_h(a,b)$ behaves smoothly and is always bounded between a and b.
2. It lies closer to the geometric mean but reflects structure induced by the exponential form of the Muirhead mean.
3. Compared to classical means, $^{i}M_h(a,b)$ is neither too aggressive (like the arithmetic mean) nor too conservative (like the harmonic mean), making it suitable for applications requiring balanced averaging with symmetric convexity.

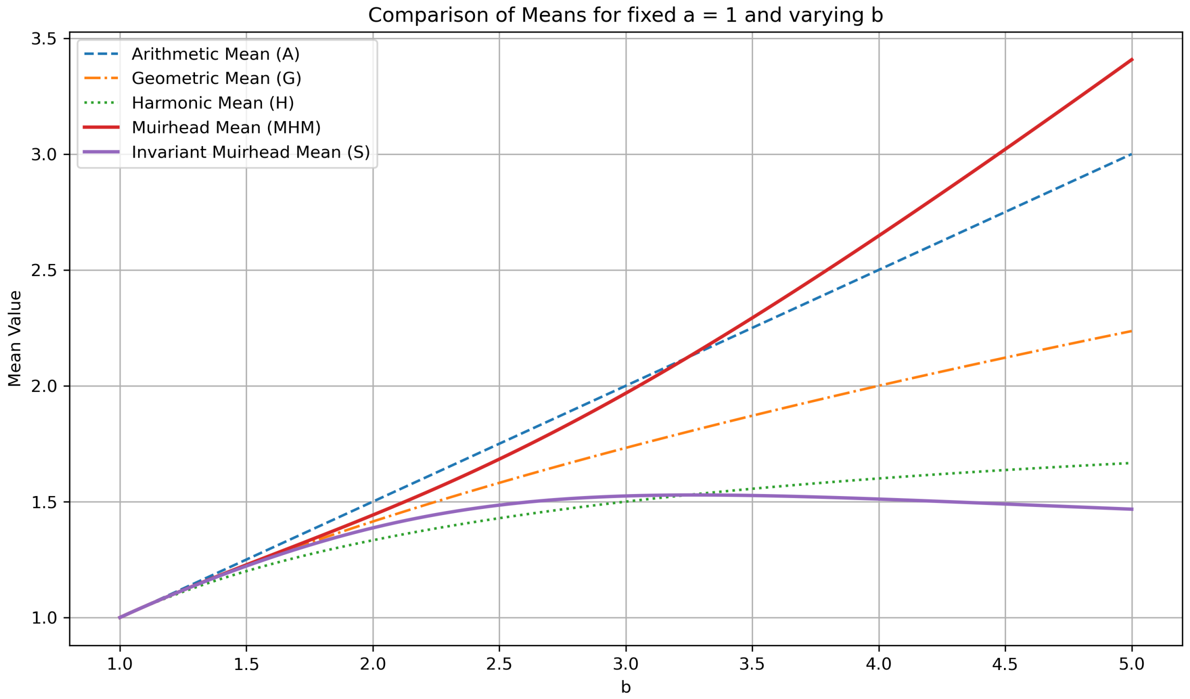

Figure 7: Comparison of ${}^{i}M_h(a, b)$ with classical means

4. Applications and Further Remarks

The study of symmetric and invariant means extends beyond classical analysis and has applications across several areas of pure and applied mathematics. In this section, we highlight the potential implications and directions for applying the invariant Muirhead mean ${}^{i}M_h(a, b)$.

4.1. Applications in Inequality Theory

The invariant Muirhead mean may be utilized in refining well-known inequalities such as Karamata's inequality, the Hardy-Littlewood-Pólya inequality, and Jensen-type inequalities for power-exponential means.

Its behavior between the geometric and arithmetic means makes it valuable when a balance between growth and stability is desired in analytic estimates.

4.2. Applications in Graph Labeling

The invariant Muirhead mean, with its structural symmetry and boundedness, could be applied in defining new labeling rules for special classes of graphs. By embedding ${}^{i}M_h(a, b)$ into labeling functions, one may obtain novel results in distinguishing edge and vertex labels, possibly enhancing labeling schemes used in network design or coding theory.

4.3. Applications in Optimization and Machine Learning

Since the invariant Muirhead mean is Schur-convex and bounded, it may be used in:

1. Constructing generalized averaging layers in neural networks,
2. Defining symmetric aggregation functions in federated learning,
3. Developing regularizers for log-linear models where classical mean penalties (like L2 norm) are too aggressive.

4.4. Possible Generalizations

While the present study focuses on the two-variable case, natural generalizations of the Muirhead mean and its invariant to n-variables are possible. For instance, one may define:

$$M_h(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n!} \sum_{\pi} \prod_{i=1}^{n} x_i^{\pi(i)} \right)^{\frac{1}{\sum x_i}},$$

and may investigate whether a corresponding invariant mean exists such that

$$M_h(x_1, \ldots, x_n) \cdot {}^{i}M_h(x_1, \ldots, x_n) = G^2(x_1, \ldots, x_n).$$

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Sowmyashree M. K., Department of Mathematics, RNS Institute of Technology, India and Affiliated to Visvesvaraya Technological University, Belagavi-590018, India.
E-mail address: `sowmyashree.mk@rnsit.ac.in`

and

Narasimhan G., Department of Mathematics, RNS Institute of Technology, India and Affiliated to Visvesvaraya Technological University, Belagavi-590018, India.
E-mail address: `narasimhan.g@rnsit.ac.in`

and

Sudhamani M. J., Department of Computer Science, RNS Institute of Technology, India and Affiliated to Visvesvaraya Technological University, Belagavi-590018, India.
E-mail address: `sudhamanimj@rnsit.ac.in`

and

K. M. Nagaraja, Department of Mathematics, JSS Academy of Technical Education, Bangalore, India.
E-mail address: `nagkmn@gmail.com`

and

SampathKumar. R, Department of Mathematics, RNS Institute of Technology, India.
E-mail address: `r.sampathkumar1967@gmail.com`