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Explicit properties of q-Cosine and q-Sine Fubini-type polynomials and numbers

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ABSTRACT: In recent years, q-special polynomials, such as q-Bernoulli, q-Genocchi, q-Euler and q-Frobenius-Genocchi have been studied and investigated by many mathematicians, as well physicists. It is important that any polynomials have explicit formulas, symmetric identities, summation formulas, and relations with other polynomials. In this work, the q-Cosine and q-Sine Fubini type polynomials are introduced and multifarious above mentioned properties for these polynomials are derived by utilizing some series manipulation methods. Moreover, several correlations related to both the q-Bernoulli, q-Euler, and q-Genocchi polynomials and the q-Stirling numbers of the second kind.

Key Words: Quantum calculus, Fubini type polynomials, Parametric q-Fubini type polynomials, generating functions, Stirling numbers of the second kind.

Contents

1. Introduction

Recently, many authors [1,2,3,5,6,35] have introduced and constructed generating functions for new families of special polynomials including two parametric kinds of polynomials as Bernoulli, Euler, Genocchi, etc. They have given fundamental properties of these polynomials. Also, they have established more identities, and relations among trigonometric functions, two parametric kinds of special polynomials by using generating functions. Special polynomials have important role in several subjects of mathematics, approximation theory, engineering and theoretical physics. Applying the partial derivative operator to these generating functions, some derivative formulae, and finite combinatorial sums involving the aforementioned polynomials and numbers are obtained. In addition, these special polynomials allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of special polynomials. The Fubini-kind polynomials seem in combinatorial mathematics and play an crucial role in the principle and applications of arithmetic, hence many wide variety idea and combinatorics experts have extensively studied their residences and received a series of exciting results (see [7,8,13,14,15,16,17]). By inspiring and motivating the above polynomials, in this study, we are purpose to define a parametric kinds of q-Fubini type polynomials by introducing the two specific q-exponential generating functions. Also, we show many formulation and family members for those polynomials, such as a few implicit summation formulas, differentiation policies, and correlations with the sooner polynomials with the aid of utilizing some collection manipulation approach.

The concern of q-calculus started performing in the 19th century due to its packages in various fields of mathematics, physics and engineering. The definitions and notations of q-calculus reviewed right here are taken from (see [4,18,19,20,21,22]):

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The q-analogue of the shifted factorial $(\alpha)_\omega$ is given by

$$(\alpha;q)_0=1,(\alpha;q)_\omega=\prod_{\gamma=0}^{\omega-1}(1-q^\gamma\alpha)\quad \omega\in\mathbb{N}.$$

The q-analogue of a complex number α and of the factorial function is given by

$$[\alpha]_q=\frac{1-q^\alpha}{1-q}\quad q\in\mathbb{C}-\{1\};\alpha\in\mathbb{C},$$

$$[\omega]_q!=\prod_{\gamma=1}^{\omega}[\gamma]_q=[1]_q[2]_q\cdots[\omega]_q=\frac{(q;q)_\omega}{(1-q)^\omega}\quad q\neq 1;\omega\in\mathbb{N},$$

$$[0]_q!=1,q\in\mathbb{C};0<q<1.$$

The Gauss q-binomial coefficient $\left(\begin{array}{c}\omega\\k\end{array}\right)_q$ is given by

$$\left(\begin{array}{c}\omega\\\gamma\end{array}\right)_q=\frac{[\omega]_q!}{[\gamma]_q![\omega-\gamma]_q!}=\frac{(q;q)_\omega}{(q;q)_\gamma(q;q)_{\omega-\gamma}}\quad \gamma=0,1,\cdots,\omega.$$

The q-analogue of the function $(x+y)_q^\omega$ is given by

$$(x+y)_q^\omega=\sum_{\gamma=0}^{\omega}\left(\begin{array}{c}\omega\\\gamma\end{array}\right)_q q^{\gamma(\gamma-1)/2}x^{\omega-\gamma}y^\gamma\quad \omega\in\mathbb{N}_0. \tag{1.1}$$

The q-analogue of exponential functions are given by [23,24,25,26,27]

$$e_q(x)=\sum_{\omega=0}^{\infty}\frac{x^\omega}{[\omega]_q!}=\frac{1}{((1-q)x;q)_\infty}\quad 0<\mid q\mid<1;\mid x\mid<\mid 1-q\mid^{-1}, \tag{1.2}$$

and

$$E_q(x)=\sum_{\omega=0}^{\infty}\frac{q^{\binom{\omega}{2}}}{[\omega]_q!}x^\omega=(-(1-q)x;q)_\infty\quad 0<\mid q\mid<1;x\in\mathbb{C}. \tag{1.3}$$

These two functions are related by the equation (see [9,10])

$$e_q(x)E_q(-x)=1.$$

The q-derivative operator is defined by

$$D_qf(z)=\frac{f(qz)-f(z)}{qz-z},0<\mid q\mid<1,$$

and $D_qf(0)=f'(0)$ provided that f is differentiable at $x=0$.

The q-derivative fulfills the following product and quotient rules

$$D_{q,z}(f(z)g(z))=f(z)D_{q,z}g(z)+g(qz)D_{q,z}f(z), \tag{1.4}$$

and

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right)=\frac{g(qz)D_{q,z}f(z)-f(qz)D_{q,z}g(z)}{g(z)g(qz)}. \tag{1.5}$$

Definition 1.1 The q-trigonometric functions are (see [11]):

$$sin_q(x)=\frac{e_q(ix)-e_q(-ix)}{2i},\quad SIN_q(x)=\frac{E_q(ix)-E_q(-ix)}{2i},$$

and

$$cos_q(x)=\frac{e_q(ix)+e_q(-ix)}{2},\quad COS_q(x)=\frac{E_q(ix)+E_q(-ix)}{2},$$

where $SIN_q(x)=sin_{q^{-1}}(x)$, $COS_q(x)=cos_{q^{-1}}(x)$.

The Apostol-type q-Bernoulli polynomials $\mathbb{B}^{(\alpha)}_{\omega,q}(x;\lambda)$ of order α, the Apostol-type q-Euler polynomials $\mathbb{E}^{(\alpha)}_{\omega,q}(x;\lambda)$ of order α and the Apostol-type q-Genocchi polynomials $\mathbb{G}^{(\alpha)}_{\omega,q}(x;\lambda)$ of order α are defined by (see [28,29,30,33]):

$$\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha}e_q(xt)=\sum_{\omega=0}^{\infty}\mathbb{B}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!}\qquad (\mid t+\log\lambda\mid)<2\pi, \tag{1.6}$$

$$\left(\frac{2}{\lambda e_q(t)+1}\right)^{\alpha}e_q(xt)=\sum_{\omega=0}^{\infty}\mathbb{E}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!}\qquad (\mid t+\log\lambda\mid)<\pi, \tag{1.7}$$

and

$$\left(\frac{2t}{\lambda e_q(t)+1}\right)^{\alpha}e_q(xt)=\sum_{\omega=0}^{\infty}\mathbb{G}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!}\qquad (\mid t+\log\lambda\mid<\pi), \tag{1.8}$$

respectively.

Clearly, we have

$$\mathbb{B}^{(\alpha)}_{\omega,q}(\lambda)=\mathbb{B}^{(\alpha)}_{\omega,q}(0;\lambda),\quad \mathbb{E}^{(\alpha)}_{\omega,q}(\lambda)=\mathbb{E}^{(\alpha)}_{\omega,q}(0;\lambda),\quad \mathbb{G}^{(\alpha)}_{n,q}(\lambda)=\mathbb{G}^{(\alpha)}_{\omega,q}(0;\lambda).$$

Kang and Ryoo [12] introduced and defined the q-Bernoulli and q-Euler polynomials of complex variable as follows:

$$\frac{t}{e_q(t)-1}e_q(xt)COS_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{B}_{j,q}((x\oplus iy)_q)+\mathbb{B}_{j}((x\ominus iy)_q)}{2}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{B}^{(C)}_{j,q}(x,y)\frac{t^j}{[j]_q!}, \tag{1.9}$$

$$\frac{t}{e_q(t)-1}e_q(xt)SIN_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{B}_{j,q}((x\oplus iy)_q)-\mathbb{B}_{j,q}((x\ominus iy)_q)}{2i}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{B}^{(S)}_{j,q}(x,y)\frac{t^j}{[j]_q!}, \tag{1.10}$$

and

$$\frac{2}{e_q(t)+1}e_q(xt)COS_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{E}_{j,q}((x\oplus iy)_q)+\mathbb{E}_{j,q}((x\ominus iy)_q)}{2}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{E}^{(C)}_{j,q}(x,y)\frac{t^j}{[j]_q!}, \tag{1.11}$$

$$\frac{2}{e_q(t)+1}e_q(xt)SIN_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{E}_{j}((x\oplus iy)_q)-\mathbb{E}_{j}((x\ominus iy))_q}{2i}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{E}^{(S)}_{j,q}(x,y)\frac{t^j}{[j]_q!}, \tag{1.12}$$

respectively.

Also they proved that (see [34]):

$$e_q(xt)COS_q(yt)=\sum_{r=0}^{\infty}C_{r,q}(x,y)\frac{t^r}{[r]_q!}, \tag{1.13}$$

and

$$e_q(xt)SIN_q(yt)=\sum_{r=0}^{\infty}S_{r,q}(x,y)\frac{t^r}{[r]_q!}, \tag{1.14}$$

where

$$C_{r,q}(x,y)=\sum_{j=0}^{[\frac{r}{2}]}(-1)^j\left(\begin{array}{c}r\\2j\end{array}\right)_q(-1)^jq^{2j-1}x^{r-2j}y^{2j}, \tag{1.15}$$

and

$$S_{r,q}(x,y)=\sum_{j=0}^{[\frac{r-1}{2}]}\left(\begin{array}{c}r\\2j+1\end{array}\right)_q(-1)^jq^{(2j+1)j}x^{r-2j-1}y^{2j+1}. \tag{1.16}$$

For $\lambda\in\mathbb{C}$, the generalized q-Stirling numbers of the second kind $S^{\omega}_{m,q}(\lambda)$ are given by (see [26]):

$$\frac{(\lambda e_q(t)-1)^m}{m!}=\sum_{\omega=0}^{\infty}S^{\omega}_{m,q}(\lambda)\frac{t^{\omega}}{[\omega]_q!}\quad m\in\mathbb{N}_0=\{0,1,2,\cdots,\}. \tag{1.17}$$

Taking $\lambda=1$, equation (1.17) reduces to the q-Stirling numbers of the second kind as follows

$$\frac{(e_q(t)-1)^m}{m!}=\sum_{\omega=m}^{\infty}S_{2,q}(\omega,m)\frac{t^{\omega}}{[\omega]_q!}. \tag{1.18}$$

The q-type Geometric polynomials of two variables are defined by (see [28]):

$$\frac{1}{1-y(e_q(t)-1)}e_q(xt)=\sum_{n=0}^{\infty}\mathbb{F}_{n,q}\left(x;y\right)\frac{t^n}{[n]_q!}. \tag{1.19}$$

When $x=0$, $\mathbb{F}_{n,q}(y)=\mathbb{F}_{n,q}\left(0;y\right)$ are called the q-type Geometric polynomials and $\mathbb{F}_{n,q}\left(0;1\right)=\mathbb{F}_{n,q}$ are called the q-type Geometric numbers.

The Fubini type polynomials of order α are defined by the following generating function [36]

$$\frac{2^{\alpha}}{(2-e^t)^{2\alpha}}e^{xt}=\sum_{n=0}^{\infty}a^{(\alpha)}_n(x)\frac{t^n}{n!},\quad \alpha\in\mathbb{N}_0;\quad |t|<\log(2). \tag{1.20}$$

When $x=0$, $a^{(\alpha)}_n=a^{(\alpha)}_n(0)$ are called the Fubini type numbers of order α.

2. Bivariate kinds of q-Cosine and q-Sine Fubini type polynomials

In this section, we consider the q-Cosine and q-Sine Fubini type polynomials of complex variable and deduce some identities of these polynomials. First, we present the following definition as.

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)E_q(ity)=\sum_{n=0}^{\infty}\mathcal{F}^{(\alpha)}_{n,q}\left((x+iy)_q\right)\frac{t^n}{[n]_q!}. \tag{2.1}$$

On the other hand, we suppose that

$$e_q(xt)E_q(ity)=e_q(xt)(COS_q(yt)+iSIN_q(yt)). \tag{2.2}$$

Thus, by (2.1) and (2.2), we have

$$\sum_{n=0}^{\infty}\mathcal{F}^{(\alpha)}_{n,q}\left((x+iy)_q\right)\frac{t^n}{[n]_q!}=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)E_q(ity)=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)(COS_q(yt)+iSIN_q(yt)), \tag{2.3}$$

and

$$\sum_{n=0}^{\infty}\mathcal{F}^{(\alpha)}_{n,q}\left((x-iy)_q\right)\frac{t^n}{[n]_q!}=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)E_q(-ity)=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)(COS_q(yt)-iSIN_q(yt)).. \tag{2.4}$$

From (2.3) and (2.4), we get

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)COS_q(yt)=\sum_{n=0}^{\infty}\left(\frac{\mathcal{F}_{n,q}^{(\alpha)}\left((x\oplus iy)_q\right)+\mathcal{F}_{n,q}^{(\alpha)}\left((x\ominus iy)_q\right)}{2}\right)\frac{t^n}{[n]_q!}, \tag{2.5}$$

and

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)SIN_q(yt)=\sum_{n=0}^{\infty}\left(\frac{\mathcal{F}_{n,q}^{(\alpha)}\left((x\oplus iy)_q\right)-\mathcal{F}_{n,q}^{(\alpha)}\left((x\ominus iy)_q\right)}{2}\right)\frac{t^n}{[n]_q!}. \tag{2.6}$$

Definition 2.1 Let $n\geq 0$. We define two parametric kinds of q-Cosine Fubini type polynomials $\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)$ and q-Sine Fubini type polynomials $\mathcal{F}_{n,q}^{(S,\alpha)}(x,y)$, for non negative integer n are defined by

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)COS_q(yt)=\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)\frac{t^n}{[n]_q!}, \tag{2.7}$$

and

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)SIN_q(yt)=\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(S,\alpha)}(x,y)\frac{t^n}{[n]_q!}, \tag{2.8}$$

respectively.

Note that

$$\mathcal{F}_{n,q}^{(C,\alpha)}(0,0)=\mathcal{F}_{n,q}^{(\alpha)},\quad \mathcal{F}_{n,q}^{(S,\alpha)}(0,0)=0.\quad (n\geq 0).$$

From (2.5)-(2.8), we have

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)=\frac{\mathcal{F}_{n,q}^{(\alpha)}\left((x\oplus iy)_q\right)+\mathcal{F}_{n,q}^{(\alpha)}\left((x\ominus iy)_q\right)}{2}, \tag{2.9}$$

and

$$\mathcal{F}_{n,q}^{(S)}(x,y)=\frac{\mathcal{F}_{n,q}^{(\alpha)}\left((x\oplus iy)_q\right)-\mathcal{F}_{n,q}\left((x\ominus iy)_q\right)}{2i}. \tag{2.10}$$

Remark 2.1 For $x=0$ in (2.7) and (2.8), we get new type of q-Cosine Fubini type polynomials $\mathcal{F}_{n,q}^{(C,\alpha)}(y)$ and q-Sine Fubini type polynomials $\mathcal{F}_{n,q}^{(S,\alpha)}(y)$ as

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}COS_q(yt)=\sum_{j=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(y)\frac{t^n}{[n]_q!}, \tag{2.11}$$

and

$$\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}SIN_q(yt)=\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(S,\alpha)}(y)\frac{t^n}{[n]_q!}, \tag{2.12}$$

respectively.

It is clear that

$$\mathcal{F}_{n,q}^{(C,\alpha)}(0)=\mathcal{F}_{n,q}^{\alpha},\quad \mathcal{F}_{n,q}^{(S,\alpha)}(0)=0,\quad n\geq 0.$$

Now, we start some basic properties of these polynomials.

Theorem 2.1 *Let n integer. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(y)=\sum_{v=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n+v \\ 2v \end{array}\right)_q(-1)^v q^{(2v-1)v}y^{2v}\mathcal{F}_{n-2v,q}^{(\alpha)}, \tag{2.13}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(y)=\sum_{v=0}^{[\frac{n-1}{2}]}\left(\begin{array}{c} n+v \\ 2v+1 \end{array}\right)_q(-1)^v q^{(2v+1)v}y^{2v+1}\mathcal{F}_{n-2v-1,q}^{(\alpha)}. \tag{2.14}$$

Proof: By (2.11) and (2.12), we can derive the following equations

$$\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(y)\frac{t^n}{[n]_q!}=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}COS_q(yt)=\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}\sum_{v=0}^{\infty}(-1)^v q^{(2v-1)v}\eta^{2v}\frac{t^v}{[2v]_q!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{v=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n+v \\ 2v \end{array}\right)_q(-1)^v q^{(2v-1)v}\eta^{2v}\mathcal{F}_{n-2v,q}^{(\alpha)}\right)\frac{t^n}{[n]_q!}, \tag{2.15}$$

and

$$\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(S,\alpha)}(y)\frac{t^n}{[n]_q!}=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}SIN_q(yt)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{v=0}^{[\frac{n-1}{2}]}\left(\begin{array}{c} n \\ 2v+1 \end{array}\right)_q(-1)^v q^{(2v+1)v}y^{2v+1}\mathcal{F}_{n-2v-1,q}^{(\alpha)}\right)\frac{t^n}{[n]_q!}. \tag{2.16}$$

Therefore, by (2.15) and (2.16), we get (2.13) and (2.14). □

Theorem 2.2 *Let n integer. Then*

$$\mathcal{F}_{n,q}^{(\alpha)}\left((x\oplus iy)_q\right)=\sum_{k=0}^{n}\binom{n}{k}_q(x\oplus iy)_q^k\mathcal{F}_{n-k,q}=\sum_{k=0}^{n}\binom{n}{k}_q(iy)^k\mathcal{F}_{n-k,q}^{(\alpha)}(x), \tag{2.17}$$

and

$$\mathcal{F}_{n,q}^{(\alpha)}\left((x\ominus iy)_q;z\right)=\sum_{k=0}^{n}\binom{n}{k}_q(x\ominus iy)_q^k\mathcal{F}_{n-k,q}=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k(iy)^k\mathcal{F}_{n,q}^{(\alpha)}(x). \tag{2.18}$$

Proof: By using (2.3) and (2.4), we obtain (2.17) and (2.18). So we omit the proof. □

Theorem 2.3 *Let n integer. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q\mathcal{F}_{k,q}^{(\alpha)}\mathcal{C}_{n-k,q}(x,y), \tag{2.19}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(x,y)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q\mathcal{F}_{k,q}^{(\alpha)}\mathcal{S}_{n-k,q}(x,y). \tag{2.20}$$

Proof: Consider

$$\left(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{k!}\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{j}a_{n-k}b_k\right)\frac{t^n}{n!}.$$

Now

$$\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)\frac{t^n}{[n]_q!}=\frac{2^{\alpha}}{(2-e_q(t))^{2\alpha}}e_q(xt)COS_q(yt)$$
$$=\left(\sum_{k=0}^{\infty}\mathbb{F}_{k,q}^{(\alpha)}\frac{t^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty}\mathcal{C}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q\mathcal{F}_{k,q}^{(\alpha)}\mathcal{C}_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!},$$

which proves (2.19). The proof of (2.20) is similar. □

Theorem 2.4 *Let $n \geq 0$. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x+r,y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q \mathcal{F}_{k,q}^{(C,\alpha)}(x,y) r^{n-k}, \tag{2.21}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(x+r,y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q \mathcal{F}_{k,q}^{(S,\alpha)}(x,y) r^{n-k}. \tag{2.22}$$

Proof: By changing x with $x+r$ in (2.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x+r,y)\frac{t^n}{[n]_q!} &= \frac{2^\alpha}{(2-e_q(t))^{2\alpha}} e_q(xt) COS_q(yt) e^{rt} \\ &= \left(\sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x,y)\frac{t^n}{[n]_q!}\right)\left(\sum_{k=0}^{\infty} r^k \frac{t^k}{[k]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array} \right)_q \mathcal{F}_{k,q}^{(C,\alpha)}(x,y) r^{n-k}\right)\frac{t^n}{[n]_q!},\end{aligned}$$

which complete the proof (2.21). The result (2.22) can be similarly proved.

□

Theorem 2.5 *Let n integer. Then*

$$\frac{\partial}{\partial x}\mathcal{F}_{n,q}^{(C,\alpha)}(x,y) = [n]_q \mathcal{F}_{n-1,q}^{(C,\alpha)}(x,y), \tag{2.23}$$

$$\frac{\partial}{\partial y}\mathcal{F}_{n,q}^{(C,\alpha)}(x,y) = -[n]_q \mathcal{F}_{n-1,q}^{(S,\alpha)}(x,qy), \tag{2.24}$$

and

$$\frac{\partial}{\partial x}\mathcal{F}_{n,q}^{(S,\alpha)}(x,y) = [n]_q \mathcal{F}_{n-1,q}^{(S,\alpha)}(x,y), \tag{2.25}$$

$$\frac{\partial}{\partial y}\mathcal{F}_{n,q}^{(S,\alpha)}(x,y) = [n]_q \mathcal{F}_{n-1,q}^{(C,\alpha)}(x,qy). \tag{2.26}$$

Proof: Equation (2.7) yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\partial}{\partial x}\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\frac{\partial}{\partial x} e_q(xt) COS_q(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x,y)\frac{t^{n+1}}{[n]_q!} \\ &= \sum_{n=1}^{\infty} \mathcal{F}_{n-1,q}^{(C,\alpha)}(x,y)\frac{t^n}{[(n-1)]_q!} = \sum_{n=1}^{\infty} [n]_q \mathcal{F}_{n-1,q}^{(C,\alpha)}(x,y)\frac{t^n}{[n]_q!},\end{aligned}$$

proving (2.23). Other (2.24), (2.25) and (2.26) can be similarly derived. □

Theorem 2.6 *Let $N \in \mathbb{N}^*$, the following formula holds true. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(2x,y) = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{F}_{n,q}^{(C,\alpha)}(x,y) x^{n-k}, \tag{2.27}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(2x,y) = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{F}_{n,q}^{(S,\alpha)}(x,y) x^{n-k}. \tag{2.28}$$

Proof: By using Definition 2.1, we can easily proof of equations (2.27) and (2.28). We omit the proof. □

Theorem 2.7 *For $n \geq 0$, we have*

$$\mathcal{F}_{n,q}^{(C,\alpha)}((1 \oplus x)_q, y) = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_q \mathcal{F}_{n-r,q}^{(C,\alpha)}(x, y), \tag{2.29}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}((1 \oplus x)_q, y) = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_q \mathcal{F}_{n-r,q}^{(S,\alpha)}(x, y). \tag{2.30}$$

Proof: Using the generating function (2.7), we have

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}((1 \oplus x)_q, y) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\tfrac{2^\alpha}{(2-e_q(t))^{2\alpha}} \right) (e_q(t) - 1) e_q(xt) COS_q(yt)$$
$$= \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!} \left(\sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} - 1 \right)$$
$$= \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} - \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_q \mathcal{F}_{n-r,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!}.$$

Finally, equating the coefficients of the like powers of t on both sides, we get (2.29). The proof of (2.30) is similar. □

Theorem 2.8 *Let n be integer. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha+\beta)}(x, y) = \sum_{k=0}^{k} \binom{n}{k}_q \mathcal{F}_{k,q}^{(\alpha)} \mathcal{F}_{n-k,q}^{(C,\beta)}(x, y) \tag{2.31}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha+\beta)}(x, y) = \sum_{k=0}^{k} \binom{n}{k}_q \mathcal{F}_{k,q}^{(\alpha)} \mathcal{F}_{n-k,q}^{(S,\beta)}(x, y) \tag{2.32}$$

Proof: By (2.7), we get

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(C,\alpha+\beta)}(x, y) \frac{t^n}{[n]_q!} = \frac{2^{\alpha+\beta}}{(2 - e_q(t))^{2(\alpha+\beta)}} e_q(xt) COS_q(yt)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{F}_{k,q}^{(\alpha)} \mathcal{F}_{n,q}^{(C,\beta)}(x, y) \frac{t^k}{[k]_q!} \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{k} \binom{n}{k}_q \mathcal{F}_{k,q}^{(\alpha)} \mathcal{F}_{n-k,q}^{(C,\beta)}(x, y) \frac{t^n}{[n]_q!},$$

which proves the result (2.31). The assertion (2.32) can be proved similarly. □

3. Relationship between q-Bernoulli, q-Euler and q-Genocchi polynomials and q-Stirling numbers of the second kind

In this section, we prove some relationships for two bivariate kind of q-Cosine Fubini type polynomials and q-Sine Fubini type polynomials related to q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and q-Stirling numbers of the second kind. We start a following theorem.

Theorem 3.1 *Each of the following relationships holds true:*

$$\mathcal{F}^{(C,\alpha)}_{n,q}(x,y)=\sum_{s=0}^{n+1}\binom{n+1}{s}_q\left[\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{B}_{s-k,q}(x)-\mathcal{B}_{s,q}(x)\right]\frac{\mathcal{F}^{(C,\alpha)}_{n+1-s,q}(y)}{[n+1]_q},\tag{3.1}$$

and

$$\mathcal{F}^{(S,\alpha)}_{n,q}(x,y)=\sum_{s=0}^{n+1}\binom{n+1}{s}_q\left[\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{B}_{s-k,q}(x)-\mathcal{B}_{s,q}(x)\right]\frac{\mathcal{F}^{(S,\alpha)}_{n+1-s,q}(y)}{[n+1]_q}.\tag{3.2}$$

Proof: By using (1.6) and (2.7), we have

$$\begin{aligned}\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)e_q(xt)COS_q(yt)&=\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)\frac{t}{e_q(t)-1}\frac{e_q(t)-1}{t}e_q(xt)COS_q(yt)\\&=\frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{s}\binom{s}{k}_q\mathbb{B}_{s-k,q}(x)p^{\binom{k}{2}}\right)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}\mathcal{F}^{(C,\alpha)}_{n,q}(y)\frac{t^n}{[n]_q!}\\&\qquad-\frac{1}{t}\sum_{s=0}^{\infty}\mathcal{B}_{s,q}(x)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}\mathcal{F}^{(C,\alpha)}_{n,q}(y)\frac{t^n}{[n]_q!}\\&=\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_q\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{B}_{s-k,q}(x)\right]\mathcal{F}^{(C,\alpha)}_{n-s,q}(y)\frac{t^n}{[n]_q!}\\&\qquad-\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_q\mathcal{B}_{s,q}(x)\right]\mathcal{F}^{(C,\alpha)}_{n-s,q}(y)\frac{t^n}{[n]_q!}.\end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the required result (3.1). The proof of (3.2) is similar. □

Theorem 3.2 *Each of the following relationships holds true:*

$$\mathcal{F}^{(C,\alpha)}_{n,q}(x,y)=\sum_{s=0}^{n}\binom{n}{s}_q\left[\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{E}_{s-k,q}(x)+\mathcal{E}_{s,q}(x)\right]\frac{\mathcal{F}^{(C,\alpha)}_{n-s,q}(y)}{[2]_q},\tag{3.3}$$

and

$$\mathcal{F}^{(S,\alpha)}_{n,q}(x,y)=\sum_{s=0}^{n}\binom{n}{s}_q\left[\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{E}_{s-k,q}(x)+\mathcal{E}_{s,q}(x)\right]\frac{\mathcal{F}^{(S,\alpha)}_{n-s,q}(y)}{[2]_q}.\tag{3.4}$$

Proof: By using definitions (1.7) and (2.7), we have

$$\begin{aligned}\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)e_q(xt)COS_q(yt)&=\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)\frac{[2]_q}{e_q(t)+1}\frac{e_q(t)+1}{[2]_q}e_q(xt)COS_q(yt)\\&=\frac{1}{[2]_q}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{E}_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!}\right]\sum_{n=0}^{\infty}\mathcal{F}^{(C,\alpha)}_{n,q}(y)\frac{t^n}{[n]_q!}\\&=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_q\sum_{k=0}^{s}\binom{s}{k}_q\mathcal{E}_{s-k,q}(x)+\sum_{s=0}^{n}\binom{n}{s}_q\mathcal{E}_{s,q}(x)\right]\mathcal{F}^{(C,\alpha)}_{n-s,q}(y)\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the desired result (3.3). The proof of (3.4) is similar. □

Theorem 3.3 *Let n be integer. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)=\sum_{s=0}^{n}\left(\begin{array}{c}n+1\\ s\end{array}\right)_q\left[\sum_{k=0}^{s}\left(\begin{array}{c}s\\ k\end{array}\right)_q\mathcal{G}_{s-k,q}(x)+\mathcal{G}_{s,q}(x)\right]\frac{\mathcal{F}_{n+1-s,q}^{(C,\alpha)}(y)}{[2]_q[n+1]_q}, \tag{3.5}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(x,y)=\sum_{s=0}^{n}\left(\begin{array}{c}n+1\\ s\end{array}\right)_q\left[\sum_{k=0}^{s}\left(\begin{array}{c}s\\ k\end{array}\right)_q\mathcal{G}_{s-k,q}(x)+\mathcal{G}_{s,q}(x)\right]\frac{\mathcal{F}_{n+1-s,q}^{(S,\alpha)}(y)}{[2]_q[n+1]_q}. \tag{3.6}$$

Proof: By (1.8) and (2.7), we have

$$\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)e_q(xt)COS_q(yt)=\left(\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}\right)e_q(xt)COS_q(yt)\frac{[2]_qt}{e_q(t)+1}\frac{e_q(t)+1}{[2]_qt}e_q(xt)COS_q(yt)$$
$$=\frac{1}{[2]_qt}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)_q\mathcal{G}_{n-k,q}(x)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x)\frac{t^n}{[n]_q!}\right]\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\ s\end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c}s\\ k\end{array}\right)_q\mathcal{G}_{s-k,q}(x)+\sum_{s=0}^{n}\left(\begin{array}{c}n\\ s\end{array}\right)_q\mathcal{G}_{s,q}(x)\right]\mathcal{F}_{n+1-s,q}^{(C,\alpha)}(y)\frac{t^n}{[n+1]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, then we have the asserted result (3.5). The proof of (3.6) is similar. □

Theorem 3.4 *Let n be integer. Then*

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x+r,y)=\sum_{k=0}^{n}\binom{n}{k}_q\sum_{m=0}^{k}\mathcal{F}_{n-k,q}^{(C,\alpha)}(y)x^mS_{2,q}(k+r,m+r), \tag{3.7}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(x+r,y)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}_q\mathcal{F}_{n-k,q}^{(S,\alpha)}(y)x^mS_{2,q}(k+r,m+r) \tag{3.8}$$

Proof: Using (1.18) and (2.7), we have

$$\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(x+r,y)\frac{t^n}{[n]_q!}=\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}e_q((r)t)(e_q(t)-1+1)^xCOS_q(yt)$$
$$=\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}e_q((r)t)COS_q(yt)\sum_{m=0}^{\infty}x^k(e_q(t)-1)^m\frac{t^m}{[m]_q!}$$
$$=\frac{2^\alpha}{(2-e_q(t))^{2\alpha}}e_q((r)t)COS_q(yt)\sum_{k=0}^{\infty}\sum_{m=0}^{k}x^kS_{2,q}(k,m)\frac{t^k}{[k]_q!}$$
$$=\sum_{n=0}^{\infty}\mathcal{F}_{n,q}^{(C,\alpha)}(y)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}x^kS_{2,q}(k,m)\frac{t^k}{[k]_q!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q\sum_{m=0}^{k}\mathcal{F}_{n-k,q}^{(C,\alpha)}(y)x^mS_{2,q}(k+r,m+r)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, then we have the asserted result (3.7). The proof of (3.8) is similar. □

Remark 3.1 Let n be integer. Then

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}_q\sum_{m=0}^{k}\mathcal{F}_{n-k,q}^{(C,\alpha)}(y)x^m S_{2,q}(k,m), \tag{3.9}$$

and

$$\mathcal{F}_{n,q}^{(S,\alpha)}(x,y)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}_q\mathcal{F}_{n-k,q}^{(S,\alpha)}(y)x^m S_{2,q}(k,m) \tag{3.10}$$

4. Conclusions

Utilizing q-numbers and q-concepts, Mahmudov [31,32] considered q-Genocchi polynomials and numbers, q-Bernoulli polynomials and numbers and q-Euler polynomials and numbers and provided many properties and formulas for these polynomials. Inspired and motivated by this consideration, many authors have introduced q-special numbers and polynomials and have described their several identities and properties. In this paper, using the q-Cosine polynomials and q-Sine polynomials, we have introduced novel kinds of q-extensions of Fubini polynomials and have acquired multifarious properties and identities by making use of some series manipulation methods. Furthermore, we have computed the q-derivative operator rules for these polynomials. Moreover, we have determined the approximate root movements of the new mentioned polynomials in a complex plane, utilizing the Newton method and illustrating them in figures. The structure of the approximate roots will come out in various ways, depending on the condition of the variables, and new methods and theorems related to this topic need to be created and proven.

Not only can the ideas presented in this paper be utilized for similar polynomials, but these polynomials may also have possible applications in other scientific areas besides the applications described at the end of the paper. We would like to continue to study this line of research in the future.

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