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Optimizing Iraq's Oil Transportation System Using Fuzzy Nonagonal Membership Functions

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ABSTRACT: In linear programming challenges, such as transportation optimization, scheduling and production operations are crucial for cost efficiency and managing uncertainty. In Iraq, oil transportation is considered one of the main issues that requires optimization to address the uncertainties in supply and demand. In this study, the fuzzy optimization method is adopted to manage uncertainties of supplies, variable costs and unstable demands that face oil transportation sector in Iraq. The use of nine-verticillium nonagonal function provides superior adaptability and inclusiveness representation compared to the conventional triangular and trapezoidal fuzzy representation. A real transportation data was obtained from Al-Basra warehouses (Khor Al-Zubair and Al-Shuaiba) serving six governorates in Iraq. The proposed nonlinear fuzzy model illustrates that there is a reduction in total transportation cost from 567.948 to 513.63 (baseline for the wavy value-added model) that significantly shows an improvement in about 9.6% in cost efficiency. Sensitivity analysis at different a-cut levels presents greater robustness yet lower susceptibility to changes in uncertain parameters. Our work presents a promising approach to optimize oil transportation in Iraq. Its impact will address supply uncertainty and improve transportation decision-making. In addition, this approach may be applied to other Iraqi industries that facing similar uncertainty challenges.

Key Words: Fuzzy optimization, nonagonal membership functions, oil transportation, uncertainty modeling, supply and demand management, transportation problem.

Contents

1. Introduction

Iraq holds a leading position among the countries that produce oil. Therefore, efficient oil transportation is essential to preserve national economic stability. Transporting oil from production areas to refineries, storage facilities, and export terminals remains a persistent logistical challenge due to vast distances, weak infrastructure, and volatile regional conditions. The oil sector contributes over 90%

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of government revenue and the majority of export earnings, yet its distribution network faces ongoing uncertainty due to market price volatility, geopolitical influences, and operational disruptions along transportation routes [1]. These factors lead to wide variations in supply and demand, making accurate modeling and optimization of transportation costs critical for policymakers.

In operations research, the optimization transportation problem aims to minimize costs on the distribution of goods [2], in this case, oil, between consumers (refineries and export markets) and suppliers (oil production fields). Classical optimization methods like Vogel's Approximation Method [3], the Northwest Corner Rule [4], and the Modified Distribution Method (MODI) [5], are typically used to solve the transportation problem. These approaches are predicated on the fixed and certain knowledge of variables like supply, demand, and transportation costs. This presumption frequently falls short of capturing the actual circumstances encountered in Iraq's oil industry, where uncertainty is rampant.

For transportation and optimization problems, Zadeh's fuzzy approach—which introduces degrees of membership instead of binary decisions—is perfect because it models uncertainty and enables flexible handling of changing supply, demand, and cost parameters [5,6].

Recent studies have looked into a variety of methods for managing supply, demand, and cost parameter uncertainties in transportation problems using fuzzy optimization techniques. Dasril et al. [8,7] used a Monte Carlo method to solve fuzzy transportation problems, offering guidance on how to deal with interval uncertainties. Kartli et al. [9] suggest a heuristic algorithm for fully fuzzy transportation problems. In order to address the complexity of fuzzy transportation networks, Aroniadi and Beligiannis [10] employ particle swarm optimization as a metaheuristic method. In optimization problems, metaheuristic approaches offer substantial benefits, especially when managing uncertainty and complex, high-dimensional spaces in many different applications [11,12,13]. These techniques are flexible, and highlight the significance of strong optimization in dynamic environments, allowing for the effective exploration of a variety of solutions, the avoidance of local optima, and the achievement of strong, nearly optimal solutions in situations with changing parameters and constraints. Through the integration of similar principles using nonagonal membership functions in a fuzzy optimization framework, this study aims to balance efficiency and flexibility while improving the flexibility and adaptability of transportation planning in uncertain environments. The generalized nonagonal fuzzy numbers with 9-tuples are used; it determines their alpha cuts and membership functions; and solves a fuzzy assignment problem to show how effective they are [14,15].

A new technique for ranking Asymmetric Pentagonal Fuzzy Numbers (APFN) by determining their centers of gravity using left and right areas is presented in [16]. Besides, it suggests a direct method for resolving the Transshipment Problem (TP), a fuzzy optimization problem in which costs are represented by APFNs to obtain the fuzzy optimal solution [17]. Pandian and Natarajan [18] proposed an approach for a fuzzy optimization transportation problem, where the transportation cost, delivery, and demand are trapezoidal fuzzy numbers. In their work, a novel set of guidelines known as the fuzzy zero factor technique is put forth in order to find an optimization fuzzy premiere answer. A trapezoidal fuzzy wide variety is the best solution to the bushy optimization transportation problem using the fuzzy zero point method. This problem may also be solved by the control and optimization approaches given in [19,20,21], by permitting adaptable changes in cost, supply, and demand in response to changing circumstances. However, these ideas can be applied to transportation optimization.

This study aims to use the nonagonal ranking function to address the data gaps that exist in real-world scenarios and deal with the fuzziness that naturally occurs in actual data.

This paper focuses on the oil transportation system in Iraq and presents a novel use of nonagonal membership functions in the fuzzy transportation problem. It offers a framework for addressing the transportation problem that is more responsive to actual circumstances by employing fuzzy numbers with nonagonal membership functions to model the uncertainties in supply, demand, and transportation costs. Our goal is to show how nonagonal membership functions can be used to solve intricate optimization problems, while simultaneously assisting in the creation of more robust and effective transportation plans for Iraq's oil industry.

This paper is organized as follows: Section 2 presents fuzzy set theory and its application to Oil Transportation problems. The formulation of the transportation problem using fuzzy optimization is covered in Section 3. Section 4 provides an overview of the use of actual data. Numerical results with their analysis are discussed in Section 5. The paper is concluded in Section 6.

2. Fuzzy Sets for Transportation Problems

Zadeh introduced the fuzzy set theory in 1965 [3]. It is a mathematical tool that is used to handle uncertainty and imprecision in decision-making processes. Unlike classical set theory, which assumes binary membership (i.e., an element either belongs to a set or does not), fuzzy sets allow for degrees of membership, which are represented by membership functions that take values between 0 and 1. Fuzzy set theory is especially helpful in modeling real-world issues where data is frequently ambiguous or imprecise because of its adaptability.

Fuzzy sets can be used to model parameters like the supply of oil at production fields, refinery demand, and transportation costs between various locations in the context of Iraq's oil optimization transportation problem. We can more accurately depict the system's inherent uncertainty by representing these parameters as fuzzy numbers instead of exact values. Several studies have demonstrated the effectiveness of fuzzy set theory in solving transportation problems under uncertain conditions [6,7]. These studies highlight the benefits of using fuzzy sets to achieve more robust and flexible solutions that can adapt to changes in supply, demand, and transportation costs.

Definition 2.1 ([16]) *A fuzzy set is identified by its membership function, using values from the domain, space or universe of discourse into the unit interval* $[0,1]$. $A = (x, \mu(x); x \in X)$ *is the definition of a fuzzy set A in the universal set X. In this case,* $\mu_A(x)$ *is the grade value of* x *in the fuzzy set A, and* μ_A *is the grade of the membership function.*

Definition 2.2 ([3]) *The* α*-cut of the fuzzy set A of the Universe of discourse X is defined as*

$$A_\alpha = \{x \in X \mid \mu_{A(x)} \geq \alpha\}, \quad \alpha \in [0,1].$$

Definition 2.3 ([16]) *The following properties are satisfied by the membership function* μ_A *for a fuzzy number A, which is a subset of real line* $\mathbb{R}$*:*

(i) μ_A *is piecewise continuous in its domain.*

(ii) A is normal, that is, there exists an $x_0 \in \mathbb{R}$ *such that* $\mu_A(x_0) = 1$.

(iii) A is convex, that is, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}\ \forall x_1, x_2 \in X$.

Definition 2.4 ([14]) *A Nonagonal fuzzy number A, denoted by* $(a_1, a_2, \ldots, a_9)$*, and its membership function is defined as:*

$$f_{\tilde{A}}(x) = \begin{cases} \frac{1}{4} \cdot \frac{(x-a_1)}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \frac{1}{4} + \frac{1}{4} \cdot \frac{(x-a_2)}{a_3-a_2} & a_2 \leq x \leq a_3 \\ \frac{1}{2} + \frac{1}{4} \cdot \frac{(x-a_3)}{a_4-a_3} & a_3 \leq x \leq a_4 \\ \frac{3}{4} + \frac{1}{4} \cdot \frac{(x-a_4)}{a_5-a_4} & a_4 \leq x \leq a_5 \\ 1 - \frac{1}{4} \cdot \frac{(x-a_5)}{a_6-a_5} & a_5 \leq x \leq a_6 \\ \frac{3}{4} - \frac{1}{4} \cdot \frac{(x-a_6)}{a_7-a_6} & a_6 \leq x \leq a_7 \\ \frac{1}{2} - \frac{1}{4} \cdot \frac{(x-a_7)}{a_8-a_7} & a_7 \leq x \leq a_8 \\ \frac{1}{4} - \frac{1}{4} \cdot \frac{(x-a_8)}{a_9-a_8} & a_8 \leq x \leq a_9 \\ 0 & o.w. \end{cases} \tag{2.1}$$

Definition 2.5 ([3]) *Using a ranking function* $\mathcal{R} : F(\mathbb{R}) \to \mathbb{R}$*, where* $F(\mathbb{R})$ *is a set of fuzzy numbers defined on a real line, is an effective way to order the fuzzy numbers. It maps each fuzzy number into the real line where a natural order exists. We define orders on* $F(\mathbb{R})$ *by:*

$$\begin{aligned} a \geq b \quad & \textit{if and only if} \quad R(a) \geq R(b) \\ a > b \quad & \textit{if and only if} \quad R(a) > R(b) \\ a = b \quad & \textit{if and only if} \quad R(a) = R(b) \end{aligned} \tag{2.2}$$

3. Formulation of the Fuzzy Optimization Transportation Problem

In this study, we address the transportation problem by modeling supply, demand, and transportation costs as fuzzy numbers, specifically represented using nonagonal membership functions. The fuzzy transportation problem can be defined as follows:

Let:

- S_i represents the fuzzy supply at the i-th supplier.
- D_j represents the fuzzy demand at the j-th destination.
- C_{ij} represents the fuzzy transportation cost from the supplier i to destination j.

We expressed these parameters as a fuzzy number where their membership is the nonagonal function, which gives a more detailed representation of uncertainty than simpler fuzzy membership functions.

The objective is to minimize the total transportation cost, such that

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}C_{ij}, \tag{3.1}$$

where x_{ij} is the transported quantity from the supplier i to destination j.

3.1. Nonagonal Membership Function Representation

The nine vertices $(a_1, a_2, \ldots, a_9)$ are used to define the nonagonal membership function. For each vertex, there is a corresponding value in $(0, 1)$. The fuzzy numbers for supply, demand, and costs are represented as follows:

$$\begin{aligned} &\text{Supply } S_i, \text{ where } S_i = (s_{i1}, s_{i2}, s_{i3}, \ldots, s_{i9}) \\ &\text{Demand } D_j, \text{ where } D_j = (d_{j1}, d_{j2}, d_{j3}, \ldots, d_{j9}) \\ &\text{Cost } C_{ij}, \text{ where } (c_{ij1}, c_{ij2}, c_{ij3}, \ldots, c_{ij9}) \end{aligned} \tag{3.2}$$

Each parameter has its entire range of membership values, which show a type of integration between the transportation model and the nonagonal fuzzy numbers. These problems are then solved iteratively by applying optimization techniques and obtaining their crisp value.

3.2. Comparison between Fuzzy and Crisp Methods

The findings show that, in contrast to more straightforward fuzzy membership functions like triangular or trapezoidal, the use of nonagonal membership functions yields a more accurate representation of uncertainty. This is especially helpful when dealing with transportation issues where input parameters, like costs, can change as a result of different market conditions.

1. **Improved Solution Robustness:** The nonagonal fuzzy approach yielded solutions that were more robust in handling uncertainty in supply, demand, and transportation costs. Traditional methods, which assume precise data, tend to produce less flexible solutions that may not perform well under uncertain conditions.

2. **Reduced Sensitivity to Parameter Variability:** One significant advantage of using nonagonal membership functions is the reduced sensitivity to changes in input parameters. Since the membership function allows for a wider and more detailed range of values, the solution is less affected by small changes in supply, demand, or cost data.

3. **Comparison with Other Fuzzy Membership Functions:** Compared to triangular and trapezoidal membership functions, the nonagonal membership function offers a higher level of precision in modeling uncertainty. This precision results in a more accurate defuzzified value, leading to better optimization results.

4. The Implementation

This part takes us to a company that distributes petroleum products in Dura. Data regarding the cost of transportation was gathered in the contracts division, and it was noted that the company uses tanker trucks under contract with civil companies, which include the following: the Iraq Jawwal company, the Treasures of the Earth company, the Ghadeer Zahra company, the Maali Burj company, the smooth gloss gold company, and the Ali Dar El Oyoun company.

In the governorates of Nasiriyah, Amara, and Basra, transportation is carried out via pipeline and is not contracted with any civil companies, according to the manager of the department of contracts; the cost of transportation is zero. It is also done by special cars to the Ministry of Oil, the department of distribution of petroleum products company or by the official agents of the ministry. Since all transportation is done within the region, the governorates of Dohuk, Sulaymaniyah, and Erbil are not included in the data.

There has also been an exception to the rule that no complete transportation data is available due to security concerns in Diyala a, Anbar, Salah Al Dien and Mosul. The distribution company's equipped organizations were also examined, and information about actual petroleum product sales and warehouse inventory levels was gathered by the assets division. The manager has told me that the quantity of supply and demand increased by 4% and 5%, respectively, due to transportation costs. We will now summarize the information gathered in Tables 1, 2, and 3, which belongs to 2020, processed exclusively from the governorate of Basra.

The transportation expenses from oil warehouses to different Iraqi governorates are shown in Table 1. The expenses of shipping oil to some governorates from the two major warehouses, Khor-Alzuber and Shuaiba, are listed in this table. The distance and logistical considerations are parameters that affect the prices.

Table 1: Data transport costs from warehouses to governorates

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna
Khor-Alzuber	62000	23000	32000	27000	23000	21000
Shuaiba	60000	20000	30000	25000	21000	20000

The daily demand for petroleum products in the aforementioned governorates is displayed in Table 2. It represents the daily quantity of petroleum needed to satisfy local demands.

Table 2: The quantity of information provided to governorates and the demand for petroleum products m^3/ day.

Governorates	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna
Demand	10248.8	1644.18	1687.4	1882.22	1465.21	845.09

The daily supply available in Khor-Alzuber and Shuaiba is shown in Table 3. According to Table 2, these figures show each warehouse's supply capacity to achieve the demands of various governorates given in Table 2.

Table 3: The amount of data to supply in the warehouses of oil products m^3/day

Warehouse	Supply
Khor-Alzuber	7318.25
Shuaiba	6607.14

5. Results and Analysis

5.1. The Traditional Numerical Results

This section solves the optimization transportation problem by using Vogel's Approximation Method (VAM). Transportation costs from the two sample warehouses, Khor-Alzuber and Shuaiba, to several Iraqi governorates (Baghdad, Wasit, Karbala, Najaf, Diwaniya, and Muthanna), are summarized in Table 4, which also ensures that the total supply and demand are balanced in the optimization model. For the model to accurately reflect real-world conditions while meeting the petroleum demand of each governorate, balanced transportation problem data is required.

Table 4: Special data problem products (white oil)

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	62000	23000	32000	27000	23000	21000	7318.25
Shuaiba	60000	20000	30000	25000	21000	20000	6607.14
Demand	10248.8	1644.18	1687.4	1882.22	1465.21	845.09	13925.39

To guarantee that all demands are satisfied without an excess of supply, it displays the transportation expenses, including the adjustments from Khor-Alzuber and Shuaiba to the governorates.

Consider that $\sum_{i=1}^{2} S_i = 13925.39$, $\sum_{j=1}^{6} D_j = 17772.9$, where $S_i \neq D_j$, then we added a slack variable to the supply with zero cost and taking $\sum_{i=1}^{2} S_i - \sum_{j=1}^{6} D_j = 3847.51$. The resulting values are abstracted in Table 5.

Table 5: Special data problem products balanced (white oil, gasoline and gasoil))

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	62000	23000	32000	27000	23000	21000	7318.25
Shuaiba	60000	20000	30000	25000	21000	20000	6607.14
Warehouse *	0	0	0	0	0	0	3847.51
Demand	10248.8	1644.18	1687.4	1882.22	1465.21	845.09	17772.9 \ 17772.9

Note 1 *A dummy warehouse (slack row) with zero transportation cost has been added to balance total supply and demand. This step ensures the feasibility of the transportation matrix without affecting the optimal cost.*

The final balanced transportation data, including the adjusted costs and transportation allocations between the governorates and the warehouses, is shown in Table 6. Using Vogel's Approximation Method (VAM) for optimal cost minimization, this table shows the final distribution plan, with all demands based on supply availability.

Table 6: Special data problem products the balanced (white oil)

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	62 6.40129	23 —	32 —	27 —	23 0.91701	21 —	7.31825
Shuaiba	60 —	20 1.64418	30 1.6874	25 1.88228	21 0.54820	20 0.84509	6.60714
Warehouse	0 3.84751	0 —	0 —	0 —	0 —	0 —	3.84751
Demand	10.2488	1.64418	1.6874	1.88222	1.46521	0.84509	17772.9 \ 17772.9

The total optimization transportation cost = 576.94851

Note 2 *Table 6 presents an initial feasible plan obtained by Vogel's approximation method using Crispr data. These results serve as a baseline for comparison with the fuzzy nonlinear model.*

5.2. The Generalized Numerical Results

To transform the crisp transportation problem into a fuzzy transportation problem, we define each transportation cost ($\tilde{A}$) as a fuzzy nonagonal number with the values given for c_{ij} (the crisp transportation cost between suppliers and destinations) and the $\Delta_1 = \Delta_6$, $\Delta_2 = \Delta_7$, $\Delta_3 = \Delta_8$, $\Delta_4 = \Delta_9$.

5.3. Fuzzy Cost Representation

Each transportation cost $\tilde{A}$ is represented as a fuzzy number with nine values:

$$\tilde{A} = (c_{ij} - \Delta_4,\, c_{ij} - \Delta_3,\, c_{ij} + \Delta_2,\, c_{ij} + \Delta_1,\, c_{ij},\, c_{ij} - \Delta_6,\, c_{ij} - \Delta_7,\, c_{ij} - \Delta_8,\, c_{ij} - \Delta_9)$$

where:

$$\Delta_1 = \Delta_6 = 1, \quad \Delta_2 = \Delta_7 = 2, \quad \Delta_3 = \Delta_8 = 3, \quad \Delta_4 = \Delta_9 = 4.$$

5.3.1. Matrix of Fuzzy Transportation Costs. This transformation leads to a matrix where each entry is a nonagonal fuzzy number representing the transportation cost between suppliers and destinations. Example of transformed costs for suppliers and destinations Using the provided data, here's how the costs would look for each supplier-destination combination:

1. From Khor-Al-Zuber:
   - Baghdad: (58, 59, 60, 61, 62, 63, 64, 65, 66)
   - Wasit: (19, 20, 21, 22, 23, 24, 25, 26, 27)
   - Karbala: (28, 29, 30, 31, 32, 33, 34, 35, 36)
   - Najaf: (23, 24, 25, 26, 27, 28, 29, 30, 31)
   - Diwaniya: (19, 20, 21, 22, 23, 24, 25, 26, 27)
   - Muthanna: (17, 18, 19, 20, 21, 22, 23, 24, 25)
2. From Shuaiba:
   - Baghdad: (56, 57, 58, 59, 60, 61, 62, 63, 64)
   - Wasit: (16, 17, 18, 19, 20, 21, 22, 23, 24)
   - Karbala: (26, 27, 28, 29, 30, 31, 32, 33, 34)
   - Najaf: (21, 22, 23, 24, 25, 26, 27, 28, 29)
   - Diwaniya: (17, 18, 19, 20, 21, 22, 23, 24, 25)
   - Muthanna: (16, 17, 18, 19, 20, 21, 22, 23, 24)

5.3.2. Demand Representation. Transportation costs, supply, and demand values are represented as nonagonal fuzzy numbers in Table 7, which presents the fuzzy transportation data. The data is prepared for optimization under fuzzy conditions by this transformation, which takes into account the uncertainties in costs and quantities.

Table 7: Special data petroleum product of fuzzy transportation

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Al-Zuber	58,59,60,61, 62,63,64,65,66	19,20,21,22, 23,24,25,26,27	28,29,30,31, 32,33,34,35,36	23,24,25,26, 27,28,29,30,31	19,20,21,22, 23,24,25,26,27	17,18,19,20, 21,22,23,24,25	3.71825,4.31825, 6.31825,7.31825, 8.31825,9.31825, 10.31825,11.31825, 12.31825
Shuaiba	56,57,58,59, 60,61,62,63,64	16,17,18,19, 20,21,22,23,24	26,27,28,29, 30,31,32,33,34	21,22,23,24, 25,26,27,28,29	17,18,19,20, 21,22,23,24,25	16,17,18,19, 20,21,22,23,24	1.66714,2.66714, 3.66714,4.66714, 6.60714,7.66714, 8.66714,9.66714, 10.66714
Warehouse	0	0	0	0	0	0	-3.15491,-2.15491 ,-1.15491,-0.15491, 0.84509,2.84509, 3.84509,4.84509, 5.84509
Demand	6.2488,7.2488, 8.2489,9.2488 ,10.2488,11.2489, 12.2488,13.2488, 14.2488	-2.35583,-1.35583, -0.35583,0.64418, 1.64418,2.64418 ,3.64418,4.64418, 5.64418	-2.3126,-1.3126, -0.3126,0.6874, 1.6874,2.6844, 3.6844,4.6844, 5.6844	-3.11778,-2.11775, -1.11778,0.88222, 1.88222,2.88222, 3.88222,4.88222, 5.88222	-2.53479,-1.53479, -0.53479,0.46521, 1.46521,2.46521, 3.46521,4.46521, 5.46521	-3.15491,-2.15491, -1.15491,0.15491, 0.84509,1.84509, 2.84509,3.84509, 4.84509	

Note 3 *Each cost, supply and demand value in Table 7 is expressed as a non-angular fuzzy number to represent the layered uncertainty in oil transportation parameters. The α-cuts obtained from these fuzzy numbers are used in the subsequent optimization step.*

5.4. New Ranking Function

In this paper, we define three types of α-cut. The α-cut of a fuzzy number is defined as $A(\alpha) = \{x : \mu(x) \geq \alpha, \alpha \in [0, 1]\}$.

1- $R(\tilde{A}_N) = \int_0^1 \frac{1}{8} [R_1(\alpha) + R_2(\alpha)] \, d\alpha$

2- $R(\tilde{A}_N) = \int_0^1 \frac{1}{4} [R_1(\alpha) + R_2(\alpha)] \, d\alpha$

3- $R(\tilde{A}_N) = \int_0^1 \frac{1}{2} [R_1(\alpha) + R_2(\alpha)] \, d\alpha$, where

$$R_1(\alpha) = \inf_1(\alpha) + \inf_2(\alpha) + \inf_3(\alpha) + \inf_4(\alpha)$$
$$R_2(\alpha) = \text{sub}_1(\alpha) + \text{sub}_2(\alpha) + \text{sub}_3(\alpha) + \text{sub}_4(\alpha)$$
$$\inf_1(\alpha) = a_1 + 4\alpha(a_2 - a_1)$$
$$\inf_2(\alpha) = a_2 + (4\alpha - 1)(a_3 - a_2)$$
$$\inf_3(\alpha) = a_3 + (4\alpha - 2)(a_4 - a_3)$$
$$\inf_4(\alpha) = a_4 + (4\alpha - 3)(a_5 - a_4)$$
$$\text{sub}_1(\alpha) = a_5 + (4 - 4\alpha)(a_6 - a_5)$$
$$\text{sub}_2(\alpha) = a_6 + (3 - 4\alpha)(a_7 - a_6)$$
$$\text{sub}_3(\alpha) = a_7 + (2 - 4\alpha)(a_8 - a_7)$$
$$\text{sub}_4(\alpha) = a_9 + (-4\alpha)(a_9 - a_8)$$

Consider an optimization transportation problem with sex suppliers and three destinations. The supply, demand, and transportation costs are modeled as fuzzy numbers with nonagonal membership functions.
The defuzzified values are obtained using the centroid method, and the transportation problem is solved using the MODI method. The optimal transportation plan is determined, and the total transportation cost is calculated.

When using the first ranking function the total cost of optimizing the fuzzy transportation cost is (513.6331755), see Table 8.
The results of the optimized transportation plan using the fuzzy model's first ranking function are displayed in Table 8. The total transportation cost is optimized through fuzzy nonagonal membership functions, which provide the optimal costs and allocations between the warehouses and the governorates.

Table 8: Special data petroleum products of fuzzy transportation when $\alpha - cat\ \frac{1}{8}$(white oil)

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	— 58.625	1.644168 23	1.6774 32	1.88247 27	1.46521 23	0.311502 21	6.98075
Shuaiba	6.134126 53.25	— 20	— 30	— 25	— 21	0.533014 20	6.66714
Warehouse	4.114724 0	— 0	— 0	— 0	— 0	— 0	4.114724
Demand	10.24885	1.644168	1.6774	1.88247	1.46521	0.84509	17.763188 \ 17.763188

Total optimization transportation Cost = 513.6331755.
When using the second-ranking function the total cost of the fuzzy transportation problem is (1979.8208), see Table 9. In contrast to Table 8, it presents an alternative optimized cost value and a different viewpoint on cost minimization under fuzzy conditions. Total optimization transportation cost = 1979.8208.

Table 9: Special data petroleum products of fuzzy transportation when $\alpha - cat\ \frac{1}{4}$(white oil)

From \ to	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	— 124.5	3.2883375 46	3.3548 64	3.76494 54	2.93042 46	0.6385425 42	13.9615
Shuaiba	12.26726 106	— 40	— 60	— 50	— 42	1.06702 40	13.33428
Warehouse	8.23044 0	— 0	— 0	— 0	— 0	— 0	8.23044
Demand	20.4977	3.2883375	3.3548	3.76494	2.93042	1.69018	35.52578 \ 35.52578

The outcomes of the fuzzy model's third-ranking function are shown in Table 10. With a different ranking method, it offers an additional level of analysis for the transportation problem and demonstrates how well fuzzy nonagonal membership functions work in further reducing costs under a different ranking approach.

Table 10: Special data petroleum products of fuzzy transportation when $\alpha - cat\ \frac{1}{2}$(white oil)

From \ To	Baghdad	Wasit	Karbala	Najaf	Diwaniya	Muthanna	Supply
Khor-Alzuber	— 248	6.576675 92	6.7096 128	7.52989 108	5.86089 92	1.251955 84	27.929
Shuaiba	24.540205 231	— 80	— 120	— 100	— 84	2.128405 80	26.66856
Warehouse	16.455195 0	— 0	— 0	— 0	— 0	— 0	16.455195
Demand	40.9954	6.576675	6.7096	7.52988	5.86084	3.38036	71.052755 \ 71.052755

The optimization transportation cost = 8760.535795 The total transportation cost was found to be more efficient than traditional crisp methods. Additionally, the solution was more adaptable to changes in supply and demand constraints, demonstrating the effectiveness of the fuzzy nonagonal membership approach.

5.5. Numerical Analyses

1. **Applying a fuzzy approach vs. conventional approach:** In the conventional optimization method, such as Vogel's Approximation Method (VAM), we use crisp (fixed) values for demand, supply, and transportation costs from the baseline data (e.g., Tables 1, 2, and 3). Then we solve the transportation problem using VAM to find the optimal distribution of oil resources from warehouses to governorates. As a result of such an interpretation, we obtain the total transportation cost and note it as a baseline. Whereas, when we use the fuzzy approach with nonagonal membership functions to handle uncertainties in costs, demand, and supply. We first convert crisp transportation costs into fuzzy nonagonal numbers with a range of values, such as:

   - For transportation costs: we represent costs as nonagonal fuzzy numbers, e.g., Baghdad: (58, 59, 60, . . . , 66) for Khor-Alzuber.
   - For demand and supply: we represent these as fuzzy nonagonal numbers to capture uncertainty in daily demand and supply values.

   Then the centroid method is applied to the fuzzy costs to resolve the transportation problem and calculate the total transportation cost under the fuzzy model. This value should ideally be lower than the crisp solution, reflecting cost efficiency under uncertainty.

2. **Analysis with different α-Cuts:** using different α-cut levels, we can assess the model's sensitivity to varying degrees of uncertainty. It can be implemented by choosing three α-cut levels (e.g., α = 0.2, 0.5, 0.8) and calculating the membership functions for each level. For each α-cut, defuzzify the nonagonal membership functions and solve the transportation problem using these values. It shows the change in the total transportation cost with different α-cuts. Lower α values represent higher uncertainty (wider cost ranges), while higher α values narrow the range.

3. **3. Comparison between the Nonagonal Fuzzy Model with normal Fuzzy Models:** Compared to triangular or trapezoidal fuzzy numbers, the nonagonal model shows greater precision and robustness across α-cuts.

We have chosen one of the normal fuzzy models (e.g., triangular or trapezoidal fuzzy numbers) to solve the transportation problem by applying α-cuts and defuzzification similarly, then by evaluating the results and comparing them in terms of total transportation cost and robustness (i.e., cost stability across different α-cuts with the nonagonal results using the same data. It shows precision advantages with greater accuracy in uncertain conditions. From the results, we observe that the fuzzy nonagonal model maintains cost efficiency over time. However, the detailed assessment of the fuzzy nonagonal approach demonstrates its strengths in cost efficiency, sensitivity management, and robustness over time.

6. Conclusion

This study concludes by demonstrating how well nonagonal membership functions work for Iraq's oil transportation system when applied within a fuzzy optimization framework. According to the comparative analysis, the fuzzy method offers notable cost advantages over traditional approaches. The utilization of fuzzy nonagonal membership functions reduced the transportation costs from 567.948 by the crisp method (Vogel's approximation) to a minimum total cost of 513. This decreased cost illustrates the improvement in the uncertainties (supply, demand, and cost), over simpler fuzzy methods and ensures reliable solutions if the market conditions are changed. However, the obtained results show the ability of the fuzzy method to model complex real-world problems and improve some decision-making processes. This approach satisfies a balance between flexible decision-making and optimization accuracy. Hence, it has proved to be suitable to solve dynamic transportation problems. The improvement achieved by this approach in managing uncertainties encourages future research to be applied in other applications.

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