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Binary Recurrence Arrays in Dimension d

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ABSTRACT: In this article, we consider d-dimensional arrays generated by suitable initial values and a binary recurrence rule. The principal result is the explicit formula for the general entry of a given array. This work extends the results dealing with the cases $d = 2$ and $d = 3$ in general or in some particular cases.

Keywords: Binary recurrence, multidimensional arrays, explicit formula.

Contents

1. Introduction

One way for extending the notion of linear recursions is to introduce 2-dimensional (or d-dimensional) arrays. The classical example is Pascal's triangle. Assume that n and k are non-negative integers. The k^{th} entry of row n (of the array) is $\binom{n}{k}$ with the condition $\binom{n}{k} = 0$ if $n < k$. One can easily see that

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

holds for $k \geq 1$ (and for arbitrary n). This is a first order recurrence valid in the rows of the array, and it has a bivariate rational function coefficient. For the columns we have a similar recursion

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k},$$

where $n \geq 1$ and k is arbitrary except $k = n$. Although the order of the recursions is the smallest possible, the rules are not simple because of the coefficients.

Binary recurrences as generators of a 2-dimensional array were probably used first by Berzsenyi [1]. His work was motivated partially by [9]. Assuming $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ he introduced the definition

$$F_{n+mi} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k}, \tag{1.1}$$

where F_{n-k} is the $(n-k)$-th Fibonacci number. Clearly, F_{n+mi} give values for the Gaussian integers at the integer points of the upper half-plan of the complex plane. Note that this definition comes from a problem of the so-called monodiffric functions. Berzsenyi showed that if we fix m, then

$$F_{n+mi} = F_{(n-1)+mi} + F_{(n-2)+mi} \tag{1.2}$$

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holds, which is the Fibonacci recursion for the rows of the array. Then he turned his attention for determining an explicit formula for F_{n+mi}, and found

$$F_{n+2mi} = (1+2i)^m F_{n-m},$$
$$F_{n+(2m+1)i} = (1+2i)^m (F_{n-m} + iF_{n-1-m}).$$

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ be the two zeros of the characteristic polynomial of the Fibonacci sequence. Note that $\alpha - \beta = \sqrt{5}$. By Definition 1.1, Lemma 2.1 and Corollary 2.1, we can even provide an other explicit formula for F_{n+mi}. A straightforward calculation admits

$$F_{n+mi} = \frac{1}{i\sqrt{5}} \left(i\alpha^n \left(1 + \frac{i}{\alpha}\right)^m - i\beta^n \left(1 + \frac{i}{\beta}\right)^m \right).$$

This formula shows, via Lemma 2.1 that if we fix n, then

$$F_{n+mi} = (2-i)F_{n+(m-1)i} + (2-i)F_{n+(m-2)i} \tag{1.3}$$

holds for the columns of the array. Observe that comparing (1.2) and (1.3), the coefficients of two recurrence rules do not coincide. Not long after, in his paper, Harman [8] constructed a 2-dimensional array $[G]$ using uniformly the Fibonacci recurrence for the horizontal and vertical directions with the initial values $G(0,0) = 0$, $G(1,0) = 1$, $G(0,1) = i$, and $G(1,1) = 1+i$. The explicit formula

$$G(n,m) = F_n F_{m-1} + iF_{n+1} F_m$$

was proved, and was used for justifying different identities.

Recently, some papers are related to similar arrays, where the recurrence rule associated to the two directions are the same. These papers study the 2- and 3-dimensional arrays. Kolarec [10] introduced arrays which were generated by arithmetic progressions. Since all such progressions satisfy the binary recurrence rule $x_n = 2x_{n-1} - x_{n-2}$, then her work is a specific case of the paper [2].

A bidimensional extension of the balancing and Lucas-balancing numbers was introduced in [4], along with some of its properties and summation identities. In [5], the bidimensional variation of the cobalancing and Lucas-cobalancing numbers was presented.

Later, the Binet's formula for bidimensional binary recurrence arrays was investigated [3], and examples were provided, particularly for bidimensional balancing and Lucas-balancing-like sequences.

Subsequently, we studied the tridimensional version of the four sequences above, which resulted in the production of different works. In [6], the tridimensional extension of the balancing numbers was introduced, and some of their properties were analyzed. In [7], the tridimensional recurrence relations of the Lucas-balancing numbers were introduced, and we studied some of their properties as well as their summation identities.

However, a valid bidimensional generalization was considered, defined by a second-order recurrence relation. For this purpose, an article was submitted (see [2]). Our current interest is to generalize this concept to a dimension d ($d \geq 2$), valid for any second-order recurrence relation.

This paper analyze the d-dimensional array $[T]$ built up by given initial values belonging to the d-dimensional unit cube, and by the coefficient A and B of binary recurrence relation. The principal result is Theorem 5.1 which gives an explicit formula for the entry $T(n_0, n_1, \ldots, n_{d-1})$ for the non-negative integers n_i.

2. Preliminaries

Now we turn our attention to some facts concerning binary recurrences. These details will be used later in another section. Let A and $B \neq 0$ be two arbitrary complex numbers, and consider the set of recurrent sequences

$$\Gamma_{A,B} = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^\infty \mid x_n = Ax_{n-1} + Bx_{n-2},\ n \geq 2\}.$$

Clearly, the sequences of $\Gamma_{A,B}$ satisfy a common recurrence rule, but they differ in initial values. It is known that the set $\Gamma_{A,B}$ forms a vector space over $\mathbb{C}$ for the operations addition and multiplication by scalar. One basis of this vector space consists of the two sequences (g_n) and (h_n) with the initial values $g_0 = 0$, $g_1 = 1$, and $h_0 = 1$, $h_1 = 0$.

Put $D = A^2 + 4B$. The common characteristic polynomial of all the sequences in $\Gamma_{A,B}$ is $c(x) = x^2 - Ax - B = (x - \alpha)(x - \beta)$. Here $\alpha = (A + \sqrt{D})/2$ and $\beta = (A - \sqrt{D})/2$ are not necessarily distinct, but non-zero complex numbers. Distinguish two cases in order to get explicit formula for G_n as for a general term of an arbitrary sequence from $\Gamma_{A,B}$.

Lemma 2.1. *If $D \neq 0$, then $\alpha \neq \beta$ and*

$$G_n = \frac{(G_1 - \beta G_0)\alpha^n - (G_1 - \alpha G_0)\beta^n}{\alpha - \beta}.$$

For the case $D = 0$ we have $\alpha = \beta$ and

$$G_n = G_1 n\alpha^{n-1} - G_0(n-1)\alpha^n.$$

These results are well known in the theory of linear recurrences. One can easily see that $\alpha + \beta = A$, $\alpha\beta = -B$, and $\alpha - \beta = \sqrt{D}$. Lemma 2.1 implies the following specific forms immediately.

Corollary 2.1. *If $D \neq 0$, then*

$$g_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad h_n = \frac{-\beta\alpha^n + \alpha\beta^n}{\alpha - \beta} = B\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = Bg_{n-1},$$

while

$$g_n = n\alpha^{n-1}, \qquad h_n = -(n-1)\alpha^n = B(n-1)\alpha^{n-2} = Bg_{n-1}$$

hold for $D = 0$.

Using the vector space property of $\Gamma_{A,B}$ it follows right away that

$$(G_n) = G_1 \cdot (g_n) + G_0 \cdot (h_n), \qquad (H_n) = H_1 \cdot (g_n) + H_0 \cdot (h_n), \tag{2.1}$$

i.e. the coordinates of the sequences (G_n), and (H_n) with respect to the basis $((g_n), (h_n))$ are (G_1, G_0), and (H_1, H_0), respectively.

It is important to clarify that if $d = 2$ we have two initial sequences to generate the whole 2-dimensional array. If they are from $\Gamma_{A,B}$, and they are linearly independent, then they form a basis, and the corresponding result of Theorem 5.1 can be converted via the inverse of (2.1) to use the two basic sequences in expressing the general entry of the array. That happened in the paper [2].

When $d \geq 3$, then we have $2^{d-1} = 4, 8, 16, \ldots$ initial sequences, and it is natural to keep (g_n) and (h_n) as the two basis sequences. This is the reason why Theorem 5.1 uses them. Of course, if there is a specific request, one can replace (g_n) and (h_n) by two linearly independent sequences. For example, if $A = 6$, and $B = -1$ then we can use the sequence of balancing numbers $(g_n) = (B_n)$, and its shifted version $(h_n) = (-B_{n-1})$ (see Example 5.2).

3. Infinite Arrays in Dimension d

Suppose that d is a positive integer. We define a map $T : \mathbb{N}^d \to \mathbb{C}$ as follows. If we glue the values of T to the location of the corresponding points in $\mathbb{N}^d$, the geometrical arrangement can be considered as an infinite array. This array is denoted by $[T]$. The coordinate axes of the d-dimensional Cartesian system are assigned by $x_0, x_1, \ldots, x_{d-1}$.

First we fix the values belonging to the vertices of the d-dimensional unit cube, i.e. we fix $T(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1})$ as arbitrary complex numbers for all $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}) \in \{0,1\}^d$. We even use another notation for the values of the vertices of the unit cube because we consider them in pairs as initial values of some carefully chosen binary recurrences from $\Gamma_{A,B}$. In the virtue of the previous argument put

$$G^{(\epsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1})} = T(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}) \qquad \text{for} \qquad (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}) \in \{0,1\}^d.$$

At the first sight, of course, it seems to be an unnecessary duplicate. For fixed $\varepsilon_1, \ldots, \varepsilon_{d-1}$ the points $G^{(0,\varepsilon_1,\ldots,\varepsilon_{d-1})}$ and $G^{(1,\varepsilon_1,\ldots,\varepsilon_{d-1})}$ are said to be *neighbour* points.

In the next step, we define 2^{d-1} binary recurrences of $\Gamma_{A,B}$. Clearly, it sufficies to give their inital values. We denote the sequences by

$$G^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})}, \qquad (\varepsilon_1, \ldots, \varepsilon_{d-1}) \in \{0,1\}^{d-1}$$

such that the initial values are the neighbour points

$$G_0^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})} = G^{(0,\varepsilon_1,\ldots,\varepsilon_{d-1})} \qquad \text{and} \qquad G_1^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})} = G^{(1,\varepsilon_1,\ldots,\varepsilon_{d-1})}.$$

Clearly, the general term for all sequences above is

$$G_{n_0}^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})} = AG_{n_0-1}^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})} + BG_{n_0-2}^{(\star,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{d-1})}$$

for any $n_0 \geq 2$. Using the values of these binary recurrences we can obtain the values $T(n_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{d-1})$ of the function T (and array $[T]$) for any $n_0 \geq 0$.

We say that the sequences $G^{(\star,0,\varepsilon_2,\ldots,\varepsilon_{d-1})}$ and $G^{(\star,1,\varepsilon_2,\ldots,\varepsilon_{d-1})}$ are *neighbours.* Then two neighbour sequences $G^{(\star,0,\varepsilon_2,\ldots,\varepsilon_{d-1})}$ and $G^{(\star,1,\varepsilon_2,\ldots,\varepsilon_{d-1})}$ constitute the plain $G^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})}$ in $[T]$. It means that

$$G_{n_0,0}^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})} = G_{n_0}^{(\star,0,\varepsilon_2\ldots,\varepsilon_{d-1})} \qquad \text{and} \qquad G_{n_0,1}^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})} = G_{n_0}^{(\star,1,\varepsilon_2\ldots,\varepsilon_{d-1})},$$

and using them we have

$$G_{n_0,n_1}^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})} = AG_{n_0,n_1-1}^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})} + BG_{n_0,n_1-2}^{(\star,\star,\varepsilon_2,\ldots,\varepsilon_{d-1})}$$

for any $n_1 \geq 2$ (and $n_0 \geq 0$).

We say that the plains $G^{(\star,\star,0,\varepsilon_3,\ldots,\varepsilon_{d-1})}$ and $G^{(\star,\star,1,\varepsilon_3,\ldots,\varepsilon_{d-1})}$ are *neighbours.* Continuing the procedure we can define the values $G_{n_0,n_1,n_2}^{(\star,\star,\star,\varepsilon_3,\ldots,\varepsilon_{d-1})}$ of the hyperplain $G^{(\star,\star,\star,\varepsilon_3,\ldots,\varepsilon_{d-1})}$ (i.e. hyperplane with non-negative integer coordinates), with the aid of the corresponding recurrences of $\Gamma_{A,B}$, and so on. Via the u-dimensional hyperplains

$$G^{(\overbrace{\star,\ldots,\star}^{u},\varepsilon_u,\ldots,\varepsilon_{d-1})}$$

(notation $\overbrace{\star,\ldots,\star}^{u}$ above and later means that the number of stars ($\star$) appearing in the list is exactly u) and suitable binary recurrences from $\Gamma_{A,B}$ finally we populate the whole d-dimensional space $G^{(\star,\ldots,\star)}$ with values

$$T(n_0, n_1, \ldots, n_{d-1}) = G_{n_0,n_1,\ldots,n_{d-1}}^{(\overbrace{\star,\ldots,\star}^{d})}.$$

That is, for any $(n_0, n_1, \ldots, n_{d-1}) \in \mathbb{N}^d$ we have determined the array $[T]$ when we computed $T(n_0, n_1, \ldots, n_{d-1})$.

Observe that the previous derivation of $T(n_0, n_1, \ldots, n_{d-1})$ depends seemingly on the order $x_0, x_1, \ldots, x_{d-1}$ of the coordinate axes. We will show that if we use any permutation of the coordinate axes (and vary the definition of neighbourhood) in the procedure, there is no change in the value $T(n_0, n_1, \ldots, n_{d-1})$. This is an important observation since the main goal of the present paper is to determine the arbitrary element $T(n_0, n_1, \ldots, n_{d-1})$ of the d-dimensional array explicitly. We note in advance that we will use the basis sequences (g_n) and (h_n) in the solution.

4. Independence

Theorem 4.1. *The value $T(n_0, n_1, \ldots, n_{d-1})$ is invariant for the order of use of coordinate axes $x_0, x_1, \ldots, x_{d-1}$ in constructing the d-dimensional array $[T]$.*

Proof. It is sufficient to show that $T(2,2,\ldots,2)$ is independent of the way we reach the point $(2,2,\ldots,2)$. For example, if $d=2$, then we have 2 ways to construct $T(2,2)$, namely

$$T(2,2) = AT(2,1) + BT(2,0), \quad \text{from} \quad \begin{cases} T(2,0) = AT(1,0) + BT(0,0), \\ T(2,1) = AT(1,1) + BT(0,1), \end{cases}$$

and

$$T(2,2) = AT(1,2) + BT(0,2), \quad \text{from} \quad \begin{cases} T(0,2) = AT(0,1) + BT(0,0), \\ T(1,2) = AT(1,1) + BT(1,0). \end{cases}$$

We can say briefly, that in the first case we used the order x_0, x_1 of the coordinate axes, while in the second case the order was x_1, x_0. Clearly, the values $T(0,0)$, $T(0,1)$, $T(1,0)$, and $T(1,1)$ of the 2-dimensional unit cube (in fact, this is the unit square in 2-dimension) are given in advance.

Going back to dimension d, assume that $x_{\pi_0}, x_{\pi_1}, \ldots, x_{\pi_{d-1}}$ is an arbitrary permutation of the coordinate axes. Taking the first one,

$$\begin{multline} T(\varepsilon_0,\ldots,\varepsilon_{\pi_0-1},2,\varepsilon_{\pi_0+1},\ldots,\varepsilon_{d-1}) = \\ AT(\varepsilon_0,\ldots,\varepsilon_{\pi_0-1},1,\varepsilon_{\pi_0+1},\ldots,\varepsilon_{d-1}) + BT(\varepsilon_0,\ldots,\varepsilon_{\pi_0-1},0,\varepsilon_{\pi_0+1},\ldots,\varepsilon_{d-1}) \tag{4.1} \end{multline}$$

holds for arbitrary $\varepsilon_i \in \{0,1\}$ values. In the next step we find

$$\begin{multline} T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},2,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},2,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1}) = \\ AT(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},1,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},2,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1})+ \\ BT(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},0,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},2,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1}). \tag{4.2} \end{multline}$$

Now apply (4.1) for the two summands in (4.2). It implies that

$$\begin{multline} T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},2,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},2,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1}) = \\ A^2T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},1,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},1,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1})+ \\ AB(T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},0,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},1,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1})+ \\ T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},1,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},0,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1}))+ \\ B^2T(\varepsilon_0,\ldots,\varepsilon_{\pi_1-1},0,\varepsilon_{\pi_1+1},\ldots,\varepsilon_{\pi_0-1},0,\varepsilon_{\pi_0+1}\ldots\varepsilon_{d-1}). \end{multline}$$

Proceedings the method by going along the given permutation of the coordinate axes, finally we find that

$$\begin{aligned} T(2,2,\ldots,2) = {} & A^d\, T(1,1,\ldots,1) \\ & + A^{d-1}B\big(T(0,1,\ldots,1) + T(1,0,1,\ldots,1) + \cdots + T(1,\ldots,1,0)\big) \\ & \vdots \\ & + A^{d-j}B^j \sum_{\ell=1}^{\binom{d}{j}} T\Bigg(\overbrace{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{d-1}}^{0:\ j \text{ pieces}}\Bigg) \\ & \vdots \\ & + B^d\, T(0,0,\ldots,0). \end{aligned} \tag{4.3}$$

and this is the end of the proof since the formula for $T(2,2,\ldots,2)$ does not depend on the permutation $\pi_0, \pi_1, \ldots, \pi_{d-1}$. □

5. Explicit Formula for $T(n_0, n_1, \ldots, n_{d-1})$

Theorem 5.1. *We have*

$$T(n_0, n_1, \ldots, n_{d-1}) = \sum_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}} G^{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1})} \prod_{j=0}^{d-1} x_{n_j},$$

where

$$x_{n_j} = \begin{cases} g_{n_j}, & \text{if } \varepsilon_j = 1; \\ h_{n_j}, & \text{if } \varepsilon_j = 0. \end{cases}$$

Proof. We prove the statement by induction on d.

Let us first consider $d = 1$. Then we have only one recurrence $G^\star$ with initial values $G_0^\star = G^{(0)}$ and $G_1^\star = G^{(1)}$. Applying Corollary 2.1 for the statement of the theorem, we have

$$\begin{aligned} T(n_0) &= \sum_{\varepsilon_0} G^{(\varepsilon_0)} x_{n_0} = G^{(0)} h_{n_0} + G^{(1)} g_{n_0} = G^{(0)} B g_{n_0 - 1} + G^{(1)} g_{n_0} \\ &= \begin{cases} G^{(0)} B \frac{\alpha^{n_0-1} - \beta^{n_0-1}}{\alpha - \beta} + G^{(1)} \frac{\alpha^{n_0} - \beta^{n_0}}{\alpha - \beta} = \frac{(G^{(1)} - \beta G^{(0)})\alpha^{n_0} - (G^{(1)} - \alpha G^{(0)})\beta^{n_0}}{\alpha - \beta} & \text{if } D \neq 0, \\ G^{(0)} B (n_0 - 1)\alpha^{n_0-2} + G^{(1)} n_0 \alpha^{n_0-1} = G^{(1)} n_0 \alpha^{n_0-1} - G^{(0)}(n_0 - 1)\alpha^{n_0} & \text{if } D = 0. \end{cases} \end{aligned}$$

In each case, the result coincides with the corresponding argument of Lemma 2.1 if the initial values are $G^{(0)}$ and $G^{(1)}$. Note that we used $B = -\alpha\beta$ when $D \neq 0$, and $B = -\alpha^2$ otherwise.

Now assume $d = 2$. According to the theorem, we must show that

$$\begin{aligned} T(n_0, n_1) &= \sum_{\varepsilon_0, \varepsilon_1} G^{(\varepsilon_0, \varepsilon_1)} \prod_{j=0}^{1} x_{n_j} \\ &= G^{(0,0)} h_{n_0} h_{n_1} + G^{(0,1)} h_{n_0} g_{n_1} + G^{(1,0)} g_{n_0} h_{n_1} + G^{(1,1)} g_{n_0} g_{n_1}. \end{aligned}$$

But this is precisely equation (6) in [2] with the notation correspondences $G^{(0,0)} = G_0$, $G^{(0,1)} = G_1$, $G^{(1,0)} = H_0$, $G^{(1,1)} = H_1$, $n_0 = k$, and $n_1 = n$.

Suppose now that the statement of the theorem is true in dimension d, and we must prove the statement for dimension $d+1$. Since the notation of the theorem is fitted to dimension d, we transform it to dimension $d+1$. This is only to introduce $\varepsilon_d \in \{0, 1\}$, and write (for any of the two ε_d)

$$T(n_0, n_1, \ldots, n_{d-1}, \varepsilon_d) = G_{n_0, n_1, \ldots, n_{d-1}}^{(\overbrace{\star, \ldots, \star}^{d}, \varepsilon_d)} = \sum_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}} G^{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_d)} \prod_{j=0}^{d-1} x_{n_j},$$

which is the current form of the theorem describing the d-dimensional case in dimension $d+1$. We apply the recurrence rule to finish the proof. We have, via (2.1), the equalities

$$\begin{aligned} T(n_0, n_1, \ldots, n_{d-1}, n_d) &= G_{n_0, n_1, \ldots, n_{d-1}, n_d}^{(\overbrace{\star, \ldots, \star}^{d+1})} \\ &= G_{n_0, n_1, \ldots, n_{d-1}}^{(\overbrace{\star, \ldots, \star}^{d}, 0)} h_{n_d} + G_{n_0, n_1, \ldots, n_{d-1}}^{(\overbrace{\star, \ldots, \star}^{d}, 1)} g_{n_d} \\ &= \sum_{\varepsilon_d} G_{n_0, n_1, \ldots, n_{d-1}}^{(\overbrace{\star, \ldots, \star}^{d}, \varepsilon_d)} x_{n_d} \\ &= \sum_{\varepsilon_d} \left(\sum_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}} G^{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1})} \prod_{j=0}^{d-1} x_{n_j} \right) x_{n_d} \\ &= \sum_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_d} G^{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_d)} \prod_{j=0}^{d} x_{n_j}. \end{aligned}$$

And the proof is complete. □

Example 5.1. For $d = 3$. In this case, Theorem 5.1 yields

$$\begin{aligned} T(n_0, n_1, n_2) = {} & G^{(0,0,0)} h_{n_0} h_{n_1} h_{n_2} \\ & + G^{(1,0,0)} g_{n_0} h_{n_1} h_{n_2} + G^{(0,1,0)} h_{n_0} g_{n_1} h_{n_2} + G^{(0,0,1)} h_{n_0} h_{n_1} g_{n_2} \\ & + G^{(1,1,0)} g_{n_0} g_{n_1} h_{n_2} + G^{(1,0,1)} g_{n_0} h_{n_1} g_{n_2} + G^{(0,1,1)} h_{n_0} g_{n_1} g_{n_2} \\ & + G^{(1,1,1)} g_{n_0} g_{n_1} g_{n_2}. \end{aligned}$$

Example 5.2. Look at the previous example, and let $g_n = B_n$, and $h_n = B_{n-1}$.
Now, we have

$$\begin{aligned} T(n_0, n-1, n_2) = {} & G^{(0,0,0)} (-B_{n_0-1})(-B_{n_1-1})(-B_{n_2-1}) \\ & + G^{(0,0,1)} (-B_{n_0-1})(-B_{n_1-1}) B_{n_2} + \cdots + G^{(1,1,1)} B_{n_0} B_{n_1} B_{n_2}. \end{aligned}$$

Example 5.3. Since $g_2 = A$, $h_2 = B$, therefore Theorem 5.1 provides

$$\begin{aligned} T(2, 2, \ldots, 2) &= \sum_{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}} G^{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1})} \prod_{j=0}^{d-1} x_2 \\ &= \sum_{j=0}^{d} A^{d-j} B^j \sum_{1}^{\binom{d}{j}} T(\overbrace{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}}^{0:\, j \text{ pieces}}), \end{aligned}$$

which coincides with (4.3).

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