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On Certain Closeness Spectra of One-Point Union of Complete Graphs K_n *

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ABSTRACT: The closeness eigenvalues of a graph G are the eigenvalues of its closeness matrix. The closeness spectrum of a graph G is the set of closeness eigenvalues with their multiplicities. In this paper the closeness spectra, closeness Laplacian spectra and closeness Q-Laplacian spectra of one-point union of m copies of complete graphs on n vertices are computed.

Keywords: Closeness matrix, closeness eigenvalues, closeness Laplacian matrix, closeness Q-Laplacian matrix.

Contents

1. Introduction

The graphs considered in this article are simple, connected and undirected. Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. There are multiple ways in which a matrix can represent a graph. The most studied matrix associated with a graph G is the adjacency matrix $A = (a_{ij})$ whose entries are defined as follows

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of G is the characteristic polynomial of the adjacency matrix A and is denoted by $p_G(\lambda)$. The eigenvalues of G are the zeros of the characteristic polynomial and the spectrum of G is the multiset of eigenvalues of G. The information that connects a graph with the spectrum of the matrix is provided by spectral graph theory [1,2,3,4,9]. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest $u - v$ path in G. The closeness matrix [6] of order $p \times p$ associated with G is given by $C(G) = (c_{uv})$ whose (u, v)-th entry is

$$c_{uv} = \begin{cases} 2^{-d(u,v)} & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

The closeness spectrum of G is the multiset of eigenvalues of closeness matrix of G denoted by $CSpec(G)$. We write

$$CSpec(G) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_p \\ m_1 & m_2 & m_3 & \cdots & m_p \end{pmatrix}$$

where $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_p$ are the eigenvalues and m_i is the multiplicity of $\alpha_i, 1 \leq i \leq p$.

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Let $D_C(G)$ be the diagonal matrix with (u,u) entry to be

$$c_G(u) = \sum_{w \in V \setminus \{u\}} 2^{-d(u,w)}$$

for each $u \in V(G)$. Motivated by [7], the closeness Laplacian is defined as the matrix $L_C(G) = D_C(G) - C(G)$, and the closeness Q-Laplacian is defined as the matrix $Q_C(G) = D_C(G) + C(G)$. That is, for $u, v \in V(G)$, the (u,v)-entry of

$$L_C(G) = \begin{cases} -2^{-d_G(u,v)} & \text{if } u \neq v, \\ c_G(u) & \text{otherwise.} \end{cases}$$

and the (u,v)-entry of

$$Q_C(G) = \begin{cases} 2^{-d_G(u,v)} & \text{if } u \neq v, \\ c_G(u) & \text{otherwise.} \end{cases}$$

A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph. The graph $G^{(m)}$ obtained by identifying the roots of m copies of G is called the one-point union of m copies of the graph G [5,8].

The one-point union of graphs is a natural graph operation that combines multiple graph structures through a common vertex. In particular, the one-point union of complete graphs yields a structured yet non-trivial class of graphs, where the high symmetry within each component and the connectivity through a single vertex make them suitable for spectral analysis. Studying the closeness spectra of such graphs helps in understanding how distance-based interactions behave under graph operations and provides insights into the effect of structural composition on spectral properties.

In this paper the closeness spectra, closeness Laplacian spectra and closeness Q-Laplacian spectra of one-point union of m copies of complete graphs on n vertices are computed.

2. Results and discussion

2.1. Closeness spectrum of $K_n^{(m)}$, $m \geq 2, n \geq 2$

Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. The closeness matrix A of the complete graph K_{n-1} is a square matrix of order $n-1$ given by

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{pmatrix}$$

Hence the closeness matrix A_G of G is given by

$$A_G = \begin{pmatrix} 0 & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & A & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & A \end{pmatrix}$$

Clearly A_G is a square matrix of order $((n-1)m+1)$ with leading term 0; $\frac{1}{2}R$ is a row vector of $n-1$ number of $\frac{1}{2}'s$; $\frac{1}{2}C$ is the transpose of $\frac{1}{2}R$. The matrix A on the leading diagonal is the closeness matrix of K_{n-1} and $\mathbf{\frac{1}{4}}$ is a square matrix of order $n-1$ whose entries are $\frac{1}{4}$.

Theorem 2.1 *Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. Then*

$$CSpec(G) = \begin{pmatrix} -\frac{1}{2} & \frac{n-3}{4} & \alpha & \beta \\ (n-2)m & (m-1) & 1 & 1 \end{pmatrix}$$

where

$$\alpha = \frac{1}{8}\left(m(n-1) + (n-3) + \sqrt{(m(n-1))^2 + (n-3)^2 + 2m(n^2+4n-5)}\right)$$

$$\beta = \frac{1}{8}\left(m(n-1) + (n-3) - \sqrt{(m(n-1))^2 + (n-3)^2 + 2m(n^2+4n-5)}\right)$$

Proof: The characteristic polynomial of G is

$$|A_G - \lambda I| = \begin{vmatrix} -\lambda & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A-\lambda I & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & A-\lambda I & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & A-\lambda I \end{vmatrix}$$

where $\frac{1}{2}R$, $\frac{1}{2}C$, A and $\mathbf{\frac{1}{4}}$ are defined earlier. We name the rows and columns of $A_G-\lambda I$ as R_1, R_i^t, C_1, C_i^t for $1 \leq t \leq m$ and $1 \leq i \leq n-1$ and perform certain row and column operations.

Step 1: For every t, $1 \leq t \leq m$, replace R_i^t by $R_i^t - R_1^t, 2 \leq i \leq n-1$

$$|A_G - \lambda I| = \left(\tfrac{1}{2}+\lambda\right)^{(n-2)m} \begin{vmatrix} -\lambda & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ H_1 & A_1 & B_1 & \cdots & B_1 \\ H_1 & B_1 & A_1 & \cdots & B_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_1 & B_1 & B_1 & \cdots & A_1 \end{vmatrix}$$

where

$$H_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -\lambda & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Step 2: For every t, $1 \leq t \leq m$, replace C_1^t by $\sum_{i=1}^{n-1} C_i^t$

$$|A_G - \lambda I| = \left(\tfrac{1}{2}+\lambda\right)^{(n-2)m} \begin{vmatrix} -\lambda & J_1 & J_1 & \cdots & J_1 \\ H_1 & A_2 & B_2 & \cdots & B_2 \\ H_1 & B_2 & A_2 & \cdots & B_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_1 & B_2 & B_2 & \cdots & A_2 \end{vmatrix}$$

where

$$J_1 = \begin{pmatrix} \frac{n-1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{n-2}{2} - \lambda & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} \frac{n-1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Step 3: Replace R_t^1 by R_1^t, $2 \leq t \leq n-1$ and C_t^1 by C_1^t, $2 \leq t \leq n-1$

$$|A_G - \lambda I| = \left(\tfrac{1}{2} + \lambda\right)^{(n-2)m} \begin{vmatrix} -\lambda & J_2 & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A_3 & S_t & \cdots & S_t \\ \mathbf{0} & \mathbf{0} & -I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

where

$$J_2 = \begin{pmatrix} \frac{n-1}{2} & \frac{n-1}{2} & \frac{n-1}{2} & \cdots & \frac{n-1}{2} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} \frac{n-2}{2} - \lambda & \frac{n-1}{4} & \frac{n-1}{4} & \cdots & \frac{n-1}{4} \\ \frac{n-1}{4} & \frac{n-2}{2} - \lambda & \frac{n-1}{4} & \cdots & \frac{n-1}{4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n-1}{4} & \frac{n-1}{4} & \frac{n-1}{4} & \cdots & \frac{n-2}{2} - \lambda \end{pmatrix}$$

and, for every t, $2 \leq t \leq n-1$, $S_t = (s_{ij})$ is a square matrix of size $(n-1) \times (n-1)$ with entries $s_{11} = s_{t2} = s_{t3} = ... = s_{t(n-1)} = \frac{1}{2}$ and all other remaining entries are $\frac{1}{4}$.

Step 4: For each i, $2 \leq i \leq n-1$ replace R_i^1 by $R_i^1 - R_1^1$

$$|A_G - \lambda I| = \left(\tfrac{1}{2} + \lambda\right)^{(n-2)m} \begin{vmatrix} -\lambda & J_2 & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ H_1 & A_4 & S_t' & \cdots & S_t' \\ \mathbf{0} & \mathbf{0} & -I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

$$A_4 = \begin{pmatrix} \frac{n-2}{2} - \lambda & \frac{n-1}{4} & \frac{n-1}{4} & \cdots & \frac{n-1}{4} \\ \frac{3-n}{4} + \lambda & -\left(\frac{3-n}{4} + \lambda\right) & 0 & \cdots & 0 \\ \frac{3-n}{4} + \lambda & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{3-n}{4} + \lambda & 0 & 0 & \cdots & -\left(\frac{3-n}{4} + \lambda\right) \end{pmatrix}$$

and, for every t, $2 \leq t \leq n-1$, $S_t{}' = (s_{ij})$ is a square matrix of size $(n-1) \times (n-1)$ with leading term $\frac{1}{2}$; $s_{12} = s_{13} = ... = s_{1(n-1)} = s_{t2} = s_{t3} = ... = s_{t(n-1)} = \frac{1}{4}$; $s_{21} = s_{31} = ... = s_{(n-1)1} = -\frac{1}{4}$ and the remaining entries are 0.

Step 5: Replace C_1^1 by $\sum_{i=1}^{n-1} C_i^1$

$$|A_G - \lambda I| = \left(\frac{1}{2}+\lambda\right)^{(n-2)m}\left(\frac{3-n}{4}+\lambda\right)^{(m-1)} \begin{vmatrix} -\lambda & J_3 & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ H_1 & A_5 & S_t & S_t & \cdots & S_t \\ \mathbf{0} & \mathbf{0} & -I & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

where

$$J_3 = \begin{pmatrix} \frac{m(n-1)}{2} & \frac{n-1}{2} & \frac{n-1}{2} & \frac{n-1}{2} & \cdots & \frac{n-1}{2} \end{pmatrix}$$

$$A_5 = \begin{pmatrix} \frac{n-2}{2}+\frac{(m-1)(n-1)}{4}-\lambda & \frac{n-1}{4} & \frac{n-1}{4} & \cdots & \frac{n-1}{4} \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Hence

$$|A_G - \lambda I| = \left(\frac{1}{2}+\lambda\right)^{(n-2)m}\left(\frac{3-n}{4}+\lambda\right)^{(m-1)} \begin{vmatrix} -\lambda & \frac{m(n-1)}{2} \\ \frac{1}{2} & \frac{n-2}{2}+\frac{(m-1)(n-1)}{4}-\lambda \end{vmatrix}$$

$$CSpec(G) = \begin{pmatrix} -\frac{1}{2} & \frac{n-3}{4} & \alpha & \beta \\ (n-2)m & (m-1) & 1 & 1 \end{pmatrix}$$

where

$\alpha = \frac{1}{8}\left(m(n-1)+(n-3)+\sqrt{(m(n-1))^2+(n-3)^2+2m(n^2+4n-5)}\right)$

$\beta = \frac{1}{8}\left(m(n-1)+(n-3)-\sqrt{(m(n-1))^2+(n-3)^2+2m(n^2+4n-5)}\right)$ □

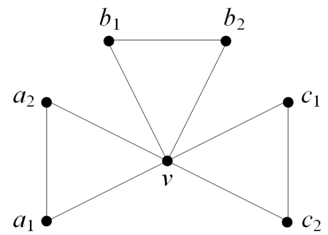

Figure 1: The graph $K_3^{(3)}$

We illustrate Theorem 2.1 with the following example.

Example 2.1 *Consider the graph $G = K_3^{(3)}$ with vertex set $V(G) = \{v, a_1, a_2, b_1, b_2, c_1, c_2\}$ as shown in Figure 1.*

The closeness matrix of G is

$$C(G) = \begin{pmatrix} 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} \\ 2^{-1} & 0 & 2^{-1} & 2^{-2} & 2^{-2} & 2^{-2} & 2^{-2} \\ 2^{-1} & 2^{-1} & 0 & 2^{-2} & 2^{-2} & 2^{-2} & 2^{-2} \\ 2^{-1} & 2^{-2} & 2^{-2} & 0 & 2^{-1} & 2^{-2} & 2^{-2} \\ 2^{-1} & 2^{-2} & 2^{-2} & 2^{-1} & 0 & 2^{-2} & 2^{-2} \\ 2^{-1} & 2^{-2} & 2^{-2} & 2^{-2} & 2^{-2} & 0 & 2^{-1} \\ 2^{-1} & 2^{-2} & 2^{-2} & 2^{-2} & 2^{-2} & 2^{-1} & 0 \end{pmatrix}.$$

The closeness spectrum of G is

$$CSpec(G) = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{3+\sqrt{33}}{4} & \frac{3-\sqrt{33}}{4} \\ 3 & 2 & 1 & 1 \end{pmatrix}$$

2.2. Closeness Laplacian spectrum of $K_n^{(m)}$, $m \geq 2, n \geq 2$

Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. The closeness matrix A_G of G is given by

$$A_G = \begin{pmatrix} 0 & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & A & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & A \end{pmatrix}$$

where A is the closeness matrix of the complete graph K_{n-1}. The diagonal matrix $D_C(G)$ is given by

$$D_C(G) = \begin{pmatrix} \frac{m(n-1)}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{(m+1)(n-1)}{4} & 0 & \cdots & 0 \\ 0 & 0 & \frac{(m+1)(n-1)}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(m+1)(n-1)}{4} \end{pmatrix}$$

The closeness Laplacian matrix $L_C(G) = D_C(G)$ - A_G

$$= \begin{pmatrix} \frac{m(n-1)}{2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_1 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & A & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & A \end{pmatrix}$$

where

$$D_1 = \begin{pmatrix} \frac{(m+1)(n-1)}{4} & 0 & 0 & \cdots & 0 \\ 0 & \frac{(m+1)(n-1)}{4} & 0 & \cdots & 0 \\ 0 & 0 & \frac{(m+1)(n-1)}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(m+1)(n-1)}{4} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{pmatrix}$$

$$L_C(G) = \begin{pmatrix} \frac{m(n-1)}{2} & -\frac{1}{2}R & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ -\frac{1}{2}C & D_1 - A & -\mathbf{\frac{1}{4}} & \cdots & -\mathbf{\frac{1}{4}} \\ -\frac{1}{2}C & -\mathbf{\frac{1}{4}} & D_1 - A & \cdots & -\mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}C & -\mathbf{\frac{1}{4}} & -\mathbf{\frac{1}{4}} & \cdots & D_1 - A \end{pmatrix}$$

Theorem 2.2 *Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. Then*

$$CLSpec(G) = \begin{pmatrix} \frac{(m+1)(n-1)+2}{4} & \frac{m(n-1)+2}{4} & \frac{m(n-1)+1}{2} & 0 \\ m(n-2) & (m-1) & 1 & 1 \end{pmatrix}$$

Proof: The characteristic polynomial of G is

$$|L_C(G) - \lambda I| = \begin{vmatrix} \frac{m(n-1)}{2} - \lambda & -\frac{1}{2}R & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ -\frac{1}{2}C & D_2 - \lambda I & -\mathbf{\frac{1}{4}} & \cdots & -\mathbf{\frac{1}{4}} \\ -\frac{1}{2}C & -\mathbf{\frac{1}{4}} & D_2 - \lambda I & \cdots & -\mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}C & -\mathbf{\frac{1}{4}} & -\mathbf{\frac{1}{4}} & \cdots & D_2 - \lambda I \end{vmatrix}$$

where

$$D_2 = \begin{pmatrix} \frac{(m+1)(n-1)}{4} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & \frac{(m+1)(n-1)}{4} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{(m+1)(n-1)}{4} & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & \frac{(m+1)(n-1)}{4} \end{pmatrix}$$

Step 1: For every t, $1 \leq t \leq m$ replace ${R_i}^t$ by ${R_i}^t - {R_1}^t$, $2 \leq i \leq n-1$

$$|L_C(G) - \lambda I| = \left(-\frac{(m+1)(n-1)+2}{4} + \lambda \right)^{(n-2)m} \begin{vmatrix} \frac{m(n-1)}{2} - \lambda & -\frac{1}{2}R & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ H_1 & D_3 & B_1 & \cdots & B_1 \\ H_1 & B_1 & D_3 & \cdots & B_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_1 & B_1 & B_1 & \cdots & D_3 \end{vmatrix}$$

where

$$H_1 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} \frac{(m+1)(n-1)}{4} - \lambda & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \cdots & -\frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Step 2: For every t, $1 \leq t \leq m$, replace C_1^t by $\sum_{i=1}^{n-1} C_i^t$

$$|L_C(G) - \lambda I| = \left(-\frac{(m+1)(n-1)+2}{4} + \lambda \right)^{(n-2)m} \begin{vmatrix} \frac{m(n-1)}{2} - \lambda & J_1 & J_1 & \cdots & J_1 \\ H_1 & D_4 & B_2 & \cdots & B_2 \\ H_1 & B_2 & D_4 & \cdots & B_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_1 & B_2 & B_2 & \cdots & D_4 \end{vmatrix}$$

where

$$J_1 = \begin{pmatrix} -\frac{n-1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \end{pmatrix}$$

$$D_4 = \begin{pmatrix} \frac{m(n-1)-(n-3)}{4} - \lambda & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -\frac{n-1}{4} & -\frac{1}{4} & -\frac{1}{4} & \cdots & -\frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Step 3: Replace R_t^1 by R_1^t, $2 \leq t \leq n-1$ and C_t^1 by C_1^t, $2 \leq t \leq n-1$

$$|L_C(G) - \lambda I| = \left(-\frac{(m+1)(n-1)+2}{4} + \lambda \right)^{(n-2)m} \begin{vmatrix} \frac{m(n-1)}{2} - \lambda & J_2 & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ -\frac{1}{2}C & D_5 & S_t & \cdots & S_t \\ \mathbf{0} & \mathbf{0} & -I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

where

$$J_2 = \begin{pmatrix} -\frac{n-1}{2} & -\frac{n-1}{2} & \cdots & -\frac{n-1}{2} \end{pmatrix}$$

$$D_5 = \begin{pmatrix} \frac{m(n-1)-(n-3)}{4} - \lambda & -\frac{n-1}{4} & -\frac{n-1}{4} & \cdots & -\frac{n-1}{4} \\ -\frac{n-1}{4} & \frac{m(n-1)-(n-3)}{4} - \lambda & -\frac{n-1}{4} & \cdots & -\frac{n-1}{4} \\ -\frac{n-1}{4} & -\frac{n-1}{4} & \frac{m(n-1)-(n-3)}{4} - \lambda & \cdots & -\frac{n-1}{4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{n-1}{4} & -\frac{n-1}{4} & -\frac{n-1}{4} & \cdots & \frac{m(n-1)-(n-3)}{4} - \lambda \end{pmatrix}$$

and, for every t, $2 \leq t \leq n-1$, $S_t = (s_{ij})$, is a square matrix of size $(n-1) \times (n-1)$ with leading term $-\frac{1}{2}$; $s_{t2} = s_{t3} = s_{t4} ... = s_{t(n-1)} = -\frac{1}{2}$ and all other remaining entries are $-\frac{1}{4}$.

Step 4: For each i, $2 \leq i \leq n-1$ replace R_i^1 by $R_i^1 - R_1^1$

$$|L_C(G) - \lambda I| = \left(-\frac{(m+1)(n-1)+2}{4} + \lambda \right)^{(n-2)m} \begin{vmatrix} \frac{m(n-1)}{2} - \lambda & J_2 & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ H_1 & D_6 & S_t' & \cdots & S_t' \\ \mathbf{0} & \mathbf{0} & -I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

$$D_6 = \begin{pmatrix} \frac{m(n-1)-(n-3)}{4} - \lambda & -\frac{n-1}{4} & -\frac{n-1}{4} & \cdots & -\frac{n-1}{4} \\ -\frac{m(n-1)+2}{4} + \lambda & \frac{m(n-1)+2}{4} - \lambda & 0 & \cdots & 0 \\ -\frac{m(n-1)+2}{4} + \lambda & 0 & -\frac{m(n-1)+2}{4} + \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{m(n-1)+2}{4} + \lambda & 0 & 0 & \cdots & \frac{m(n-1)+2}{4} - \lambda \end{pmatrix}$$

and, for every t, $2 \leq t \leq n-1$, $S_t' = (s_{ij})$, is a square matrix of size $(n-1) \times (n-1)$ with leading term $-\frac{1}{2}$; $s_{12} = s_{13} = ... = s_{1(n-1)} = -\frac{1}{4}$; $s_{21} = s_{31} = ... = s_{(n-1)1} = \frac{1}{4}$; $s_{t2} = s_{t3} = ... = s_{t(n-1)} = -\frac{1}{4}$ and all other remaining entries are 0.

Step 5: Replace C_1^1 by $\sum_{i=1}^{n-1} C_i^1$

$$|L_C(G)-\lambda I| = \Big(-\frac{(m+1)(n-1)+2}{4}+\lambda\Big)^{(n-2)m}\Big(-\frac{m(n-1)+2}{4}+\lambda\Big)^{(m-1)} \begin{vmatrix} \frac{m(n-1)}{2}-\lambda & J_3 & -\frac{1}{2}R & \cdots & -\frac{1}{2}R \\ H_1 & D_7 & S_t^{1'} & \cdots & S_t^{1'} \\ \mathbf{0} & \mathbf{0} & -I & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -I \end{vmatrix}$$

where

$$J_3 = \begin{pmatrix} -\frac{m(n-1)}{2} & -\frac{n-1}{2} & \cdots & -\frac{n-1}{2} \end{pmatrix}$$

$$D_7 = \begin{pmatrix} \frac{m(n-1)-(n-3)}{4} - \frac{(m-1)(n-1)}{4} - \lambda & -\frac{n-1}{4} & -\frac{n-1}{4} & \cdots & -\frac{n-1}{4} \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Hence

$$|L_C(G)-\lambda I| = \Big(-\frac{(m+1)(n-1)+2}{4}+\lambda\Big)^{(n-2)m}\Big(-\frac{m(n-1)+2}{4}+\lambda\Big)^{(m-1)} \begin{vmatrix} \frac{m(n-1)}{2}-\lambda & -\frac{m(n-1)}{2} \\ -\frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix}$$

$$CLSpec(G) = \begin{pmatrix} \frac{(m+1)(n-1)+2}{4} & \frac{m(n-1)+2}{4} & \frac{m(n-1)+1}{2} & 0 \\ m(n-2) & (m-1) & 1 & 1 \end{pmatrix}$$

□

2.3. Closeness Q-Laplacian spectrum of $K_n^{(m)}$, $m \geq 2, n \geq 2$

Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. The closeness Q-Laplacian matrix $Q_C(G) = D_C(G) + A_G$

$$= \begin{pmatrix} \frac{m(n-1)}{2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & A & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & A & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & A \end{pmatrix}$$

where

$$D_1 = \begin{pmatrix} \frac{(m+1)(n-1)}{4} & 0 & 0 & \cdots & 0 \\ 0 & \frac{(m+1)(n-1)}{4} & 0 & \cdots & 0 \\ 0 & 0 & \frac{(m+1)(n-1)}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(m+1)(n-1)}{4} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{pmatrix}$$

$$Q_C(G) = \begin{pmatrix} \frac{m(n-1)}{2} & \frac{1}{2}R & \frac{1}{2}R & \cdots & \frac{1}{2}R \\ \frac{1}{2}C & D_1 + A & \mathbf{\frac{1}{4}} & \cdots & \mathbf{\frac{1}{4}} \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & D_1 + A & \cdots & \mathbf{\frac{1}{4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}C & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{4}} & \cdots & D_1 + A \end{pmatrix}$$

We state the result for closeness Q-Laplacian spectrum of $K_n^{(m)}$, $m \geq 2, n \geq 2$ without proof.

Theorem 2.3 *Let G be $K_n^{(m)}$, $m \geq 2, n \geq 2$. Then*

$$CQSpec(G) = \begin{pmatrix} \frac{(m+1)(n-1)-2}{4} & \frac{m(n-1)+2(n-2)}{4} & \alpha & \beta \\ m(n-2) & (m-1) & 1 & 1 \end{pmatrix}$$

where

$\alpha = \frac{1}{4}\left((2mn - 2m + n - 2) + \sqrt{(n-2)^2 + 4m(n-1)}\right)$

$\beta = \frac{1}{4}\left((2mn - 2m + n - 2) - \sqrt{(n-2)^2 + 4m(n-1)}\right)$

3. Conclusion

In this paper, we determine three types of closeness spectra for the one-point union of m copies of complete graphs on n vertices. The connections between the spectral properties and energies of closeness Laplacian and closeness Q-Laplacian matrices remain an interesting direction for further investigation.

The results obtained in this paper suggest several directions for future research. One possible extension is to investigate the closeness spectra of other graph operations such as vertex corona, edge corona, and rooted product of graphs. It would also be of interest to study algorithmic aspects related to the efficient computation of closeness spectra for larger classes of graphs. Furthermore, similar analysis can be carried out for other distance-based matrices, leading to a broader understanding of spectral properties associated with graph structures.

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