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Impact of Levels Taxation on Fisheries Management

I. EL HAKKI, A. BERGAM and R. MCHICH

ABSTRACT: In this study, we present two stock-effort mathematical models for fishery management, governing four ordinary differential equations that describe the dynamics of the fish stock and the fishing efforts. The initial model pertains to constant displacement rates of fishing effort, while the second model involves displacement rates that depend on fish stock. The first two equations track the changes in two fish populations, which migrate and grow across two distinct regions and are subject to exploitation by two separate fishing fleets, each with its corresponding fishing effort. Assuming that the growth of fish and changes in fishing effort occur over a slower time scale, while fish migration and vessel movement take place over a faster time scale. Subsequently, an aggregation of variables approach is used to get an aggregated model. This aggregated model is analyzed mathematically. The influence of taxation on the population of fish and the income generated by the fishery is being investigated. Numerical simulations are showcased to demonstrate the results.

Key Words: Fishery model, ODE, aggregation of variables, equilibria, stability, MSY, optimal taxation.

Contents

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1. Introduction

Numerous mathematical models have been constructed to explore the complexities inherent in fishing systems [18,24,25]. The objective of these models is to contribute to the development of decision-support tools in the context of global challenges in fishing, in order to ensure sustainable development of the resource, balancing the maximization of fishing revenue and the sustainability of stocks [10,13] .

In the most basic model [5,8,24], a logistic equation is suggested to describe the fish population, where the term for the fishing catch is directly proportional to both the current stock of fish "n" and the fishing effort "E" (which is determined by either the number of boats engaged in fishing or the duration of fishing at sea).

Considering variations in investment in fisheries management models is crucial for sustainable resource management. Fishery rent, defined as the disparity between the catch's value derived from the catch volume multiplied by the landing unit price "p", and the costs incurred from fishing efforts, is a pivotal concept [26,27]. The unit cost "c" of fishing effort encompasses expenses such as fuel procurement, fishermen's wages, and assorted taxes [20]. Consequently, profitability dictates the intensity of fishing effort; a positive rent encourages increased effort, while a negative rent prompts its reduction. This dynamic mirrors the dynamics of the Lotka-Volterra predator-prey model, coupled with the logistic growth pattern of prey populations [8,9]. This analogy underscores the intricate interplay between economic incentives and ecological dynamics within fisheries management.

In their studies [15,16], researchers delved into a dynamic model elucidating the interplay between stock and effort within fisheries. This model delineates a scenario where fish populations migrate between distinct zones, subject to harvesting by various fishing fleets. Notably, The dynamics of fish and vessel movements are presumed to transpire swiftly in comparison to fluctuations in stock abundance and fleet size. It is conjectured that vessel displacements between fishing areas hinge on stock dynamics. By harnessing the integration of both rapid and gradual time scales, the researchers derived a concise model governing the dynamics of total harvested stock and harvesting effort [1,2].

Depending on specific assumptions, their findings revealed the model's inclination to demonstrate either a steady balance or a consistent limit cycle, involving noticeable cyclic oscillations. This applies to both the overall fish stock and harvesting effort. This model underscores the pivotal role of different fishing strategies operating at a rapid time scale in shaping the long-term trajectory of fisheries dynamics on a global scale.

Controlling the exploitation of biological resources has become a significant concern due to diminishing resource stocks and environmental degradation. Specifically, regulating marine fisheries poses challenges in terms of law enforcement. The economic aspects of enforcing regulations in marine fisheries were examined in [29]. Various regulatory control options have been proposed, such as imposing taxes, licensing, property rights leasing, seasonal harvesting, and direct control. In [30], authors explored the complexities of selecting the optimal regulatory instrument and ensuring its enforcement in fishery management. One potential regulatory instrument is the imposition of a tax based on the quantity of landed fish biomass, as explained by Clark in [8]. Economists find taxation appealing because of its flexibility and its ability to uphold the advantages of a competitive economic system. Analyzing tax-based control provides a benchmark for evaluating other regulatory methods, such as licensing and seasonal restrictions, which can approximate the effort calculated from a tax policy model. The concept of using taxation as a control measure in a single-species fishery model was initially introduced by Clark in [8].

In our present study, we delve into the examination and discussion of two distinct models grounded in the bioeconomic framework of fisheries. These models intricately consider the interplay between fish stocks and fishing efforts across two delineated zones. Moreover, our investigation incorporates the crucial aspect of taxation on fishing behavior, aiming to uncover the circumstances wherein taxation can either foster the conservation of fish populations and promote the sustainability of fisheries or precipitate a limit cycle within the stock-effort model. Through rigorous analysis, we aim to shed light on the complex dynamics governing fisheries management and the potential implications of policy interventions on long-term sustainability.

The structure of this paper is organized into eight sections. Section 2 introduces a bioeconomic model described by a system of four ordinary differential equations (ODEs), involving two primary variables: n, representing the fish population, and E, denoting the fishing effort. Assuming the presence of two distinct time scales, the model is simplified in Section 3 through the application of the variable aggregation technique. Section 4 provides a detailed mathematical analysis of the reduced system. The focus of Section 5 is on determining the maximum sustainable yield. Section 6 explores how taxation policies affect the total catch. Finally, Section 7 extends the study to a more realistic scenario, in which fleet displacement rates depend on the state of the fish stock.

2. Presentation of the Fishery Model

2.1. Complete Model

In this study, we examine a spatial fisheries model that characterizes the interactions of two separate fish populations. Let's denote by $n_1(t)$ and $n_2(t)$ the density of fish in site i at time t, respectively, and by $E_i(t)$ the fishing effort on site i at time t, where $i = 1, 2$. The natural variation of the stock follows a logistic law. It is posited that the system operates under a constant global limit capacity denoted as K.

The dynamics of the fish stock follow the well-known logistic function $f(n) = rn(1 - \frac{n}{k})$ [3,6], where r denotes the rate of fish biomass growth and k represents the carrying capacity. The fish stock is depleted due to the stock-effort harvesting function, which tracks the evolution of fishing efforts in each of the two zones. We assume that the harvesting function in the two distinct zones follows a S Schaefer function, which can be written as $h_i = q_i E_i n_i$, where $i = 1, 2$. We suppose also that a constant tax T is imposed on all harvested fish [28] .

We posit that the dynamics of fish movements and boat maneuvers unfold swiftly, characterized by a time scale denoted as τ. Conversely, the processes governing fish growth and the broader dynamics of the fishery operate at a comparatively slower pace, delineated by a time scale $t = \varepsilon\tau$, where ε is a minuscule dimensionless parameter ($\varepsilon << 1$). This assumption allows us to segregate the system's evolution into distinct temporal regimes. Specifically, at the fast time scale, the following system encapsulates the intricate interplay of factors shaping the trajectory of the fishery.

In our model, the system consists of four equations and takes the form as follows:

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) + \varepsilon[r_1 n_1(1 - \frac{n_1}{k_1}) - q_1 n_1 E_1] \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) + \varepsilon[r_2 n_2(1 - \frac{n_2}{k_2}) - q_2 n_2 E_2] \\ \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [mE_2 - m^{'}E_1] + \varepsilon E_1(-c_1 + q_1 n_1(p - T)) \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}E_1 - mE_2] + \varepsilon E_2(-c_2 + q_2 n_2(p - T)) \end{cases} \tag{2.1}$$

r_i and k_i denote positive real values, indicating the intrinsic growth rate and the carrying capacity of the population in zone 1 and zone 2 respectively, and distinct catchability coefficients for the fishing fleet in each zone denoted by q_1 and q_2, the system presents notable disparities. Oversight of fishery exploitation falls under the purview of a regulatory agency, which imposes a tax T (> 0) per unit biomass [20]. The parameter p denotes the price of the catch per unit biomass, while the constant c encapsulates the cost of fishing effort per unit.

To investigate and analyze the model's dynamics, We use the method of aggregation of variables [1], which will reduce the dimension of this system to two equations. The method of aggregation of variables is based on the existence of multiple time scales, slow and fast, in a dynamic system. It is a complexity reduction method that allows for the construction of a global model governing only a few global variables that describe the dynamics of the system at a slow time scale [14].

2.2. Fast Model

Firstly, we neglect the terms that appear at a slow time scale by setting $\varepsilon = 0$ in all equations of (7.1) [4,17,23]. The fast model for fish will take the form as follows:

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) \\ \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) \end{cases} \tag{2.2}$$

In our context, the fast time scale is represented by τ, whereas (t) refers to the slow time scale, with the relationship ($t = \varepsilon\tau$). The fast dynamics describe a conservative system in which the total biomass—the sum of the two stock components ($n_1(t)$) and ($n_2(t)$) remains invariant over the fast time scale. In other words, $n(t) = n_1(t) + n_2(t)$ is constant during this phase.

Accordingly, the fast subsystem governing the evolution of fishing efforts can be expressed as follows:

$$\begin{cases} \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [mE_2 - m^{'}E_1] \\ \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}E_1 - mE_2] \end{cases} \tag{2.3}$$

With a constant total number of vessels, the fast model also conserves quantities. Additionally, at the rapid time scale, the total stock, represented as $E(t) = E_1(t) + E_2(t)$, remains constant.

A straightforward analysis reveals that, for any positive initial condition, the system admits a unique positive and stable equilibrium point.

The corresponding steady-state expressions describing the fast equilibria of both the fish populations and fishing efforts are given by:

$$\begin{cases} n_1^* = v_1^* n, \quad \text{and} \quad n_2^* = v_2^* n \\ E_1^* = \mu_1^* E, \quad \text{and} \quad E_2^* = \mu_2^* E \end{cases} \tag{2.4}$$

The equilibrium proportions of fish, denoted by v_i^*, together with the corresponding equilibrium shares of fishing effort, μ_i^*, within each zone i, are defined as follows:

$$\begin{cases} v_1^* = \frac{k}{k+k'}, \quad \text{and} \quad v_2^* = \frac{k'}{k+k'} \\ \mu_1^* = \frac{m}{m+m'}, \quad \text{and} \quad \mu_2^* = \frac{m'}{m+m'} \end{cases} \tag{2.5}$$

3. Aggregated Model

To derive the aggregated (reduced) model, the rapid dynamics and corresponding steady-state equilibrium values of fish and fleet movements (see Eq. 2.4) are substituted into the original system (Eq. 7.1) [21].

This procedure yields the following stable reduced model:

$$\begin{cases} \frac{\mathrm{d}n}{\mathrm{d}t} = n[r(1 - \frac{n}{K}) - QE] \\ \\ \frac{\mathrm{d}E}{\mathrm{d}t} = E[Q(p - T)n - c] \end{cases} \tag{3.1}$$

Where:

$$\begin{cases} r = v_1^* r_1 + v_2^* r_2 \\ Q = q_1 \mu_1^* v_1^* + q_2 \mu_2^* v_2^* \\ c = c_1 \mu_1^* + c_2 \mu_2^* \\ \frac{r}{K} = \frac{r_1 v_1^{*2}}{k_1} + \frac{r_2 v_2^{*2}}{k_2} \end{cases} \tag{3.2}$$

3.1. Positivity and Boundedness of Reduced System

Theorem 3.1 *For any initial condition $(n(0), E(0)) \in \mathbf{R}_+^2$, every corresponding solution $(n(t), E(t))$ of the reduced system (3.1) remains strictly positive for all $t \geq 0$.*

Proof. From the first equation of the aggregated system (3.1), it follows that the set $n = 0$ is invariant. Consequently, whenever $n(0) > 0$, we have $n(t) > 0$ for all $t \geq 0$.

Similarly, examining the second equation of (3.1) shows that at $E = 0$, the derivative of E is positive, implying that $E(t)$ increases near this boundary. Hence, if $E(0) \geq 0$, it follows that $E(t) > 0$, for all $t \geq 0$.

Theorem 3.2 *All solutions of the aggregated model of (3.1) are uniformly bounded.*

Proof. Considering

$$B(t) = B(n(t), E(t)) = n(t) + \frac{1}{(p-T)} E(t) \tag{3.3}$$

With

$B(0,0) = 0,\ B(n(t), E(t)) > 0$

The time derivative of equation (3.3), as it evolves over the solutions corresponding to the reduced system (3.1) can be articulated in the following manner:

$$\begin{aligned} \frac{dB}{dt} &= nr(1 - \frac{n}{K}) - \frac{Ec}{p-T} \\ &\leq nr(1 - \frac{n}{K}) + cn - c(n + \frac{E}{p-T}) \end{aligned}$$

So the expression $[nr(1 - \frac{n}{K}) + cn]$ reaches a maximum value of $\frac{K(r+c)^2}{4r}$ then:

$$\begin{aligned} \frac{dB}{dt} &\leq \frac{K(r+c)^2}{4r} - c(n + \frac{E}{p-T}) \\ &\leq \mu - cB \end{aligned}$$

Where $\mu = \frac{K(r+c)^2}{4r}$ then

$$\frac{dB}{dt} + cB \leq \mu$$

By utilizing the differential inequality theorem proposed by [11], we acquire:

$$0 < B(t) < B(0)e^{-ct} + \frac{\mu}{c}(1 - e^{-ct})$$

As $t \to \infty$, we observe

$$B(t) \leq \frac{\mu}{c}$$

Finally, all solutions of the reduced model (3.1) which initiate at positive conditions are confined in the compact

$$\Phi = \left\{ (n, E) \in \mathbf{R}_+^2 \;\middle|\; n(t) + \frac{E(t)}{p - T} \leq \frac{\mu}{c} \right\}$$

4. Analysis of the Aggregated Model

4.1. Existence of Equilibria

By examining the reduced system (3.1), the *n-nullclines* are identified as the union of $n = 0$ and $E = \frac{r}{Q}\left(1 - \frac{n}{K}\right)$, while the *E-nullclines* correspond to $E = 0$ and $n = \frac{c}{Q(p-T)}$. From the intersection of these nullclines, we obtain three equilibrium points:

$$(0,0), \quad (K,0), \quad \text{and} \quad (n^*, E^*) = \left(\frac{c}{Q(p-T)}, \frac{r}{Q}\left(1 - \frac{c}{(p-T)K}\right)\right).$$

The *interior equilibrium* (n^*, E^*) is positive whenever

$$Q(p-T)K - c > 0,$$

which is equivalent to

$$T < p - \frac{c}{QK} = T_1.$$

4.2. Stability of Equilibria and Discussion

The Jacobian matrix associated with system (3.1), evaluated at an arbitrary point (n, E), takes the following form:

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{K} - QE & -Qn \\ E(p-T) & Qn(p-T) - c \end{pmatrix} \tag{4.1}$$

- **Local Stability of the Origin**. The Jacobian matrix at $(0,0)$ can be expressed as

$$Jac_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & -c \end{pmatrix}$$

The corresponding eigenvalues are $\lambda_1 = r$ and $\lambda_2 = -c$. Since $r > 0$ and $-c < 0$, the equilibrium point $(0,0)$ is a *saddle point.*

- **Local Stability of the Fishing Free Equilibrium (FFE)**. The Jacobian matrix at the over-exploitation equilibrium $(K, 0)$ can be expressed as

$$Jac_{(K,0)} = \begin{pmatrix} -r & -QK \\ 0 & QK(p-T) - c \end{pmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = -r$ and $\lambda_2 = QK(p-T)-c$. Hence, $(K,0)$ is *locally asymptotically stable* if and only if

$$T > T_1 = p - \frac{c}{QK}.$$

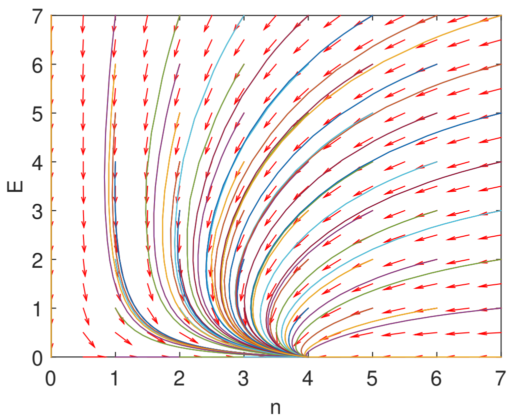

Figure 1: Phase plane for the case corresponds to FFE with parameters values $c = 1.2$, $T = 4.7$, $K = 4$, $p = 5$, $r = 0.9$, $Q = 0.2$.

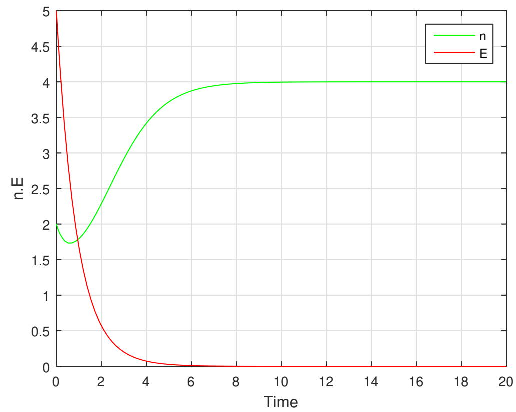

Figure 2: Illustration of the case of the FFE$(4, 0)$ witch obtained by $c = 1.2$, $T = 4.7$, $K = 4$, $p = 5$, $r = 0.9$, $Q = 0.2$, $n(0) = 2$, $E(0) = 5$.

Remark 1. If $T > T_1$ that means when the government increases taxes in an unlimited way on boat owners to cover costs, the rate of harvesting will decrease and tend to zero and the fish stock tends to its carrying capacity K see Figure 2. Figure 1 illustrates typical scenarios depicting the phase portrait in the context of a "Fishing Free Equilibrium" $(K, 0)$, leading to the closure of the fishery. This means the fishery is not profitable. The high taxes make the fishery economically unviable, leading to reduced harvesting and eventually, the fishery is closed due to a lack of profitability. In this scenario, there is an absence of a sustainable equilibrium for fishery operations, with only the equilibrium representing a state of no fishing activity remaining stable. Regardless of the initial conditions with positive values, the solutions rapidly converge toward this "Fishing Free Equilibrium" (FFE), resulting in a significant reduction in fishing efforts, which eventually dwindled to nothing.

- **Local Stability of the Interior Equilibrium**.
  The Jacobian matrix at (n^*, E^*) yields

$$Jac_{(n^*,E^*)} = \begin{pmatrix} -\frac{rn^*}{K} & -Qn^* \\ E^*(p-T)Q & 0 \end{pmatrix}$$

The formulas for the Trace and Determinant of the Jacobian matrix are given by: $Trac = -\frac{rc}{QK(p-T)}$ and $det = cr(1 - \frac{c}{K(p-T)})$. The equilibrium state (n^*, E^*) is locally asymptotically stable if $T < T_1$ and $(K, 0)$ become a saddle point.

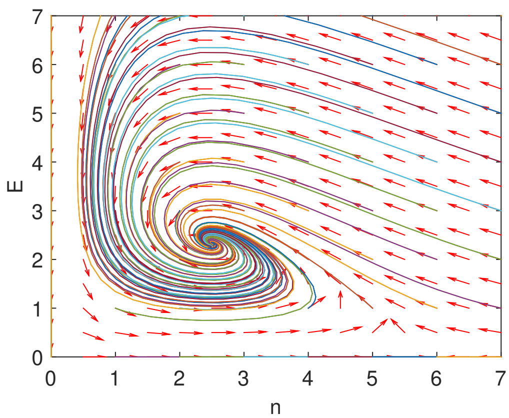

Figure 3: Phase plane for the case corresponds to interior equilibrium with parameters values $c = 1$, $T = 6.2$, $K = 6$, $p = 7$, $r = 2$, $Q = 0.5$.

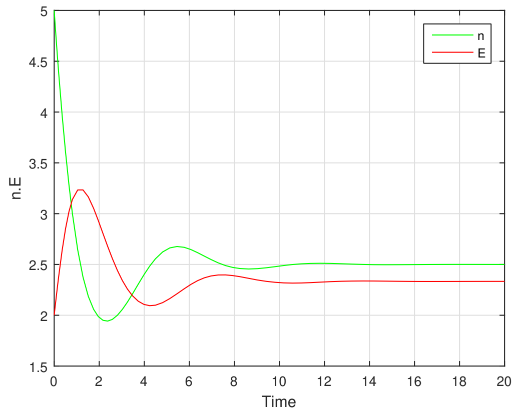

Figure 4: Illustration of the case corresponds to the interior equilibrium, which obtained by $c = 1$, $T = 6.2$, $K = 6$, $p = 7$, $r = 2$, $Q = 0.5$, $n(0) = 5$, $E(0) = 2$.

Remark 2. If $T < T_1$ that means, the government imposes taxes on boat owners, but the tax rates are kept at a reasonable level and do not surpass the threshold T_1. The fishing effort may still be profitable and sustainable at these lower tax levels. The harvesting rate will not approach zero. Since the fishing effort remains profitable and sustainable, the fishery may continue to operate, and the FFE $(K, 0)$ (representing unharvested fish) become unstable and the interior equilibrium tends to a sustainable state, see Figures 3 and 4. In other words, small perturbations or changes in the system may lead to different outcomes, and the fishery may not necessarily close. If the government keeps taxes at a reasonable level (not exceeding T_1), the fishery may remain open, profitable, and sustainable, without necessarily leading to fish stock reaching its carrying capacity or fishery closure.

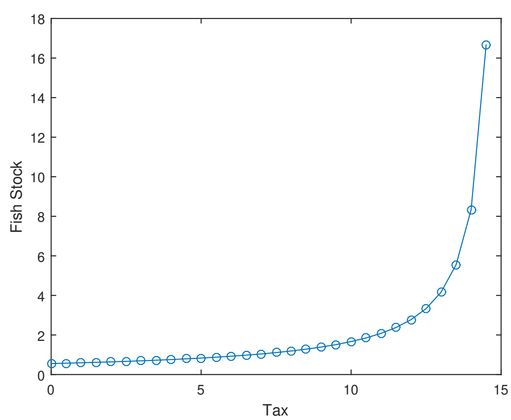

Figure 5: Dynamical behavior of the fish stock with Tax parameter, which obtained by $c = 10$, $p = 15$, $Q = 1.2$.

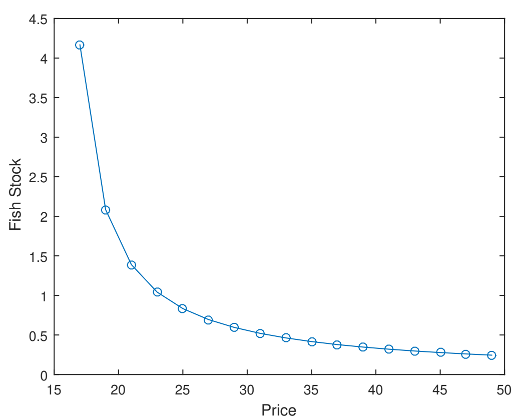

Figure 6: Dynamical behavior of the fish stock with price parameter, which obtained by $c = 10$, $T = 15$, $Q = 1.2$.

Remark 3. Figure 5 provides a visual representation of a notable trend: as the tax level increases, there is a corresponding rise in the level of fish stock. In simpler terms, this means that when the government imposes higher taxes on fishing activities, it is associated with an increase in the abundance of fish within the ecosystem. This phenomenon can be explained by the fact that higher taxes often lead to reduced fishing efforts, which, in turn, allows fish populations to thrive and grow. Figure 6 shows that as fish prices increase, there is a corresponding decrease in fish stock levels. This pattern underscores the economic dynamics of supply and demand, where higher fish prices incentivize increased fishing efforts in pursuit of greater profits, and that lead to a decrease in fish biomass.

It's apparent that leveraging taxation policies can serve as an efficient regulatory tool to safeguard predator species from extinction and ensure the sustainability of fisheries, and high taxation will cause a closed fishery (case of the FFE).

5. Maximum Sustainable Yield

An important concept in resource management is the Maximum Sustainable Yield (MSY), which refers to the maximum amount of a resource that can be harvested while still maintaining its population at a sustainable level. However, if the harvesting rate exceeds the MSY, it can lead to the depletion of the resource and ultimately result in its extinction.

This is because, at an equilibrium state, the population of the resource is balanced with the rate of harvesting. If the harvesting rate exceeds the MSY, the population will decline at a faster rate than it can replenish itself, leading to a decline in the overall size of the population. As the population decreases, it becomes more difficult to maintain the resource at a sustainable level, and it may eventually become extinct.

The following discussion focuses on analyzing the scenario of a dynamic system in equilibrium at point (n^*, E^*) under the condition that $T > T_1$.

The total catch function at equilibrium is:

$$Y^* = Qn^*E^* = rn^*(1 - \frac{n^*}{K}) \tag{5.1}$$

So that

$$\frac{\mathrm{d}Y^*}{\mathrm{d}n^*} = r(1 - 2\frac{n^*}{K})$$

Then $\frac{\mathrm{d}Y^*}{\mathrm{d}n^*} = 0$ means:

$$n^* = \frac{K}{2}$$

Consequently, the value $n^* = \frac{K}{2}$, gives the maximum catch and it is presented in the following manner:

$$Y^*_{MSY} = \frac{rK}{4} \ \ at \ \ n^* = \frac{K}{2} \tag{5.2}$$

Then, Y^* achieves a maximum equal to $\frac{rK}{4}$ for $n^* = \frac{K}{2}$ and it relates to the Maximum Sustainable Yield (MSY) (see Figure 7).

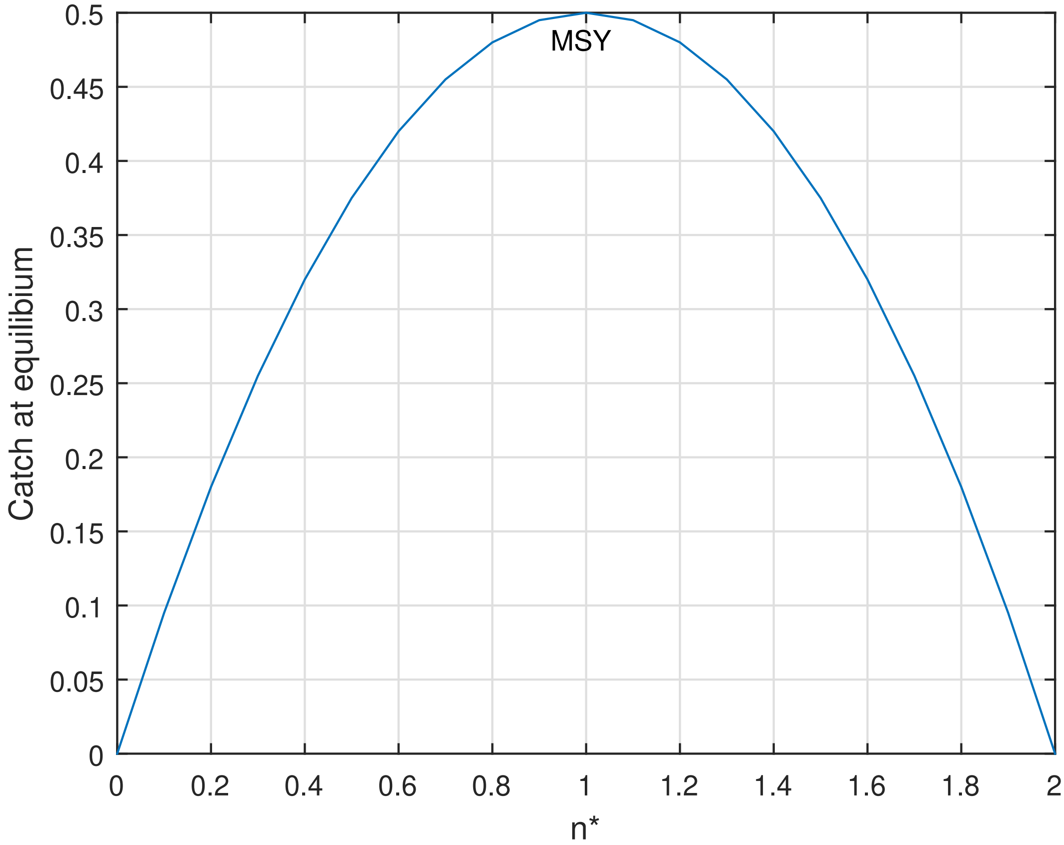

Figure 7: Illustration of the MSY.

6. Effect of Taxation on the Catch of a Sustainable Fishery

In this section, we delve into the analysis of a durable fishery. Specifically, we focus on scenarios where $T < T_1 = p - \frac{c}{QK}$ hence we have one positive stable equilibrium which is (n^*, E^*) with:

$$n^* = \frac{c}{Q(p-T)} \tag{6.1}$$

$$E^* = \frac{r}{Q}(1 - \frac{c}{(p-T)K}) \tag{6.2}$$

The harvest per time unit at balance is:

$$Y^* = Qn^*E^* \tag{6.3}$$

Substituting n^* and E^* in (6.1) gives

$$Y^* = \frac{rc}{Q(p-T)}(1 - \frac{c}{(p-T)K})$$

So that Then

$$\frac{\mathrm{d}Y^*}{\mathrm{d}T^*} = \frac{rc}{Q(p-T)^2}(1 - \frac{2c}{K(p-T)})$$

Then $\frac{dY^*}{dT} = 0$ means:

$$T_{opt} = p - \frac{2c}{K} \tag{6.4}$$

T_{opt} represents the optimal Taxation that will maximize the total yield at equilibrium to prevent the Fish Extinction state.

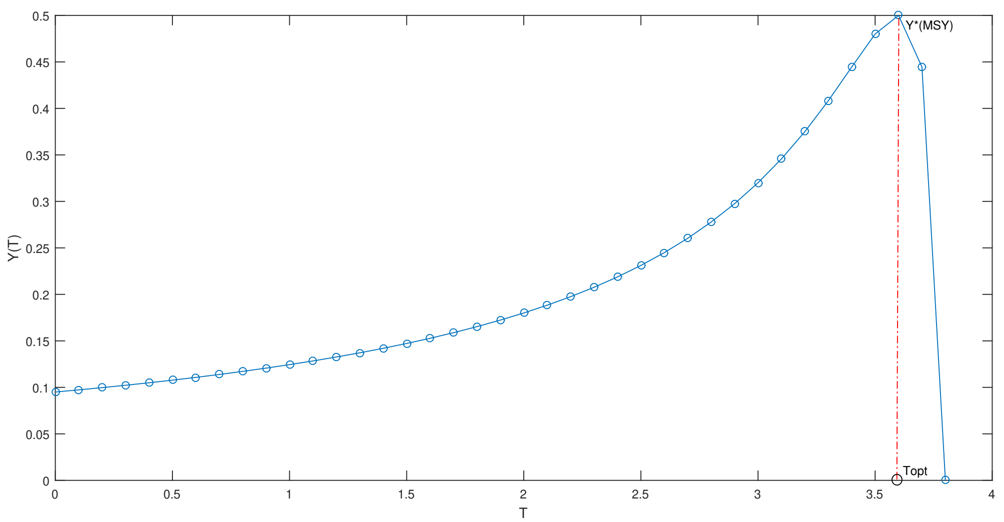

Figure 8: Effect of the taxation on the catch of a sustainable fishery with parameter values $p = 4$, $c = 0.4$, $K = 2$, $r = 1$.

The impact of taxation on total yield is thoroughly examined and illustrated in Figure 8. Initially, with the imposition of higher taxes, there is a discernible uptick in yield ($\dot{Y}^* > 0$). However, as taxation levels continue to rise significantly, a subsequent downturn is observed, resulting in a decrease in yield ($\dot{Y}^* < 0$).

In Figure 8, it is evident that at $T = 3.6$, the yield reaches its pinnacle, signifying the attainment of the maximum sustainable yield at $Y^*_{MSY} = 0.5$. This underscores the existence of an optimal positive tax rate conducive to maximizing total yield and can be derived in the following manner:

$$T_{opt} = 3.6$$

Substituting $n^* = \frac{K}{2}$ in (6.1) gives T_{opt} as

$$T = T_{opt} = p - \frac{2c}{K}$$

Moreover, it is apparent that when $T = 3.8$, as shown in Figure 8, the overall yield decreases to zero. This underscores the significant impact of heightened taxation, ultimately leading to the inevitable closure of the fishery.

We can observe that through the implementation of a well-calibrated catch tax T, where $0 < T < p$, as depicted in Figure 5, notable adjustments can be achieved. Indeed, with a sufficiently elevated tax rate, it becomes conceivable to contemplate the complete suspension of fishery operations. Such strategic tax interventions hold the potential to fundamentally alter the dynamics of the fishery sector, ensuring sustainability and equilibrium.

7. Effect of Taxation Between Two Zones with Stock Dependant

7.1. Complet model

Consider a diverse habitat partitioned into two distinct regions, each characterized by its unique carrying capacity, denoted as K_i $(i = 1, 2)$. At t, let $n_i(t)$, $(i = 1, 2)$ represent the density of fish within the ith zone, such that the total fish density is given by $n(t) = n_1(t) + n_2(t)$. Similarly, denote E_i, $(i = 1, 2)$ as the fishing exertion within each zone at time t, with the total fishing endeavor defined as $E(t) = E_1(t) + E_2(t)$. We assume uniformity among fishing vessels, enabling us to quantify fishing endeavors solely in terms of the number of vessels. These vessels traverse swiftly between zones, driven by the goal of optimizing their returns. Within this framework, species within both zones are presumed to engage in apparent rivalry, influenced by the activities of the same harvesting vessel. Subsequently, we present a system of four ordinary differential equations encapsulating biomass of fish populations, the dynamics of fishing vessel numbers, fish migrations, and the movements of fishery vessels [16].

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) + \varepsilon[r_1 n_1(1 - \frac{n_1}{k_1}) - q_1 n_1 E_1] \\ \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) + \varepsilon[r_2 n_2(1 - \frac{n_2}{k_2}) - q_2 n_2 E_2] \\ \\ \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [m(n_2)E_2 - m^{'}(n_1)E_1] + \varepsilon E_1(-c + q_1 n_1(p - T)) \\ \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}(n_1)E_1 - m(n_2)E_2] + \varepsilon E_2(-c + q_2 n_2(p - T)) \end{cases} \tag{7.1}$$

Within zone i $(i = 1, 2)$, r_i and K_i signify the inherent growth rate and maximum capacity of the population, respectively. q_i denotes the catchability coefficient for the fleet in zone i, while p represents the market price per catch unit. T denotes the tax rate per unit, and c stands for the fishing effort cost per unit. k and k' are coefficients delineating the per capita fish migration rates between the both zones

Furthermore, the functions of migration for fishing vessels, $m'(n_1)$ and $m(n_2)$, are influenced by the fish stock within their respective zones. These functions, illustrating vessel movement between the zones, are expected to decrease monotonically as fish population density increases. Therefore, higher fish populations lead to lower migration rates.

Explicitly, the migration functions are described as stated below:

$$m^{'}(n_1) = \frac{1}{\alpha n_1 + \alpha_0} \quad \text{and} \quad m(n_2) = \frac{1}{\beta n_2 + \beta_0}$$

The complete model (7.1) showcases robust properties. The functions featured on the right-hand sides of the equations in (7.1) demonstrate continuity and smoothness across $\mathbb{R}^4_+ = \{n_1(t), n_2(t), E_1(t), E_2(t)\} \in \mathbb{R}^4$, with each variable constrained to be non-negative: $n_1(t) \geq 0$, $n_2(t) \geq 0$, $E_1(t) \geq 0$, and $E_2(t) \geq 0$. Moreover, these functions exhibit Lipschitz continuity on $\mathbb{R}^4_+$. Consequently, the solution to system (7.1), initiated with non-negative initial conditions, not only exists but is also uniquely determined. Furthermore, it is evident that these solutions persist for all $t > 0$ while maintaining non-negativity throughout. Thus, the interior of $\mathbb{R}^4_+$ emerges as an invariant set for system (7.1). This robustness underscores the stability and dependability of the model's dynamics, ensuring its suitability for capturing real-world phenomena accurately.

7.2. The fast system

The rapid system, regarding τ as a swift time scale when ε is extremely small, obtains its swift dynamics by disregarding terms of ε order:

$$\begin{cases} \frac{dn_1}{d\tau} &= (kn_2 - k^{'}n_1), \\ \frac{dn_2}{d\tau} &= (k^{'}n_1 - kn_2), \\ \frac{dE_1}{d\tau} &= m(n_2)E_2 - m^{'}(n_1)E_1, \\ \frac{dE_2}{d\tau} &= m^{'}(n_1)E_1 - m(n_2)E_2. \end{cases} \tag{7.2}$$

The rapid system delineates the intricate movement of fish and vessels traversing between two distinct zones. During the rapid time scale, the total population size n and fishing effort E exhibit constancy, illustrated by the equations $n_1(\tau) + n_2(\tau) = n$ and $E_1(\tau) + E_2(\tau) = E$. However, at the slower time scale, fluctuations occur in both the overall fish stock and the number of vessels. The ensuing equilibria are derived for the swift component of the system:

$$(n_1^*, n_2^*, E_1^*, E_2^*) = (v_1^* n, v_2^* n, \eta_1^*(n)E, \eta_2^*(n)E). \tag{7.3}$$

Where :

$$\begin{cases} v_1^* &= \frac{k}{k+k'}, \\ v_2^* &= \frac{k'}{k+k'}, \\ \eta_1^*(n) &= \frac{m(n_2)}{m(n_2)+m'(n_1)} = \frac{\alpha v_1^* n+\alpha_0}{(\alpha v_1^*+\beta v_2^*)n+(\alpha_0+\beta_0)}, \\ \eta_2^*(n) &= \frac{m'(n_1)}{m(n_2)+m'(n_1)} = \frac{\beta v_2^* n+\beta_0}{(\alpha v_1^*+\beta v_2^*)n+(\alpha_0+\beta_0)}. \end{cases} \tag{7.4}$$

At the rapid equilibrium, v_1^* and v_2^* signify the proportions of stock present on individual patches, while $\eta_1^*(n)$ and $\eta_2^*(n)$ similarly indicate the distribution of fishing effort across these patches.

Investigating the stability of rapid equilibria within each zone-i entails a substitution: $n_2 = n - n_1$ and $E_2 = E - E_1$. Consequently, the system in patch-1 during the swift phase adopts the following format, allowing for a deeper analysis of its dynamics:

$$\begin{cases} \frac{dn_1}{d\tau} &= k(n-n_1) - k' n_1, \\ \frac{dE_1}{d\tau} &= m(n-n_1)(E-E_1) - m'(n_1)E_1. \end{cases} \tag{7.5}$$

Fast equilibria must adhere to the subsequent equations:

$$k(n-n_1^*) - k' n_1^* = 0,$$
$$m(n-n_1^*)(E-E_1^*) - m'(n_1^*)E_1^* = 0.$$

For each combination of (n, E), there exists a distinct equilibrium point (n_1^*, E_1^*) within the system (7.5). Assessing the stability of (n_1^*, E_1^*) necessitates evaluating the Jacobian matrix at this specific equilibrium, providing deeper insight into its dynamical behavior and is represented by:

$$J(n_1^*, E_1^*) = \begin{pmatrix} -(k+k') & 0 \\ -m'(n-n_1^*)(E-E_1^*) - m'(n_1^*) & -m(n-n_1^*) - m'(n_1^*) \end{pmatrix}.$$

The functions $m'(n_1^*)$ and $m(n-n_1^*)$ demonstrate positive behavior and decreasing trends, implying that their derivatives are negative. This leads to the conclusion that the trace of the Jacobian matrix at the swift equilibrium (n_1^*, E_1^*) is negative $(\mathrm{tr}(J(n_1^*, E_1^*)) < 0)$, while the determinant is positive $(\det(J(n_1^*, E_1^*)) > 0)$, affirming its regional asymptotic stability. Similarly, the analysis confirms the regional asymptotic stability of the alternative rapid equilibrium (n_2^*, E_2^*).

As a result, each pair (n_i^*, E_i^*) $(i = 1, 2)$ for the rapid segment exhibits local asymptotic stability. This indicates that any deviations from these equilibrium points will eventually converge back to their respective steady states, underscoring the system's resilience and its capability to maintain stability in the presence of minor disturbances.

Hence, by incorporating the solution derived from (7.2) as dictated by (7.3) into the overarching model (7.1), we streamline the comprehensive system (7.1), which encompasses four equations, into a consolidated system of two equations during the leisurely time scale. This amalgamated model will be further elucidated in the subsequent section.

7.3. Aggregated system

The following aggregated model emerges from a systematic integration of fast equilibria in (7.1), coupled with the incorporation of fish stock and harvesting effort equations. Through meticulous algebraic manipulation, this synthesis yields a consolidated system comprising two equations, ultimately delineating the dynamics of the total fish stock denoted by n and the exerted fishing effort represented by E, at slow time scale [7,19,22].

$$\begin{cases} \frac{dn}{dt} &= rn\left(1-\frac{n}{K}\right) - Q(n)nE, \\ \frac{dE}{dt} &= E\left(Q(n)(p-T)n - c\right), \end{cases} \tag{7.6}$$

where

$$\begin{cases} r &= r_1 v_1^* + r_2 v_2^*, \\ \frac{r}{K} &= r_1 \frac{{v_1^*}^2}{k_1} + r_2 \frac{{v_2^*}^2}{k_2}, \\ Q(n) &= q_1 v_1^* \eta_1^*(n) + q_2 v_2^* \eta_2^*(n) = \frac{Q_1 n + Q_0}{Q_{11} n + Q_{00}}, \end{cases} \tag{7.7}$$

and,

$$\begin{cases} Q_1 &= q_1 \alpha {v_1^*}^2 + q_2 \beta {v_2^*}^2, \\ Q_0 &= q_1 v_1^* \alpha_0 + q_2 v_2^* \beta_0, \\ Q_{11} &= \alpha v_1^* + \beta v_2^*, \\ Q_{00} &= \alpha_0 + \beta_0. \end{cases} \tag{7.8}$$

The constants r and K, within the context of the complete model (7.1), signify positive parameters defined in relation to the local parameters.

The aggregated system (7.6) provides a simplified representation of the comprehensive system (7.1).

7.4. Equilibrium States Existence of the Aggregate Model

The $\dot{n} = 0$ and $\dot{E} = 0$ nullclines lead us to show that the aggregated system (7.6) has 3 equilibrium points; $E_0(0,0)$, $E_1(K,0)$ and the interior equilibrium point $E_2(n^*, E^*)$ of (7.6) is established through this process:

$$E^* = \frac{r}{Q(n^*)}\left(1 - \frac{n^*}{K}\right),$$

$$n^* Q(n^*)(p-T) - c = 0.$$

The determination of n^* is derived from the subsequent equation:

$$n^* Q(n^*)(p-T) - c = 0. \tag{7.9}$$

Upon solving Equation (7.9), we can formulate a quadratic expression as follows:

$${n^*}^2 (p-T) Q_1 + n^*(Q_0(p-T) - cQ_{11}) - cQ_{00} = 0. \tag{7.10}$$

$$n^* = \frac{-((p-T)Q_0 - cQ_{11}) + \sqrt{((p-T)Q_0 - cQ_{11})^2 + 4cQ_1(p-T)Q_{00}}}{2(p-T)Q_1}.$$

The positive value of E^* indicates that for

$$n^* < K. \tag{7.11}$$

The expression (7.11) gives the following condition

$$T < T_2; \quad T_2 = p - \frac{c(Q_{11}K + Q_{00})}{K(Q_1 K + Q_0)} = p - p_2. \tag{7.12}$$

With $p_2 = \frac{c(Q_{11}K+Q_{00})}{K(Q_1K+Q_0)}$.

Hence, the interior equilibrium point $E_2(n^*, E^*)$ of (7.6) is achievable within the provided constraints:

$$T < T_2.$$

Nevertheless, if $p > p_2$, the existence of $E_2(n^*, E^*)$ requires that $T < T_2$ (or $p > p_2$).

7.5. Analyzing the Stability of Equilibrium Points

To analyze the local stability of the system comprehensively, we meticulously compute the Jacobian matrix $J(n, E)$ for system (7.6) at every conceivable point (n, E) within its state space. This matrix, is formally expressed as:

$$J(n,E) = \begin{pmatrix} r - \frac{2r}{K} - EQ'(n)n - EQ(n) & -Q(n)n \\ E(p-T)(nQ'(n) - Q(n)) & (p-T)nQ(n) - c \end{pmatrix}.$$

- **Local Stability of the Origin**. $E_0(0, 0)$ eigenvalues are as follows:

  $\lambda_1 = r$ and $\lambda_2 = -c$.

  Therefore, the equilibrium point $(0, 0)$ consistently takes the form of a saddle.

- **Local Stability of the Fishing Free Equilibrium (FFE)** $E_1(K, 0)$, the eigenvalues of the Jacobian matrix are:

  $\lambda_1 = -r < 0$ and $\lambda_2 = Q(K)(p - T)K - c$.

  Hence, $E_1(K, 0)$ displays local asymptotic stability when

$$T > T_2; \quad T_2 = p - \frac{c(Q_{11}K + Q_{00})}{K(Q_1K + Q_0)} = p - p_2. \tag{7.13}$$

  If condition (7.13) is violated, the equilibrium point $E_1(K, 0)$ transforms into a saddle. When $p < p_2$, condition (7.13) is naturally met, indicating that the fishery is economically unviable and will be closed. Conversely, if $p > p_2$ and tax T is sufficiently high to meet (7.13), the fishery remains economically unviable, leading to its closure.

- **Local Stability of the Interior Equilibrium**. To analyze the proximity stability, the Jacobian matrix of the model (7.6) at an inside point (n^*, E^*) is expressed as

$$J(n^*,E^*) = \begin{pmatrix} -\frac{n^*r}{K} - E^*Q'(n^*)n^* & -Q(n^*)n^* \\ E^*(p-T)(Q'(n^*)n^* + Q(n^*)) & 0 \end{pmatrix}.$$

The stability conditions found within the interior point $E_2(n^*, E^*)$ present as follows

$$\begin{aligned} \operatorname{tr}(J(n^*,E^*)) &= -\frac{rn^*}{K} - E^*Q'(n^*)n^* < 0, \\ &= n^* \left(-\frac{r}{K} - E^* \frac{(Q_1Q_{00} - Q_0Q_{11})}{(Q_{11}n^* + Q_{00})} \right). \end{aligned}$$

$$\det(J(n^*,E^*)) = E^*Q(n^*)n^*(p-T)[Q'(n^*)n^* + Q(n^*)] > 0.$$

Thus, the equilibrium state (n^*, E^*) achieves local asymptotic stability if

$$n^*[-\frac{r}{K} - E^*Q'(n^*)] < 0, \tag{7.14}$$

When the condition (7.14) is breached, the internal state $E_{(n^*,E^*)}$ destabilizes and may exhibit limit cycles (periodic solutions).

It's worth noting that if $Q_1Q_{00} > Q_0Q_{11}$, the trace will consistently be negative, ensuring the stability of the interior point. However, in cases where $Q_1Q_{00} < Q_0Q_{11}$, the stability of the interior state is guaranteed if

$$\frac{r}{K} > E^* \frac{(Q_1Q_{00} - Q_0Q_{11})}{(Q_{11}n^* + Q_{00})}.$$

Figure 9 illustrates the case where dynamics tend towards a stable limit cycle. In this scenario, both the total fishing effort and total biomass vary periodically in the long term. Figure 10 depicts variations over time in total stock and overall effort.

The cyclic fluctuations align with the dynamics of the aggregate model, which roughly characterizes the system's dynamics over slow time scale.

In this model 7.6, the limit cycle results from the stock-dependence of fishing effort rates. We assumed that fishing boats quickly leave areas with low fish density because their income becomes very small as well. Therefore, even if the stock becomes quite small, it cannot tend toward extinction because in such a case, when the number of fishing boats decreases drastically, the stock can regenerate (see FIG. 10). Thus, the fishery income grows, and new boats participate in the fishery, starting a new cycle.

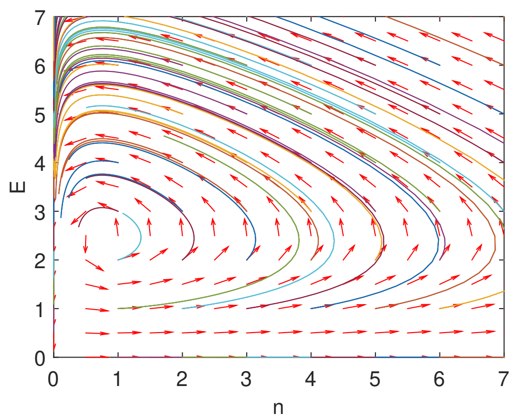

Figure 9: Phase depiction in the instance of a stable limit cycle, with parameters: $q_1 = 1$, $\alpha = 0,2$, $v_1^* = 0,2$, $q_2 = 1$, $v_2^* = 0,2$, $r = 0,5$, $\beta = 0,2$, $\beta_0 = 0,2$, $c = 0,06$, $\alpha_0 = 0,2$, $K = 100$, $p = 0,4$.

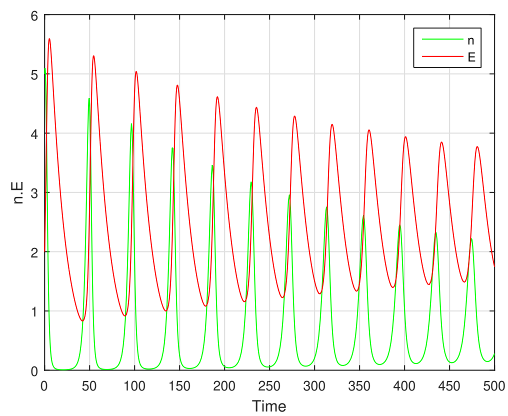

Figure 10: Case of a stable limit cycle: the changes over time of the overall fish population and harvesting fleets.

7.6. Maximizing Sustainable Yield through Taxation Optimization

As demonstrated in the previous section, the function representing the total catch, at the balance state is expressed as :

$$Y^* = H(n^*, E^*) = Q(n^*)n^*E^* = rn^* \left(1 - \frac{n^*}{K}\right). \tag{7.15}$$

Thus

$$\frac{\partial Y^*}{\partial n^*} = 0 \quad \text{and} \quad \frac{\partial^2 Y^*}{\partial n^{*2}} < 0.$$

As a result,

$$n^* = \frac{K}{2} \quad \text{and} \quad \frac{\partial^2 Y^*}{\partial n^{*2}} = -2\frac{r}{K}.$$

Hence, the quantity $n^* = \frac{K}{2}$ represents the maximum catch, which is formulated as follows:

$$Y^*_{\text{MSY}} = \frac{rK}{4} \quad \text{at} \quad n^* = \frac{K}{2}. \tag{7.16}$$

The analysis of taxation's impact on the total catch, as depicted in Figure 11, reveals notable patterns. Initially, with increasing taxation, the yield experiences a positive trend ($\frac{dY^*}{dT} > 0$). However, as taxation continues to rise, this trend eventually reverses ($\frac{dY^*}{dT} < 0$).

Figure 11 highlights a significant observation: at $T = 3.8$, the yield reaches its maximum, coinciding with the maximum sustainable yield $Y^*_{\text{MSY}} = 0.5$. This implies the existence of an optimal positive tax rate that maximizes total yield. The precise value of this optimal taxation level is determined as follows:

$$T_{\text{opt}} = 3.8. \tag{7.17}$$

By substituting $n^* = \frac{K}{2}$ into (7.10), T_{opt} is obtained as

$$T = T_{\text{opt}} = p - \frac{(2KQ_{11} + 4cQ_{00})}{K(KQ_1 + 2Q_0)}. \tag{7.18}$$

Validation of (7.17) is done by substituting parameter values into (7.18). Additionally, the total harvest becomes zero at $T = 3.9$. This implies that higher taxation would result in the closure of the fishery.

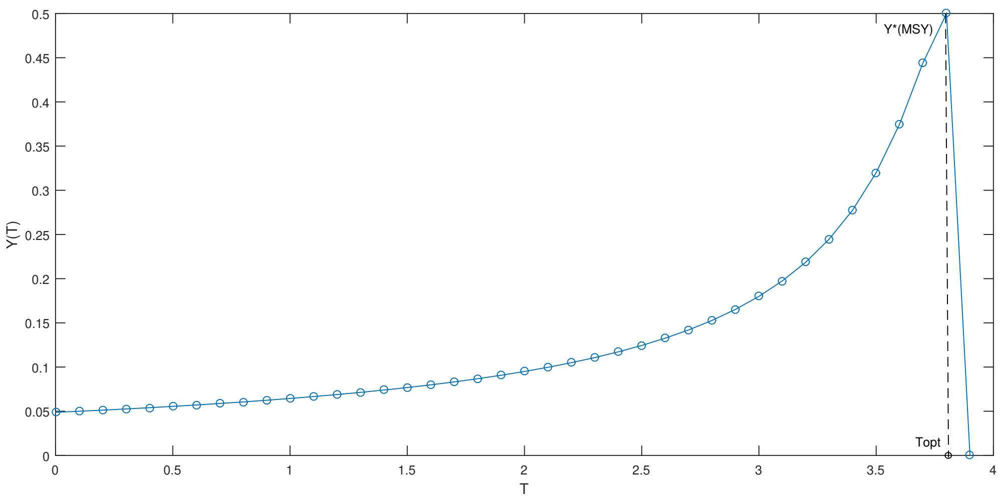

Figure 11: Effect of the taxation on the catch of a sustainable fishery for the stock-dependent case with parameter values $p = 4$, $c = 0.4$, $K = 2$, $r = 1$, $Q_{11} = 0.1$, $Q_{00} = 0.5$, $Q_1 = 0.25$, $Q_0 = 0.25$.

When juxtaposed with Figure 8, it becomes apparent that the occurrence of maximum sustainable yield (MSY) is situated at a higher taxation level ($T_{\text{opt}} = 3.8$) in the context of density-dependent migration rates. This observation underscores the pronounced influence of migration dynamics on the optimal tax regime for resource management.

8. Conclusion and Perspective

In this study, we introduce a stock-effort spatial fishery model that integrates economic considerations. Our approach involves a comprehensive model where we incorporate the intricate dynamics between fish migration, boat movement, and economic factors. Specifically, we operate under the assumption that the processes of fish migration and boat movement unfold at a significantly faster time scale compared to fish growth and fishery dynamics. Leveraging this assumption, we derive a streamlined, two-dimensional model that lends itself to analytical treatment, enabling deeper insights into the dynamics of the system.

The analysis of the reduced model reveals two primary scenarios contingent upon parameter values: the presence of either a fish-free equilibrium or an interior equilibrium, with the taxation level playing a pivotal role in determining which equilibrium state predominates. The existence of a MSY is also examined. Moreover, it has highlighted the existence of an optimal Taxation that maximizes captures at equilibrium. The variation of taxation has important consequences on fisheries dynamics. High levels of taxation can lead to closer fisheries, as compared to a sustainable one which can be achieved when the level of taxation is fusible. Then we explored the impacts of taxation in the case of stock-dependent vessel migration rates. Beyond the previously discussed scenarios of extinction and coexistence, we unveiled a novel case characterized by a stable limit cycle.

As a perspective, there are two main approaches to managing and regulating fishing activities: the implementation of taxes on the effort, and the introduction of individual catch or effort quotas. Alternatively, these approaches can be used in combination to achieve more comprehensive and effective fisheries management.

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Ismail EL HAKKI
Department of Mathematics
Applied Mathematics and Computer Sciences Research Team, Higher Normal School of Tétouan, Abdelmalek Essaâdi University
Tétouan, Morocco
E-mail address: ismaillakki2@gmail.com

and

Amal BERGAM,
Laboratoire de Simulation et Modélisation Mathématique, Analyse de Données, Chimie des Substances Naturelles,
Faculté Polydisciplinaire à Larache (FPL), Université Abdelmalek Essaâdi,
Larache, Morocco
E-mail address: abergam@uae.ac.ma

and

Rachid MCHICH,
Laboratoire de Gouvernance des Organisations et des Territoires,
École Nationale de Commerce et de Gestion (ENCG), Université Abdelmalek Essaâdi,
Tanger, Morocco
E-mail address: rmchich@uae.ac.ma