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Some New Congruences For Partitions With Monochromatic Even Parts and Multichromatic Odd Parts

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ABSTRACT: Let $a(n)$ denote the number of integer partitions of a positive integer n wherein even parts appear in only one color (i.e. monochromatic) while the odd parts may appear in one of three colors (i.e. trichromatic). Hirschhorn and Sellers (2025) introduced a generalised version of $a(n)$ and defined the partition function $a_t(n)$ wherein even parts appear in one colour and odd parts may occur with one of the t colours for any fixed positive integer t. They also proved some congruences modulo 7 for $a_{7j+1}(n)$ for any integer $j \geq 1$. In this paper, we prove some new particular and infinite families of congruences for $a_t(n)$ by using q-series identities.

Keywords: Chromatic partitions, partition congruence, q-series identities.

Contents

1. Introduction

A partition of positive integer n is a sequence of integers $(\nu_1, \nu_2, ..., \nu_k)$ such that $\nu_1 \geq \nu_2 \geq ... \geq \nu_k \geq 1$ and $\sum_{j=1}^{k} \nu_j = n$, where the integers ν_j are called parts or summands of the partition. For example, distinct partition of $n = 3$ are $3, 2+1$, and $1+1+1$. If $p(n)$ counts the total number of distinct partitions of n with $p(0) = 1$, then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where for any complex number c and $|q| < 1$,

$$(c;q)_\infty = \prod_{n=0}^{\infty} (1 - cq^n). \tag{1.2}$$

Throughout the paper we will use the notation, $f_\ell := (q^\ell; q^\ell)_\infty$ for any positive integer ℓ. Ramanujan [3,10,11] offered following three beautiful congruence of $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Amdeberhan and Merca [2] considered the partition function $a(n)$, which counts the number of integer partitions of a positive integer n wherein even parts come in only one color (i.e. monochromatic), while the odd parts appear in three colors (i.e. trichromatic), say black(b), white(w) and green(g). For instance, $a(2) = 7$ with relevant partitions given by $2, 1_b + 1_w, 1_w + 1_g, 1_g + 1_b, 1_b + 1_b, 1_w + 1_w$, and $1_g + 1_g$. The generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f_2^2}{f_1^3}. \tag{1.4}$$

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Amdeberhan and Merca [2] also proved some congruences for $a(n)$. For example, they proved that, for all $n \geq 0$, $a(7n+2) \equiv 0 \pmod 7$.

Recently, Hirschhorn and Sellers [9] introduced a generalised form of $a(n)$ and defined the partition fucntion $a_t(n)$ wherein even parts appear in one colour and odd parts may occur with one of the t colours for any fixed positive integer t. For example, if odd parts appear in $t = 4$ colors, say black(b), white(w), green(g) and purple(p), then $a_4(2) = 11$ with partitions given by, $2, 1_b + 1_b, 1_b + 1_w, 1_b + 1_g, 1_b + 1_p, 1_w + 1_w, 1_w + 1_g, 1_w + 1_p, 1_g + 1_g, 1_g + 1_p$, and $1_p + 1_p$. The generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{f_2^{t-1}}{f_1^t}. \tag{1.5}$$

Using the generating function, Hirschhorn and Sellers [9] also proved some congruences modulo 7 of $a_{7j+1}(n)$ for any integer $j \geq 1$. In sequel, we will prove some infinite families of congruences satisfy by $a_t(n)$ in this paper. To prove the congruences, we employ some q-series identities listed in Sect. 2. In Sect. 3, we prove congruences for $a_t(n)$.

2. Preliminaries

Ramanujan's general theta-function $F(x, y)$ is defined by

$$F(x,y) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2}, \ |xy| < 1. \tag{2.1}$$

The three special cases of $F(x, y)$ [4, p. 36, Entry 22 (i), (ii))] are the theta-functions $\phi(q)$, $\psi(q)$ and $f(-q)$ given by

$$\phi(q) := F(q,q) = \sum_{\nu=0}^{\infty} q^{\nu^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.2}$$

$$\psi(q) := F(q,q^3) = \sum_{\nu \geq 0} q^{\nu(\nu+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{2.3}$$

and

$$f(-q) := F(-q,-q^2) = \sum_{u=-\infty}^{\infty} (-1)^u q^{u(3u+1)/2} = f_1. \tag{2.4}$$

Employing elementary q-operations, it is easily seen that

$$\phi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \frac{f_1^2}{f_2}. \tag{2.5}$$

Lemma 2.1 *[6] We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.6}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.7}$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}}, \tag{2.8}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}. \tag{2.9}$$

Lemma 2.2 *[6] We have*

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \Big(E(q^5)^4 + qE(q^5)^3 + 2q^2 E(q^5)^2 + 3q^3 E(q^5) + 5q^4$$

$$-\frac{3q^5}{E(q^5)}+\frac{2q^6}{E(q^5)^2}-\frac{q^7}{E(q^5)^3}+\frac{q^8}{E(q^5)^4}\Bigg), \tag{2.10}$$

where

$$E(q)=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Lemma 2.3 *[12, Eq (3.1)] We have*

$$\frac{f_2^3}{f_1^3}=\frac{f_6}{f_3}+3q\frac{f_6^4f_9^5}{f_3^8f_{18}}+6q^2\frac{f_6^3f_9^2f_{18}^2}{f_3^7}+12q^3\frac{f_6^2f_{18}^5}{f_3^6f_9}. \tag{2.11}$$

Lemma 2.4 *[7] We have*

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{2.12}$$

Lemma 2.5 *[8] We have*

$$f_1f_2=\frac{f_6f_9^4}{f_3f_{18}^2}-qf_9f_{18}-2q^2\frac{f_3f_{18}^4}{f_6f_9^2}. \tag{2.13}$$

Lemma 2.6 *[5, Theorem 2.2] For any prime $p\geq 5$, we have*

$$f_1=\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^kq^{(3k^2+k)/2}F\left(-q^{(3p^2+(6k+1)p)/2},-q^{(3p^2-(6k+1)p)/2}\right)$$

$$+(-1)^{(\pm p-1)/6}q^{(p^2-1)/24}f_{p^2}, \tag{2.14}$$

where

$$\frac{\pm p-1}{6}=\begin{cases}\dfrac{(p-1)}{6}, & if\ p\equiv 1 \pmod 6\\ \dfrac{(-p-1)}{6}, & if\ p\equiv -1 \pmod 6).\end{cases}$$

Furthermore, if $\dfrac{-(p-1)}{2}\leq k\leq\dfrac{(p-1)}{2}$ *and* $k\neq\dfrac{(\pm p-1)}{6}$, *then*

$$\frac{3k^2+k}{2}\not\equiv\frac{p^2-1}{24} \pmod p.$$

Lemma 2.7 *[1, Lemma 2.3] For any prime $p\geq 3$, we have*

$$f_1^3=\sum_{\substack{l=0\\ l\neq(p-1)/2}}^{(p-1)}(-1)^lq^{(l(l+1))/2}\sum_{n=0}^{\infty}(-1)^n(2pn+2l+1)q^{pn\cdot(pn+2l+1)/2}$$

$$+p(-1)^{(p-1)/2}q^{(p^2-1)/8}f_{p^2}^3. \tag{2.15}$$

Furthermore, if $l\neq\dfrac{(p-1)}{2}$, $0\leq l\leq(p-1)$, *then*

$$\frac{(l^2+l)}{2}\not\equiv\frac{(p^2-1)}{8} \pmod p.$$

Lemma 2.8 *[5, Theorem 2.1] For any prime $p > 2$, we have*

$$\psi(q) = \sum_{j=0}^{(p-3)/2} q^{(j^2+j)/2} F\left(q^{(p^2+(2j+1)p)/2}, q^{(p^2-(2j+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}). \tag{2.16}$$

Furthermore, $(j^2+j)/2 \not\equiv (p^2-1)/8 \pmod{p}$ for $0 \le j \le (p-3)/2$.

To end this section, we record the following congruences which can be easily proved using the Binomial theorem: For any positive integer h, m and prime p, we have

$$f_{pm}^{p^{h-1}} \equiv f_m^{p^h} \pmod{p^h}, \tag{2.17}$$

$$f_h^p \equiv f_{hp} \pmod{p}. \tag{2.18}$$

3. Congruences for $a_t(n)$

In this section, Legendre's symbol will be used in the proofs of congruences of $a_t(n)$. Let p be any odd prime and λ be any integer relatively prime to p, then the Legendre symbol $\left(\frac{\lambda}{p}\right)$ is defined by

$$\left(\frac{\lambda}{p}\right) = \begin{cases} 1, & \text{if } \lambda \text{ is a quadratic residue of } p, \\ -1, & \text{if } \lambda \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 3.1 *Let $p \ge 5$ be a prime with $\left(\frac{-5}{p}\right) = -1$ and $1 \le r \le p-1$. Then for any integers $\alpha \ge 0$ and $n \ge 0$, we have*

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha}n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv f_1 \pmod{4}, \tag{3.1}$$

$$a_4\left(2\cdot p^{2\alpha+1}(pn+r) + \frac{p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{4}. \tag{3.2}$$

Proof: Setting $t = 4$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{f_2^3}{f_1^4}. \tag{3.3}$$

Employing (2.7) in (3.3), we obtain

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{f_4^{14}}{f_2^{11}f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^7}. \tag{3.4}$$

Extracting the terms involving q^{2n} and replacing q^2 by q from (3.4), we obtain

$$\sum_{n=0}^{\infty} a_4(2n)q^n = \frac{f_2^{14}}{f_1^{11}f_4^4}. \tag{3.5}$$

Employing (2.17) in (3.5), we obtain

$$\sum_{n=0}^{\infty} a_4(2n)q^n \equiv f_1 \pmod{4}, \tag{3.6}$$

which is $\alpha=0$ case of (3.1). Assume that (3.1) is true for some integer $\alpha\geq 0$. Employing (2.14) in (3.1), we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha}n+\frac{p^{2\alpha}-1}{12}\right)q^n \equiv \sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^k q^{(3k^2+k)/2}F\left(-q^{(3p^2+(6k+1)p)/2},-q^{(3p^2-(6k+1)p)/2}\right)$$
$$+(-1)^{(\pm p-1)/6}q^{(p^2-1)/24}f_{p^2} \pmod 4. \tag{3.7}$$

Extracting the terms involving $q^{pn+(p^2-1)/24}$ from (3.7), dividing $q^{(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{12}\right)q^n \equiv f_p \pmod 4. \tag{3.8}$$

Extracting the terms involving q^{pn} from (3.8) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha+2}n+\frac{p^{2\alpha+2}-1}{12}\right)q^n \equiv f_1 \pmod 4, \tag{3.9}$$

which is the $\alpha+1$ case of (3.1). Hence, the proof of (3.1) is complete. Extracting the terms involving q^{pn+r}, for $1\leq r\leq p-1$, from (3.8), we arrive at (3.2). □

Theorem 3.2 *Let $p\geq 5$ be a prime with $\left(\frac{-12}{p}\right)=-1$ and $1\leq r\leq p-1$. Then for any integers $\alpha\geq 0$ and $n\geq 0$, we have*

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha}n+\frac{13\cdot p^{2\alpha}-1}{12}\right)q^n \equiv 4f_1f_4^3 \pmod 8, \tag{3.10}$$

$$a_4\left(2\cdot p^{2\alpha+1}(pn+j)+\frac{13\cdot p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod 8. \tag{3.11}$$

Proof: Extracting the terms involving q^{2n+1} from both sides of (3.4), dividing by q, replacing q^2 by q and using (2.17), we obtain

$$\sum_{n=0}^{\infty} a_4(2n+1)q^n \equiv 4f_1f_4^3 \pmod 8, \tag{3.12}$$

which is $\alpha=0$ case of (3.10). Assume that (3.10) is true for some integer $\alpha\geq 0$. Employing (2.14) and (2.15)in (3.10), we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha}n+\frac{13\cdot p^{2\alpha}-1}{12}\right)q^n$$
$$\equiv 4\Bigg[\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^k q^{(3k^2+k)/2}F\left(-q^{(3p^2+(6k+1)p)/2},-q^{(3p^2-(6k+1)p)/2}\right)+(-1)^{(\pm p-1)/6}q^{(p^2-1)/24}f_{p^2}\Bigg]$$
$$\times\Bigg[\sum_{\substack{l=0\\ l\neq(p-1)/2}}^{(p-1)}(-1)^l q^{(4l(l+1))/2}\sum_{n=0}^{\infty}(-1)^n(2pn+2l+1)q^{4pn\cdot(pn+2l+1)/2}+p(-1)^{(p-1)/2}q^{4(p^2-1)/8}f_{4p^2}^3\Bigg]$$
$$\pmod 8. \tag{3.13}$$

Consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+4\left(\frac{l^2+l}{2}\right)\equiv 13\left(\frac{p^2-1}{24}\right) \pmod p. \tag{3.14}$$

(3.14) is similar to

$$(6k+1)^2+12(4l+1)^2 \equiv 0 \pmod{p}. \tag{3.15}$$

Since $\left(\dfrac{-12}{p}\right)=-1$, the above congruence has only solution for $k=\dfrac{(\pm p-1)}{6}$ and $m=\dfrac{(\pm p-1)}{4}$. Therefore, extracting the terms involving $q^{pn+13(p^2-1)/24}$ from both sides of (3.13), dividing throughout by $q^{13(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha+1}n+\frac{13\cdot p^{2\alpha+2}-1}{12}\right)q^n \equiv 4f_pf_{4p}^3 \pmod{8}. \tag{3.16}$$

Extracting the terms involving q^{pn} from (3.16) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_4\left(2\cdot p^{2\alpha+2}n+\frac{13\cdot p^{2\alpha+2}-1}{12}\right)q^n \equiv 4f_1f_4^3 \pmod{8}. \tag{3.17}$$

which is the $\alpha+1$ case of (3.10). Hence, the proof of (3.10) is complete. Extracting the terms involving q^{pn+r}, for $1\le r\le p-1$, from (3.16), we arrive at (3.11). □

Theorem 3.3 *For for any integer $n\ge 0$, we have*

$$a_5(27n+1) \equiv 2a_5(3n) \pmod{3}, \tag{3.18}$$

$$a_5(27n+10) \equiv a_5(3n+1) \pmod{3}, \tag{3.19}$$

$$a_5(27n+19) \equiv 0 \pmod{3}, \tag{3.20}$$

$$a_5(5n+3) \equiv 0 \pmod{5}. \tag{3.21}$$

Proof: Setting $t=5$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f_2^4}{f_1^5} = \left(\frac{f_2^3}{f_1^3}\right)\left(\frac{f_2}{f_1^2}\right). \tag{3.22}$$

Employing (2.11) and (2.12) and then using (2.17) in (3.22), we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n \equiv \frac{f_6^5f_9^6}{f_3^9f_{18}^3}+2q\frac{f_6^4f_9^3}{f_3^8}+4q^2\frac{f_6^3f_{18}^3}{f_3^7} \pmod{3}. \tag{3.23}$$

Extracting the terms involving q^{3n} and replacing q^3 by q from (3.23), we obtain

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv \frac{f_3^3}{f_2^4} \pmod{3}. \tag{3.24}$$

Again, extracting the terms involving q^{3n+1}, replacing q^3 by q and dividing by q from (3.23), we obtain

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv 2f_1f_2^4 \pmod{3}. \tag{3.25}$$

Simplifying (3.25) and then using (2.13), we obtain

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv 2\frac{f_6^2f_9^4}{f_3f_{18}^2}-2qf_6f_9f_{18}-4q^2\frac{f_3f_{18}^4}{f_9^2} \pmod{3}. \tag{3.26}$$

Extracting the terms involving q^{3n} and replacing q^3 by q from (3.26), we obtain

$$\sum_{n=0}^{\infty} a_5(9n+1)q^n \equiv 2\left(\frac{f_2^2}{f_1}\right)\left(\frac{f_3^4}{f_6^2}\right) \pmod{3}. \tag{3.27}$$

Employing (2.9) in (3.27), we obtain

$$\sum_{n=0}^{\infty} a_5(9n+1)q^n \equiv 2\frac{f_3^3 f_9^2}{f_6 f_{18}} + 2q\frac{f_3^4 f_{18}^2}{f_6^2 f_9} \pmod 3. \tag{3.28}$$

Extracting the terms involving q^{3n} and replacing q^3 by q from (3.28), we obtain

$$\sum_{n=0}^{\infty} a_5(27n+1)q^n \equiv 2\frac{f_3^3}{f_2^4} \pmod 3. \tag{3.29}$$

Combining (3.24) and (3.29), we arrived at (3.18). Extracting the terms involving q^{3n+1}, replacing q^3 by q and dividing by q from (3.28), we obtain

$$\sum_{n=0}^{\infty} a_5(27n+10)q^n \equiv 2f_1 f_2^4 \pmod 3. \tag{3.30}$$

Combining (3.25) and (3.30), we arrived at (3.19).
Extracting the terms involving q^{3n+2} from (3.28), we easily arrived at (3.20).
Rewrite (3.22), we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f_2^5}{f_1^5 f_2}. \tag{3.31}$$

Employing (2.10) and (2.18) with $p = 5$ and $h = 1$ in (3.31), we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n \equiv \frac{f_{50}^4}{f_5}\bigg(E(q^{10})^4 + q^2E(q^{10})^3 + 2q^4E(q^{10})^2 + 3q^6E(q^{10})$$

$$-\frac{3q^{10}}{E(q^{10})} + \frac{2q^{12}}{E(q^{10})^2} - \frac{q^{14}}{E(q^{10})^3} + \frac{q^{16}}{E(q^{10})^4}\bigg) \pmod 5. \tag{3.32}$$

Extracting the terms involving q^{5n+3}, replacing q^5 by q and dividing by q from (3.32), we easily arrive at (3.21). □

Theorem 3.4 *Let $p \geq 5$ be a prime with $\left(\frac{-12}{p}\right) = -1$ and $1 \leq r \leq p-1$. Then for any integers $\alpha \geq 0$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} a_6\left(4\cdot p^{2\alpha}n + \frac{13\cdot p^{2\alpha}-1}{6}\right)q^n \equiv 2f_1 f_4^3 \pmod 4, \tag{3.33}$$

$$a_6\left(4\cdot p^{2\alpha+1}(pn+r) + \frac{13\cdot p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod 4. \tag{3.34}$$

Proof: Setting $t = 6$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} a_6(n)q^n = \frac{f_2^5}{f_1^6} = f_2^5\cdot\left(\frac{1}{f_1^2}\right)^3. \tag{3.35}$$

Employing (2.6) in (3.35) and the applying (2.17), we obtain

$$\sum_{n=0}^{\infty} a_6(n)q^n \equiv \frac{f_8^{15}}{f_2^{10} f_{16}^6} + 6q\frac{f_8^9 f_4^2}{f_2^{10} f_{16}^2} \pmod 4 \tag{3.36}$$

Extracting the terms involving q^{2n} and replacing q^2 by q from (3.36), we obtain

$$\sum_{n=0}^{\infty} a_6(2n)q^n \equiv \frac{f_4}{f_1^2} \pmod 4. \tag{3.37}$$

Again, employing (2.6) in (3.37), we obtain

$$\sum_{n=0}^{\infty} a_6(2n)q^n \equiv \frac{f_4 f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^3 f_{16}^2}{f_2^5 f_8} \pmod 4. \tag{3.38}$$

Extracting the terms involving q^{2n+1}, replacing q^2 by q and dividing by q from (3.38), we obtain

$$\sum_{n=0}^{\infty} a_6(4n+2)q^n \equiv 2f_1 f_4^3 \pmod 4. \tag{3.39}$$

The remaining part of the proofs of (3.33) and (3.34) are similar to the proofs of (3.10) and (3.11), respectively and the desired results can be obtained by employing (2.14) and (2.15). □

Theorem 3.5 *Let $p>5$ be a prime with $\left(\frac{-6}{p}\right) = -1$ and $1 \le r \le p-1$. Then for any integers $\alpha \ge 0$ and $n \ge 0$, we have*

$$\sum_{n=0}^{\infty} a_6\left(4\cdot p^{2\alpha} n + \frac{7\cdot p^{2\alpha}-1}{6}\right) q^n \equiv 2f_1 f_2^3 \pmod 4, \tag{3.40}$$

$$a_6\left(4\cdot p^{2\alpha+1}(pn+r) + \frac{7\cdot p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod 4. \tag{3.41}$$

Proof: Extracting the terms involving q^{2n+1}, replacing q^2 by q and dividing by q from (3.36), we obtain

$$\sum_{n=0}^{\infty} a_6(2n+1)q^n \equiv 2f_2 f_4^3 \pmod 4. \tag{3.42}$$

Extracting the terms involving q^{2n} and replacing q^2 by q from (3.42), we obtain

$$\sum_{n=0}^{\infty} a_6(4n+1)q^n \equiv 2f_1 f_2^3 \pmod 4. \tag{3.43}$$

The remaining of the proofs of (3.40) and (3.41) are analogous to the proofs of (3.10) and (3.11) and the desired results can be obtained by using (2.14) and (2.15), respectively. □

Corollary 3.1 *For $n \ge 0$, we have*

$$a_6(4n+3) \equiv 0 \pmod 4. \tag{3.44}$$

Proof: Extracting the terms involving q^{2n+1}, replacing q^2 by q and dividing by q from (3.42),we easily arrived at (3.44). □

Theorem 3.6 *Let $p>5$ be a prime with $\left(\frac{-1}{p}\right) = -1$ and $1 \le r \le p-1$. Then for any integers $\alpha \ge 0$ and $n \ge 0$, we have*

$$\sum_{n=0}^{\infty} a_8\left(2\cdot p^{2\alpha} n + \frac{5\cdot p^{2\alpha}-1}{4}\right) q^n \equiv 8\psi(q) f_4^3 \pmod{16}, \tag{3.45}$$

$$a_8\left(2\cdot p^{2\alpha+1}(pn+r) + \frac{5\cdot p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{16}. \tag{3.46}$$

Proof: Setting $t = 8$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} a_8(n)q^n = \frac{f_2^7}{f_1^8}. \tag{3.47}$$

Employing (2.8) and (2.17) in (3.47), we obtain

$$\sum_{n=0}^{\infty} a_8(n)q^n \equiv \frac{f_4^{28}}{f_2^{21}f_8^8} + 8q\frac{f_4^{16}}{f_2^{17}} \pmod{16}. \tag{3.48}$$

Extracting the terms involving q^{2n+1}, replacing q^2 by q and dividing by q from (3.48),we obtain

$$\sum_{n=0}^{\infty} a_8(2n+1)q^n \equiv 8\frac{f_2^{16}}{f_1^{17}} \pmod{16}. \tag{3.49}$$

Simplifying (3.49) and the using (2.3), we obtain

$$\sum_{n=0}^{\infty} a_8(2n+1)q^n \equiv 8\psi(q)f_4^3 \pmod{16}, \tag{3.50}$$

which is $\alpha = 0$ case of (3.45). Assume that (3.45) is true for some integer $\alpha \geq 0$. Employing (2.15) and (2.16) in (3.45), we obtain

$$\sum_{n=0}^{\infty} a_8\left(2\cdot p^{2\alpha}n + \frac{5\cdot p^{2\alpha}-1}{4}\right)q^n$$
$$\equiv \left[\sum_{j=0}^{(p-3)/2} q^{(j^2+j)/2} F\left(q^{(p^2+(2j+1)p)/2}, q^{(p^2-(2j+1)p)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2})\right]$$
$$\times\left[\sum_{\substack{l=0\\ l\neq(p-1)/2}}^{(p-1)} (-1)^l q^{(l(l+1))/2} \sum_{n=0}^{\infty}(-1)^n(2pn+2l+1)q^{pn\cdot(pn+2l+1)/2} + p(-1)^{(p-1)/2}q^{(p^2-1)/8}f_{p^2}^3\right]$$
$$\pmod{16}. \tag{3.51}$$

Consider the congruences

$$\left(\frac{j^2+j}{2}\right) + 4\left(\frac{l^2+l}{2}\right) \equiv 5\left(\frac{p^2-1}{8}\right) \pmod{p},$$

which is similar to

$$(2j+1)^2 + (4l+l)^2 \equiv 0 \pmod{p}. \tag{3.52}$$

Since $\left(\frac{-1}{p}\right) = -1$, the above congruence has only solution for $j = l = \frac{(\pm p-1)}{2}$. Therefore, extracting the terms involving $q^{pn+5(p^2-1)/8}$ from both sides of (3.51), dividing throughout by $q^{5(p^2-1)/8}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_8\left(2\cdot p^{2\alpha+1}n + \frac{5\cdot p^{2\alpha+2}-1}{4}\right)q^n \equiv 8\psi(q^p)f_{4p}^3 \pmod{16}. \tag{3.53}$$

Extracting the terms involving q^{pn} from (3.53) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_8\left(2\cdot p^{2\alpha+2}n + \frac{5\cdot p^{2\alpha+2}-1}{4}\right)q^n \equiv 8\psi(q)f_4^3 \pmod{16}, \tag{3.54}$$

which is the $\alpha+1$ case of (3.45). Hence, the proof of (3.45) is complete. Extracting the terms involving q^{pn+r}, for $1 \leq r \leq p-1$, from (3.53), we arrive at (3.46). □

Theorem 3.7 *For all integers $j \geq 0$ and $n \geq 0$, we have*

$$a_{5j+1}(5n+4) \equiv 0 \pmod{5}, \tag{3.55}$$

$$a_{11j+1}(11n+6) \equiv 0 \pmod{11}. \tag{3.56}$$

Proof: Setting $t = 5j+1$ in (1.5), we obtain

$$\sum_{n=0}^{\infty} a_{5j+1}(n)q^n = \frac{f_2^{5j}}{f_1^{5j+1}} = \frac{f_2^{5j}}{f_1^{5j}} \cdot \frac{1}{f_1}. \tag{3.57}$$

Employing (2.18) with $p = 5$ and $h = 1$ and (1.1) in (3.57), we obtain

$$\sum_{n=0}^{\infty} a_{5j+1}(n)q^n = \frac{f_{10}^j}{f_5^j} \sum_{n=0}^{\infty} p(n)q^n \pmod{5}. \tag{3.58}$$

Now (3.55) and (3.56) follow immediately from (3.58) and (1.3). □

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Declarations

Conflict of Interest. The authors declare that there is no conflict of interest regarding the publication of this article.
Human and animal rights. The authors declare that there is no research involving human participants or animals in the contained of this paper.
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