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Asymptotic Behavior and Numerical Analysis for a Thermoelastic-Bresse System with Second Sound

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ABSTRACT: In this study, we investigate the behavior of a linear one-dimensional thermoelastic Bresse system that incorporates second sound phenomena. We begin by establishing that the system is well-posed and identifying the conditions necessary for it to demonstrate exponential stability, which depend on certain parameters of the system. Our proof utilizes semigroup theory and a hybrid methodology that combines energy techniques with frequency domain analysis. Subsequently, we introduce a finite element approximation for the system and demonstrate that the associated discrete energy decreases over time. Additionally, we derive several a priori error estimates to assess the accuracy of our approximation. Finally, we validate our theoretical findings by demonstrating that the numerical results align with our established theoretical predictions.

Key Words: Bresse system, well-posedness, semigroup theory, energy method, frequency domain approach, numerical approximation, finite element method.

Contents

1. Introduction

Over the past few decades, various dynamic equations have been employed as mathematical models to represent engineering systems. This study focuses on the circular arch problem, commonly referred to as the Bresse system (see [1] for further details), which is coupled with heat conduction phenomena characteristic of second sound. Elastic structures shaped like arches have been extensively studied across multiple fields, including engineering, architecture, marine engineering, and aeronautics. Understanding the properties that link the energy behavior of solutions to the corresponding dynamic model with the system parameters is of significant interest. The dynamic equations relevant to this research are presented as follows:

$$\begin{cases} \rho_1\phi_{tt} - k\left(\phi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\phi\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\phi_x + \psi + l\,w\right) + \gamma\theta_x = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\phi\right)_x + lk\left(\phi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_3\theta_t + q_x + \gamma\psi_{xt} = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \tau q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty)\,, \end{cases} \tag{1.1}$$

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with the initial and boundary conditions

$$\begin{cases} \phi(x,0)=\phi_0(x)\,,\ \phi_t(x,0)=\phi_1(x)\,,\ \theta(x,0)=\theta_0(x) & \text{for } x\in(0,1)\,,\\ \psi(x,0)=\psi_0(x)\,,\ \psi_t(x,0)=\psi_1(x)\,,\ q(x,0)=q_0(x) & \text{for } x\in(0,1)\,,\\ w(x,0)=w_0(x)\,,\ w_t(x,0)=w_1(x) & \text{for } x\in(0,1)\,,\\ \phi(0,t)=\psi_x(0,t)=w_x(0,t)=\theta(0,t)=0 & \forall t\geq 0,\\ \phi_x(1,t)=\psi(1,t)=w(1,t)=q(1,t)=0 & \forall t\geq 0. \end{cases} \tag{1.2}$$

where ρ_1, ρ_2, ρ_3, b, k, k_0, τ, β, γ and l are positive constants, the initial data ϕ_0, ϕ_1, ψ_0, ψ_1, w_0, w_1, θ_0 and q_0 an element of an appropriate Hilbert space, and the unknowns in the equations (1.1) – (1.2) consist of the following variables:

$$(\phi,\psi,w,\theta,q):(0,1)\times(0,\infty)\to\mathbb{R}^5. \tag{1.3}$$

In recent years, numerous researchers have investigated the well-posedness and stability of Bresse systems. Various types of feedback mechanisms have been analyzed, leading to different stability outcomes that depend on several factors. These factors include the nature and quantity of feedbacks, the regularity of initial conditions, and how the relationships between the coefficients in the dynamic system influence the overall stability of total energy.

For a comprehensive understanding, we direct readers to several key studies: references [1], [3], [9,11], and [20,28] discuss cases involving (linear or nonlinear, global or local) frictional damping. Additionally, references [22], [12], [13], and [14] focus on systems with memory effects.

Recently, Keddi et al. [17] demonstrated that the thermoelastic Bresse system characterized by second sound, as described in equations (1.1) – (1.2), exhibits exponential stability under certain conditions.

$$\xi=\left[\left(1-\frac{\tau k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)-\frac{\tau\gamma^2}{b}\right]=0\ ,\ k=k_0 \text{ and } l \text{ small}, \tag{1.4}$$

and the solutions for (1.1) – (1.2) is not exponentially stable if

$$\xi\neq 0 \ \text{ or } k=k_0. \tag{1.5}$$

Moreover, when

$$\xi\neq 0\ \ ,\ k=k_0 \text{ and } l \text{ small}, \tag{1.6}$$

they proved the polynomial decay.

In this paper, we first prove exponential stability under the following explicit condition:

$$\xi=\left(1-\frac{\tau k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)-\frac{\tau\gamma^2}{b}=0,\ k=k_0 \text{ and } l\neq\frac{\pi}{2}+n\pi\ \forall n\in\mathbb{Z}, \tag{1.7}$$

Our method of proof is based on the frequency techniques combined with the energy method without requiring the smallness condition on l as in [17]. Secondly, we introduce the numerical approximation of the solution of (1.1) using a finite element method, and then we prove the decay of the discrete energy. In addition, we establish some error estimates. It is worth mentioning that the novelty of this paper lies in the proposed approximation scheme based on finite element analysis, as well as the use of an iterative method to solve the resulting discrete system of implicit equations.

The structure of the paper is as follows: In Section 2, we present the well-posedness of the problem (1.1) – (1.2) without providing a proof. Section 3 focuses on the exponential decay of stability for (1.1) – (1.2). In Section 4, we introduce finite element analysis and provide numerical simulations to validate the accuracy of the approximation, maintaining the same parameter conditions as in the continuous model.

2. Existence, Uniqueness, and Stability of the Solution

In this section, we establish a result on the existence and uniqueness of solutions to problem (1.1)–(1.2) using semigroup theory. To this end, we define the vector function $\Psi=(\phi,u,\psi,v,w,y,\theta,q)^T$, where $u=\phi_t$, $v=\psi_t$, and $y=w_t$. Consequently, the system (1.1) – (1.2) can be reformulated as follows:

$$\begin{cases} \Psi_t=\mathcal{A}\Psi,\\ \Psi(0)=\Psi_0=(\phi_0,u_0,\psi_0,v_0,w_0,y_0,\theta_0,q_0)^T \end{cases} \tag{2.1}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Psi=\begin{bmatrix} u \\ \frac{k}{\rho_1}\left(\phi_x+\psi+l\,w\right)_x+\frac{lk_0}{\rho_1}\left(w_x-l\phi\right) \\ v \\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\phi_x+\psi+l\,w\right)-\frac{\gamma}{\rho_3}\theta_x \\ y \\ \frac{k_0}{\rho_1}\left(w_x-l\phi\right)_x-\frac{lk}{\rho_1}\left(\phi_x+\psi+l\,w\right) \\ -\frac{1}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\tau}q-\frac{1}{\tau}\theta_x \end{bmatrix} \tag{2.2}$$

We consider the following spaces

$$\begin{aligned} \mathbb{H}_*^1(0,1) &= \left\{f\in H^1(0,1)\mid f(0)=0\right\}, \\ \widetilde{\mathbb{H}}_*^1(0,1) &= \left\{f\in H^1(0,1)\mid f(1)=0\right\}, \\ \mathbb{H}_*^2(0,1) &= H^2(0,1)\cap\mathbb{H}_*^1(0,1), \\ \widetilde{\mathbb{H}}_*^2(0,1) &= H^2(0,1)\cap\widetilde{\mathbb{H}}_*^1(0,1), \end{aligned}$$

and the Hilbert space

$$\mathcal{H}=\mathbb{H}_*^1(0,1)\times L^2(0,1)\times\widetilde{\mathbb{H}}_*^1(0,1)\times L^2(0,1)\times\widetilde{\mathbb{H}}_*^1(0,1)\times\left(L^2(0,1)\right)^3, \tag{2.3}$$

equipped $<.,.>_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}}$ the inner product and the energy norm defined by

$$\begin{aligned} \langle\Psi_1,\Psi_2\rangle_{\mathcal{H}} &= k\langle\phi_{1x}+\psi_1+l\,w_1,\phi_{2x}+\psi_2+l\,w_2\rangle+k_0\langle w_{1x}-l\phi_1,w_{2x}-l\phi_2\rangle \\ &\quad+\rho_1\langle u_1,u_2\rangle+b\langle\psi_{1x},\psi_{2x}\rangle+\rho_2\langle v_1,v_2\rangle+\rho_1\langle y_1,y_2\rangle \\ &\quad+\rho_3\langle\theta_1,\theta_2\rangle+\tau\langle q_1,q_2\rangle. \\ \|\Psi\|_{\mathcal{H}}^2 &= k\|\phi_x+\psi+l\,w\|^2+k_0\|w_x-l\phi\|^2+\rho_1\|u\|^2+b\|\psi_x\|^2+\rho_2\|v\|^2 \\ &\quad+\rho_1\|y\|^2+\rho_3\|\theta\|^2+\tau\|q\|^2. \end{aligned}$$

where $<.,.>$ and $\|.\|$ denote the scalar product and the norm of $L^2(0,1)$. Then $\mathcal{A}$, formally given in (2.2), with domain

$$D(\mathcal{A})=\left\{\begin{array}{c} \Psi\in\mathcal{H}\mid\phi\in\mathbb{H}_*^2(0,1);\ \psi,w\in\widetilde{\mathbb{H}}_*^2(0,1);\ u,\theta\in\mathbb{H}_*^1(0,1); \\ v,y,q\in\widetilde{\mathbb{H}}_*^1(0,1);\ \phi_x(1)=w_x(0)=\psi_x(0)=0 \end{array}\right\}. \tag{2.4}$$

It is clear from the conditions in (1.2) that

$$\begin{aligned} 0 &= k\|\phi_x+\psi+l\,w\|^2+k_0\|w_x-l\phi\|^2+\rho_1\|u\|^2+b\|\psi_x\|^2 \\ &\quad+\rho_2\|v\|^2+\rho_1\|y\|^2+\rho_3\|\theta\|^2+\tau\|q\|^2, \end{aligned}$$

implies that

$$u=\psi=v=y=\theta=q=0, \tag{2.5}$$

and

$$\phi(x)=c\sin(lx)\ \text{ and }\ w(x)=-c\cos(lx), \tag{2.6}$$

with $c \in \mathbb{R}$. Furthermore, using the conditions in (1.2), we get $\phi = w = 0$ if

$$l \neq \frac{\pi}{2} + n\pi;\ \forall n \in \mathbb{Z}. \tag{2.7}$$

Assume that condition (2.7) holds. Consequently, we conclude that $\mathcal{H}$ qualifies as a Hilbert space and that $D(\mathcal{A})$ is dense within $\mathcal{H}$.

Theorem 2.1 *Let Ψ_0 be an element of the Hilbert space $\mathcal{H}$. Then, there exists a unique weak solution. $\Psi \in C(\mathbb{R}^+, \mathcal{H})$ of system (1.1) − (1.2). Moreover, if $\Psi_0 \in D(\mathcal{A})$, then $\Psi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof: It has been demonstrated in ([17]) that the operator $\mathcal{A}$ qualifies as a maximal dissipative operator within the energy space $\mathcal{H}$, and it is responsible for generating a C_0-semigroup of contractions in $\mathcal{H}$. □

3. Exponential Stability

In this section, we demonstrate the exponential decay based on the following specific condition

$$\begin{cases} \left(\left(1 - \frac{\tau k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\tau\gamma^2}{b}\right) = 0, \\ k = k_0, \\ l^2 \neq \frac{\pi}{2} + n\pi;\ \forall n \in \mathbb{Z}. \end{cases} \tag{3.1}$$

Theorem 3.1 *We take it as given that the conditions specified in (3.1) are satisfied. Consequently, the semigroup linked to (2.1) exhibits exponential stability.*

Proof. To establish our findings, we will adopt the methodology presented in [16] and [21], thus it is necessary to demonstrate that

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{3.2}$$

and

$$\sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\| < +\infty. \tag{3.3}$$

Given that $D(\mathcal{A})$ is compactly embedded in $\mathcal{H}$, it follows that the operator $(I - \mathcal{A})^{-1}$ is a compact. This leads to the conclusion that the spectrum of $\mathcal{A}$ is discrete. Furthermore, since the resolvent of the operator $\mathcal{A}$ is compact in $\mathcal{H}$, we can apply results from [5] work to conclude that the system described by equations (1.1) − (1.2) exhibits strong stability if and only if $\mathcal{A}$ lacks pure imaginary eigenvalues.

To verify the condition (3.2), we proceed as follow: let $a \in \mathbb{R}^*$ and let $\Psi \in D(\mathcal{A})$, with

$$\mathcal{A}\Psi = i\,a\,\Psi, \tag{3.4}$$

which is equivalent to

$$\begin{cases} u = ia\phi\,,\ v = ia\psi\,,\ y = iaw, \\ k\left(\phi_x + \psi + l\,w\right)_x + lk_0\left(w_x - l\phi\right) = ia\rho_1 u, \\ b\psi_{xx} - k\left(\phi_x + \psi + l\,w\right) - \gamma\theta_x = ia\rho_2 v, \\ k_0\left(w_x - l\phi\right)_x - lk\left(\phi_x + \psi + l\,w\right) = ia\rho_1 y, \\ -q_x - \gamma v_x = ia\rho_3\theta, \\ -\beta q - \theta_x = ia\tau q. \end{cases} \tag{3.5}$$

Computing $Re\left\langle \mathcal{A}\Psi, \Psi\right\rangle_{\mathcal{H}}$ and using (3.4), we deduce that

$$q = 0. \tag{3.6}$$

Taking into account that $\theta \in \mathbb{H}^1_*(0,1)$ and using $(3.5)_8$ and (3.6), we deduce that

$$\theta = 0. \tag{3.7}$$

By using $(3.5)_7$, (3.6) and (3.7)

$$v_x = 0, \tag{3.8}$$

and with $(3.5)_2$, we obtain that

$$\psi_x = 0. \tag{3.9}$$

As $\psi \in \widetilde{\mathbb{H}}_*^1(0,1)$, we have

$$\psi = 0 \text{ and } v = 0. \tag{3.10}$$

Using $(3.5)_5$, (3.7) and (3.10), we get

$$\phi_x + l\, w = 0. \tag{3.11}$$

Using $(3.5)_1$, $(3.5)_4$, (3.10) and (3.11), we get

$$w_x + \left(\frac{a^2\rho_1}{k_0 l} - l\right)\phi = 0. \tag{3.12}$$

Using $(3.5)_3$, $(3.5)_6$, (3.10) and (3.11), we get

$$w_{xx} + \left(\frac{a^2\rho_1}{k_0} + l^2\right) w = 0. \tag{3.13}$$

As, we have

$$w(1) = w_x(0) = 0, \tag{3.14}$$

so, we have

$$w(x) = C_1 \cos\left(\sqrt{\left(\frac{a^2\rho_1}{k_0} + l^2\right)}\; x\right). \tag{3.15}$$

Case 1: if $C_1 = 0$ then with (3.15), we have

$$w = 0, \tag{3.16}$$

from (3.11) and $(3.5)_3$ we obtain

$$\phi = 0 \text{ and } y = 0. \tag{3.17}$$

By using $(3.5)_1$we obtain

$$u = 0, \tag{3.18}$$

thus, we get

$$\Psi = 0. \tag{3.19}$$

Case 2 : if $C_1 \neq 0$ then with (3.15), using the fact that $w(1) = 0$, then we have necessary

$$\sqrt{\left(\frac{a^2\rho_1}{k_0} + l^2\right)} = \left(\frac{\pi}{2} + n\pi\right) \text{ and } n \in \mathbb{Z}, \tag{3.20}$$

which means that:

$$l^2 = -\frac{a^2\rho_1}{k_0} + \left(\frac{\pi}{2} + n\pi\right)^2 \text{ and } n \in \mathbb{Z}, \tag{3.21}$$

From (3.20), (3.11), by incorporating the boundary conditions, we obtained

$$\phi(x) = -\frac{C_1 l}{\left(\frac{\pi}{2} + n\pi\right)} \sin\left(\frac{a^2\rho_1}{kl} - l^2\right) x. \tag{3.22}$$

Using (3.12), (3.20) and (3.22), we obtain that

$$a = 0, \text{ and } l^2 = \left(\frac{\pi}{2} + n\pi\right)^2,$$

which is a contradiction with (2.7).

So as a conclusion we have $C_1 = 0$ and therefore we find $\Psi = 0$ then we obtain (3.2).

To demonstrate the validity of (3.3), we will use a proof by contradiction. We begin by assuming that the statement (3.3) is not true. Under this assumption, it follows that there exists a real sequence $(\lambda_n)_{n\in\mathbb{N}}$ and a sequence $(\Psi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{A})$, such that

$$\|\Psi_n\|_{\mathcal{H}} = 1, \tag{3.23}$$

$$|\lambda_n| \to \infty, \tag{3.24}$$

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})\Psi_n\|_{\mathcal{H}} = 0, \tag{3.25}$$

i.e., we have the following convergence:

$$\begin{cases} i\lambda_n\phi_n - u_n \to 0 \quad \text{in} \quad \mathbb{H}^1_*(0,1), \\ i\lambda_n\rho_1 u_n - k(\phi_{n,x} + \psi_n + l\, w_n)_x - lk_0(w_{n,x} - l\phi_n) \to 0 \quad \text{in} \quad L^2(0,1), \\ i\lambda_n\psi_n - v_n \to 0 \quad \text{in} \quad \widetilde{\mathbb{H}}^1_*(0,1), \\ i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k(\phi_{n,x} + \psi_n + l\, w_n) + \gamma\theta_{n,x} \to 0 \quad \text{in} \quad L^2(0,1), \\ i\lambda_n w_n - y_n \to 0 \quad \text{in} \quad \widetilde{\mathbb{H}}^1_*(0,1), \\ i\lambda_n\rho_1 y_n - k_0(w_{n,x} - l\phi_n)_x + lk(\phi_{n,x} + \psi_n + l\, w_n) \to 0 \quad \text{in} \quad L^2(0,1), \\ i\lambda_n\rho_3\theta_n + q_{n,x} + \gamma v_{n,x} \to 0 \quad \text{in} \quad L^2(0,1), \\ i\lambda_n\tau q_n + \beta q_n + \theta_{n,x} \to 0 \quad \text{in} \quad L^2(0,1), \end{cases} \tag{3.26}$$

We will now verify condition (3.3) by deriving a contradiction with (3.23). Our proof is structured into several steps:

Step 1. Taking the inner product of $(i\lambda_n I - \mathcal{A})\Psi_n$ with Ψ_n in $\mathcal{H}$, we get

$$\Re\langle(i\lambda_n I - \mathcal{A})\Psi_n, \Psi_n\rangle_{\mathcal{H}} = \beta\|q\|^2_{L^2(0,1)}, \tag{3.27}$$

using (3.25), we deduce that

$$q_n \to 0 \text{ in } L^2(0,1), \tag{3.28}$$

applying triangular inequality, we get

$$\left\|\frac{\theta_{n,x}}{\lambda_n}\right\| \le \frac{1}{|\lambda_n|}\|i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}\| + \left\|i\tau q_n + \frac{\beta}{\lambda_n}q_n\right\|. \tag{3.29}$$

From (3.24), $(3.26)_8$ and (3.28), we deduce that

$$\frac{\theta_{n,x}}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{3.30}$$

Step 2. Multiplying $(3.26)_1$ by $\overline{\dfrac{i\phi_n}{\lambda_n}}$, we obtain

$$\|\phi_n\|^2 - \frac{1}{\lambda_n}\langle u_n, i\phi_n\rangle \to 0. \tag{3.31}$$

Multiplying $(3.26)_3$ by $\overline{\dfrac{i\psi_n}{\lambda_n}}$, we get

$$\|\psi_n\|^2 - \frac{1}{\lambda_n}\langle v_n, i\psi_n\rangle \to 0. \tag{3.32}$$

Multiplyin g$(3.26)_5$ by $\overline{\dfrac{iw_n}{\lambda_n}}$, we have

$$\|w_n\|^2 - \frac{1}{\lambda_n}\langle y_n, iw_n\rangle \to 0, \tag{3.33}$$

using (3.23) and (3.24), we deduce that

$$\phi_n \to 0 \text{ in } L^2(0,1) \tag{3.34}$$

$$\psi_n \to 0 \text{ in } L^2(0,1) \tag{3.35}$$

$$w_n \to 0 \text{ in } L^2(0,1) \tag{3.36}$$

Step 3. Multiplying $(3.26)_7$ by $\overline{\dfrac{\theta_n}{\lambda_n}}$ and integrating by parts, we have

$$i\rho_3\|\theta_n\|^2 + \left[\left\langle q_n, \frac{\theta_n}{\lambda_n}\right\rangle\right]_0^1 - \left\langle q_n, \frac{\theta_{n,x}}{\lambda_n}\right\rangle + \gamma\left[\left\langle v_n, \frac{\theta_n}{\lambda_n}\right\rangle\right]_0^1 + \gamma\left\langle v_n, \frac{\theta_{n,x}}{\lambda_n}\right\rangle \to 0, \tag{3.37}$$

using boundary conditions, (3.23), (3.28) and (3.30), we find

$$\theta_n \to 0 \text{ in } L^2(0,1). \tag{3.38}$$

By applying the triangle inequality, we obtain

$$\begin{aligned}\left\|\frac{\phi_{n,xx}}{\lambda_n}\right\| \quad \le \quad & \left\|\frac{1}{k\lambda_n}\left(i\lambda_n\rho_1 u_n - k(\phi_{n,x}+\psi_n+l\,w_n)_x - lk_0(w_{n,x}-l\phi_n)\right)\right\| \\ & + \left\|\frac{i\rho_1}{k}u_n - \frac{1}{\lambda_n}(\psi_{n,x}+l\,w_{n,x}) - \frac{lk_0}{k\lambda_n}(w_{n,x}-l\phi_n)\right\|,\end{aligned}$$

by (3.23), (3.24) and $(3.26)_2$, we obtain

$$\left(\left\|\frac{\phi_{n,xx}}{\lambda_n}\right\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{3.39}$$

multiplying $(3.26)_7$ by $\overline{\dfrac{i\psi_{n,x}}{\lambda_n}}$, we have

$$\rho_3\langle\theta_n, \psi_{n,x}\rangle + \frac{1}{\lambda_n}\langle q_{n,x}, i\psi_{n,x}\rangle - \gamma\left\langle (i\lambda_n\psi_{n,x} - v_{n,x}), \frac{i\psi_{n,x}}{\lambda_n}\right\rangle + \gamma\|\psi_{n,x}\|^2 \to 0, \tag{3.40}$$

using (3.23), $(3.26)_3$ and integration by parts, we obtain

$$\rho_3\langle\theta_n, \psi_{n,x}\rangle + \frac{1}{\lambda_n}\left[\langle q_n, i\psi_{n,x}\rangle\right]_0^1 - \left\langle q_n, \frac{i\psi_{n,xx}}{\lambda_n}\right\rangle + \gamma\|\psi_{n,x}\|^2 \to 0, \tag{3.41}$$

again, using boundary conditions, (3.28), (3.38) and (3.39), we deduce from (3.41) that

$$\psi_{n,x} \to 0 \text{ in } L^2(0,1), \tag{3.42}$$

from $(3.26)_3$, we have

$$\frac{v_{n,x}}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{3.43}$$

As v_n *in* $\widetilde{\mathbb{H}}_*^1(0,1)$, then by (3.43)

$$\frac{v_n}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{3.44}$$

Step 4. Multiplying $(3.25)_2$ by $\overline{\dfrac{iv_n}{\lambda_n}}$ and integration by parts, we obtain

$$\rho_2 \|v_n\|^2 - b\left[\left\langle \psi_{n,x}, \frac{iv_n}{\lambda_n}\right\rangle\right]_0^1 + b\left\langle \psi_{n,x}, \frac{iv_{n,x}}{\lambda_n}\right\rangle + \frac{k}{\lambda_n}\left\langle (\phi_{n,x} + \psi_n + l\,w_n), iv_n\right\rangle + \gamma\left\langle \frac{\theta_{n,x}}{\lambda_n}, iv_n\right\rangle \to 0. \quad (3.45)$$

Using the boundary conditions, (3.23), (3.24) and (3.30), then we get

$$\rho_2 \|v_n\|^2 + \frac{b}{\lambda_n}\langle i\psi_{n,x}, (i\lambda_n\psi_{n,x} - v_{n,x})\rangle - b\|\psi_{n,x}\|^2 \to 0, \quad (3.46)$$

using (3.24), $(3.26)_3$ and (3.42), then we have

$$v_n \to 0 \text{ in } L^2(0,1). \quad (3.47)$$

By $(3.26)_3$ and (3.47), we deduce

$$\lambda_n\psi_n \to 0 \text{ in } L^2(0,1). \quad (3.48)$$

Step 5. We have

$$\begin{aligned}
\langle \theta_{n,x}, \phi_{n,x}\rangle &= \langle (i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}), \phi_{n,x}\rangle - \beta\langle q_n, \phi_{n,x}\rangle - \langle i\lambda_n\tau q_n, \phi_{n,x}\rangle \quad (3.49)\\
&= \langle (i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}), \phi_{n,x}\rangle - \beta\langle q_n, \phi_{n,x}\rangle + \tau\langle q_n, i\lambda_n\phi_{n,x}\rangle\\
&= \langle (i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}), \phi_{n,x}\rangle - \beta\langle q_n, \phi_{n,x}\rangle - \tau\langle q_{n,x}, (i\lambda_n\phi_n - u_n)\rangle - \tau\langle q_{n,x}, u_n\rangle\\
&= \langle (i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}), \phi_{n,x}\rangle - \beta\langle q_n, \phi_{n,x}\rangle - \tau\langle q_{n,x}, (i\lambda_n\phi_n - u_n)\rangle\\
&\quad -\tau\langle (i\lambda_n\rho_3\theta_n + q_{n,x} + \gamma v_{n,x}), u_n\rangle + \tau\rho_3\langle i\lambda_n\theta_n, u_n\rangle + \tau\gamma\langle v_{n,x}, u_n\rangle\\
&= \langle (i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}), \phi_{n,x}\rangle - \beta\langle q_n, \phi_{n,x}\rangle + \tau\langle q_n, (i\lambda_n\phi_n - u_n)_x\rangle\\
&\quad -\tau\langle (i\lambda_n\rho_3\theta_n + q_{n,x} + \gamma v_{n,x}), u_n\rangle + \frac{\tau k\rho_3}{\rho_1}\langle \theta_{n,x}, \phi_{n,x}\rangle\\
&\quad -\frac{\tau\rho_3}{\rho_1}\left\langle \theta_n, \left[i\lambda_n\rho_1 u_n - k(\phi_{n,x} + \psi_n + l\,w_n)_x - lk_0(w_{n,x} - l\phi_n)\right]\right\rangle\\
&\quad -\frac{\tau k\rho_3}{\rho_1}\langle \theta_n, (\psi_n + l\,w_n)_x\rangle - \frac{\tau lk_0\rho_3}{\rho_1}\langle \theta_n, (w_{n,x} - l\phi_n)\rangle + \tau\gamma\langle v_{n,x}, u_n\rangle,
\end{aligned}$$

using (3.23), $(3.26)_1$, $(3.26)_2$, $(3.26)_7$, $(3.26)_8$, (3.28), (3.38) and (3.49), we deduce that

$$\left(1 - \frac{\tau k\rho_3}{\rho_1}\right)\langle \theta_{n,x}, \phi_{n,x}\rangle - \tau\gamma\langle v_{n,x}, u_n\rangle \to 0 \text{ in } L^2(0,1). \quad (3.50)$$

Step 6. Multiplying $(3.26)_2$ by $\overline{\psi_{n,x}}$, and using (3.23) and (3.42), we obtain

$$\langle i\lambda_n\rho_1 u_n, \psi_{n,x}\rangle - \langle k\phi_{n,xx}, \psi_{n,x}\rangle \to 0. \quad (3.51)$$

On the other hand we have

$$\begin{aligned}
\langle i\lambda_n\rho_1 u_n,\psi_{n,x}\rangle-\langle k\phi_{n,xx},\psi_{n,x}\rangle &= -\rho_1\langle u_n,(i\lambda_n\psi_{n,x}-v_{n,x})\rangle-\rho_1\langle u_n,v_{n,x}\rangle+k\langle\phi_{n,x},\psi_{n,xx}\rangle\\
&= -\rho_1\langle u_n,(i\lambda_n\psi_{n,x}-v_{n,x})\rangle-\rho_1\langle u_n,v_{n,x}\rangle\\
&\quad -\frac{k}{b}\langle\phi_{n,x},[i\lambda_n\rho_2 v_n-b\psi_{n,xx}+k(\phi_{n,x}+\psi_n+l\,w_n)+\gamma\theta_{n,x}]\rangle\\
&\quad +\frac{k^2}{b}\langle\phi_{n,x},(\psi_n+l\,w_n)\rangle+\frac{k\rho_2}{b}\langle\phi_{n,x},i\lambda_n v_n\rangle\\
&\quad +\frac{\gamma k}{b}\langle\phi_{n,x},\theta_{n,x}\rangle+\frac{k^2}{b}\|\phi_{n,x}\|^2 \qquad (3.52)\\
&= -\rho_1\langle u_n,(i\lambda_n\psi_{n,x}-v_{n,x})\rangle+\frac{k^2}{b}\langle\phi_{n,x},(\psi_n+l\,w_n)\rangle\\
&\quad -\frac{k}{b}\langle\phi_{n,x},[i\lambda_n\rho_2 v_n-b\psi_{n,xx}+k(\phi_{n,x}+\psi_n+l\,w_n)+\gamma\theta_{n,x}]\rangle\\
&\quad -\frac{k\rho_2}{b}\langle(i\lambda_n\phi_{n,x}-u_{n,x}),v_n\rangle+\left(\frac{k\rho_2}{b}-\rho_1\right)\langle u_n,v_{n,x}\rangle\\
&\quad +\frac{\gamma k}{b}\langle\phi_{n,x},\theta_{n,x}\rangle+\frac{k^2}{b}\|\phi_{n,x}\|^2.
\end{aligned}$$

Using (3.23), $(3.26)_1$, $(3.26)_3$,$(3.26)_4$, (3.35), (3.36), (3.51) and (3.52), we obtain

$$\left(\frac{k\rho_2}{b}-\rho_1\right)\langle u_n,v_{n,x}\rangle+\frac{\gamma k}{b}\langle\phi_{n,x},\theta_{n,x}\rangle+\frac{k^2}{b}\|\phi_{n,x}\|^2\to 0, \tag{3.53}$$

with (3.50) and (3.53), we deduce

$$\left[\left(\frac{k\rho_2}{b}-\rho_1\right)\left(1-\frac{\tau k\rho_3}{\rho_1}\right)+\frac{\tau k\gamma^2}{b}\right]\Re\langle\phi_{n,x},\theta_{n,x}\rangle+\frac{\tau\gamma k^2}{b}\|\phi_{n,x}\|^2\to 0, \tag{3.54}$$

then we have

$$\left[\left(1-\frac{\tau k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)-\frac{\tau\gamma^2}{b}\right]\Re\langle\phi_{n,x},\theta_{n,x}\rangle+\frac{\tau\gamma k}{b}\|\phi_{n,x}\|^2\to 0, \tag{3.55}$$

using $(3.1)_1$ we get

$$\phi_{n,x}\to 0 \text{ in } L^2(0,1). \tag{3.56}$$

Multiplying $(3.26)_2$ by $\overline{\phi_n}$ and using (3.23), (3.35) and (3.36), we have

$$-\rho_1\langle u_n,i\lambda_n\phi_n-u_n\rangle-\rho_1\|u_n\|^2-k\langle\phi_{n,xx},\phi_n\rangle\to 0, \tag{3.57}$$

integrating by parts and using (3.56), we obtain

$$u_n\to 0 \text{ in } L^2(0,1), \tag{3.58}$$

using again $(3.1)_1$, we get

$$\lambda_n\phi_n\to 0 \text{ in } L^2(0,1). \tag{3.59}$$

Step 7. Multiplying $(3.26)_6$ by $\overline{\phi_{n,x}}$, using (3.23) and (3.56), we obtain

$$i\lambda_n\rho_1\langle y_n,\phi_{n,x}\rangle-k_0\langle w_{n,xx},\phi_{n,x}\rangle\to 0. \tag{3.60}$$

On the other hand, we have

$$\begin{aligned}
i\lambda_n\rho_1\left\langle y_n,\phi_{n,x}\right\rangle - k_0\left\langle w_{n,xx},\phi_{n,x}\right\rangle &= -\rho_1\left\langle y_n,\left(i\lambda_n\phi_{n,x}-u_{n,x}\right)\right\rangle - \rho_1\left\langle y_n,u_{n,x}\right\rangle + k_0\left\langle w_{n,x},\phi_{n,xx}\right\rangle \\
&= -\rho_1\left\langle y_n,\left(i\lambda_n\phi_{n,x}-u_{n,x}\right)\right\rangle - \rho_1\left\langle y_n,u_{n,x}\right\rangle \\
&\quad -\frac{k_0}{k}\left\langle w_{n,x}, i\lambda_n\rho_1 u_n - k\left(\phi_{n,x}+\psi_n+l\,w_n\right)_x - lk_0\left(w_{n,x}-l\phi_n\right)\right\rangle \\
&\quad -k_0\left\langle w_{n,x},\psi_{n,x}\right\rangle - \frac{l^2k_0^2}{k}\left\langle w_{n,x},\phi_n\right\rangle + \frac{k_0}{k}\left\langle w_{n,x}, i\lambda_n\rho_1 u_n\right\rangle \\
&\quad -\frac{lk_0\left(k+k_0\right)}{k}\left\|w_{n,x}\right\|^2 \qquad (3.61)\\
&= -\rho_1\left\langle y_n,\left(i\lambda_n\phi_{n,x}-u_{n,x}\right)\right\rangle - k_0\left\langle w_{n,x},\psi_{n,x}\right\rangle - \frac{l^2k_0^2}{k}\left\langle w_{n,x},\phi_n\right\rangle \\
&\quad -\frac{k_0}{k}\left\langle w_{n,x}, i\lambda_n\rho_1 u_n - k\left(\phi_{n,x}+\psi_n+l\,w_n\right)_x - lk_0\left(w_{n,x}-l\phi_n\right)\right\rangle \\
&\quad -\frac{\rho_1 k_0}{k}\left\langle\left(i\lambda_n w_{n,x}-y_{n,x}\right),u_n\right\rangle + \rho_1\left(\frac{k_0}{k}-1\right)\left\langle y_n,u_{n,x}\right\rangle \\
&\quad -\frac{lk_0\left(k+k_0\right)}{k}\left\|w_{n,x}\right\|^2.
\end{aligned}$$

Using (3.23), $(3.26)_1$,$(3.26)_2$,$(3.26)_5$, (3.34), (3.35), (3.42), $(3.1)_2$, (3.60)and (3.61), we obtain

$$w_{n,x} \to 0 \text{ in } L^2(0,1). \qquad (3.62)$$

Multiplying $(3.26)_6$ by $\overline{w_n}$, and using (3.23),(3.36), we get

$$-\rho_1\left\|y_n\right\|^2 + k_0\left\|w_{n,x}\right\|^2 \to 0, \qquad (3.63)$$

with (3.62), we have

$$y_n \to 0 \text{ in } L^2(0,1). \qquad (3.64)$$

using $(3.26)_5$ and (3.64), we obtain

$$\lambda_n w_n \to 0 \text{ in } L^2(0,1). \qquad (3.65)$$

Finally, we deduce that

$$\left\|\Phi_n\right\|_{\mathcal{H}} \longrightarrow 0. \qquad (3.66)$$

Hence, the proof is completed.

Remark 3.1 By applying the same reasoning, we can derive the exponential decay for (1.1) under Dirichlet boundary conditions.

4. Numerical Approximation

In this section, we present a finite element approximation for the system defined by (1.1), subject to given initial conditions and Dirichlet boundary conditions.

We introduce and analyze an implicit Euler-type scheme, employing finite differences for temporal discretization and finite elements for spatial discretization. We demonstrate that the discrete energy associated with the scheme decays over time.

We consider the following functions $\widetilde{\phi}=\phi_t$, $\widetilde{\psi}=\psi_t$, $\widetilde{w}=w_t$ and we rewrite the system (1.1) as follows :

$$\left\{\begin{array}{l}
\rho_1\widetilde{\phi}_t - k(\phi_x+\psi+lw)_x - lk_0(w_x-l\phi)=0,\\
\rho_2\widetilde{\psi}_t - b\psi_{xx} + k(\phi_x+\psi+lw) + \gamma\theta_x = 0,\\
\rho_1\widetilde{w}_t - k_0(w_x-l\phi)_x + lk(\phi_x+\psi+lw)=0,\\
\rho_3\theta_t + q_x + \gamma\widetilde{\psi}_x = 0,\\
\tau q_t + \beta q + \theta_x = 0.
\end{array}\right. \qquad (4.1)$$

To get the weak form associated to system (4.1), we multiply the equations by test functions $\zeta,\ \chi,\ \xi,\ \eta,\ \alpha \in H_0^1(0,1)$ and integrating by parts.

$$\begin{cases} \rho_1\left(\widetilde{\phi}_t,\zeta\right)+k\left(\phi_x+\psi+lw,\zeta_x\right)-k_0 l\left(w_x-l\phi,\zeta\right)=0,\\ \rho_2\left(\widetilde{\psi}_t,\ \chi\right)+b\left(\psi_x,\ \chi_x\right)+k\left(\phi_x+\psi+lw,\chi\right)+\gamma\left(\theta_x,\chi\right)=0,\\ \rho_1\left(\widetilde{w}_t,\xi\right)+k_0\left(w_x-l\phi,\xi_x\right)+lk\left(\phi_x+\psi+lw,\xi\right)=0,\\ \rho_3\left(\theta_t,\eta\right)+\left(q_x,\eta\right)+\gamma\left(\widetilde{\psi}_x,\eta\right)=0,\\ \tau\left(q_t,\alpha\right)+\beta\left(q,\alpha\right)+\left(\theta_x,\alpha\right)=0. \end{cases} \tag{4.2}$$

Let J be positive integer, we define the space step as $h=\frac{1}{J}$. Then the uniform mesh points are denoted by $x_j=jh, \forall j=0,...,J$ and

$$S_0^h=\left\{u\in H_0^1(0,1)\,\middle|\, u\in C([0,1]),\ u\big|_{(x_j,x_{j+1})} \textit{ is a linear polynomial } j=0,...,J-1\right\}. \tag{4.3}$$

Let N a positive integer and T the final time, we define the time step as $\triangle t=T/N$ and $t_n=n\triangle t,\ n=0,\ldots,N$.

The finite element method for (4.2) with Dirichlet homogeneous boundary conditions using the implicit Euler scheme is to find $\widetilde{\phi}_h^n,\ \widetilde{\psi}_h^n,\ \widetilde{w}_h^n,\ \theta_h^n$ and $q_h^n\in S_0^h\subset H_0^1(0,1)$ such that, for $n=1,\ldots,N$ and for all $\zeta_h,\ \chi_h,\ \xi_h,\ \eta_h,\ \alpha_h\in S_0^h$

$$\begin{cases} \frac{\rho_1}{\triangle t}\left(\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1},\zeta_h\right)+k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\zeta_{hx}\right)-lk_0\left(w_{hx}^n-l\phi_h^n,\zeta_h\right)=0,\\ \frac{\rho_2}{\triangle t}\left(\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1},\chi_h\right)+b\left(\psi_{hx}^n,\chi_{hx}\right)+k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\chi_h\right)+\gamma\left(\theta_{hx}^n,\chi_h\right)=0,\\ \frac{\rho_1}{\triangle t}\left(\widetilde{w}_h^n-\widetilde{w}_h^{n-1},\ \xi_h\right)+k_0\left(w_{hx}^n-l\phi_h^n,\xi_{hx}\right)+lk\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\ \xi_h\right)=0,\\ \frac{\rho_3}{\triangle t}\left(\theta_h^n-\theta_h^{n-1},\eta_h\right)+\left(q_{hx}^n,\eta_h\right)+\gamma\left(\widetilde{\psi}_x^n,\eta_h\right)=0,\\ \frac{\tau}{\triangle t}\left(q_h^n-q_h^{n-1},\alpha_h\right)+\beta\left(q_h^n,\alpha_h\right)+\left(\theta_{hx}^n,\alpha_h\right)=0, \end{cases} \tag{4.4}$$

where

$$\widetilde{\phi}_h^n=\frac{\phi_h^n-\phi_h^{n-1}}{\triangle t}\ ,\ \widetilde{\psi}_h^n=\frac{\psi_h^n-\psi_h^{n-1}}{\triangle t}\ ,\ \widetilde{w}_h^n=\frac{w_h^n-w_h^{n-1}}{\triangle t}, \tag{4.5}$$

are approximations to $\phi_t(t_n),\ \psi_t(t_n)$ and $w_t(t_n)$, respectively.

Here, $\phi_h^0, \widetilde{\phi}_h^0,\ \psi_h^0, , \widetilde{\psi}_h^0,\ w_h^0,\ \widetilde{w}_h^0,\ \theta_h^0$ and q_h^0 are given approximations to the initial conditions $\phi_0,\phi_1,\psi_0,\psi_1,w_0,w_1,\ \theta_0,q_0$ respectively.

The standard identity below will often be employed:

$$(a-b,a)=\frac{1}{2}\left(\|a-b\|^2+\|a\|^2-\|b\|^2\right). \tag{4.6}$$

For the discrete version of the energy decay property satisfied by the solution of system (1.1), is given by the following

Theorem 4.1 *The discrete energy*

$$\mathcal{E}_h^n=\frac{1}{2}(\rho_1\left(\left\|\widetilde{\phi}_h^n\right\|^2+\left\|\widetilde{w}_h^n\right\|^2\right)+\rho_2\left\|\widetilde{\psi}_h^n\right\|^2+k\left\|\phi_{hx}^n+\psi_h^n+lw_h^n\right\|^2+b\left\|\psi_{hx}^n\right\|^2+ \tag{4.7}$$

$$k_0\left\|w_{hx}^n-l\phi_h^n\right\|^2+\rho_3\left\|\theta_h^n\right\|^2+\tau\left\|q_h^n\right\|^2), \tag{4.8}$$

decay to 0 *as* t *goes to* ∞, *that is,*

$$\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\triangle t}\leq 0, \tag{4.9}$$

Proof: Taking $\zeta_h = \widetilde{\phi}_h^n,\ \chi_h = \widetilde{\psi}_h^n,\ \xi_h = \widetilde{w}_h^n,\ \eta_h = \theta_h^n$ and $\alpha_h = q_h^n$ in (4.4).
Recalling (4.5) and (4.6), we deduce that :

$$\frac{\rho_1}{2\triangle t}\left(\left\|\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}\right\|^2+\left\|\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}_h^{n-1}\right\|^2\right)+k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\widetilde{\phi}_{hx}^n\right)-lk_0\left(w_{hx}^n-l\phi_h^n,\widetilde{\phi}_h^n\right)=0, \quad (4.10)$$

$$\begin{aligned}&\tfrac{\rho_2}{2\triangle t}\left(\left\|\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}\right\|^2+\left\|\widetilde{\psi}_h^n\right\|^2-\left\|\widetilde{\psi}_h^{n-1}\right\|^2\right)+\tfrac{b}{2\triangle t}\left(\left\|\psi_{hx}^n-\psi_{hx}^{n-1}\right\|^2+\left\|\psi_{hx}^n\right\|^2-\left\|\psi_{hx}^{n-1}\right\|^2\right)+\\&k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\widetilde{\psi}_h^n\right)+\gamma\left(\theta_{hx}^n,\widetilde{\psi}_h^n\right)=0,\end{aligned} \quad (4.11)$$

$$\begin{aligned}&\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{w}_h^n-\widetilde{w}_h^{n-1}\right\|^2+\left\|\widetilde{w}_h^n\right\|^2-\left\|\widetilde{w}_h^{n-1}\right\|^2\right)+k_0\left(w_{hx}^n-l\phi_h^n,\widetilde{w}_{hx}^n\right)\\&+lk\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\ \widetilde{w}_h^n\right)=0,\end{aligned} \quad (4.12)$$

$$\frac{\rho_3}{2\triangle t}\left(\left\|\theta_h^n-\theta_h^{n-1}\right\|^2+\left\|\theta_h^n\right\|^2-\left\|\theta_h^{n-1}\right\|^2\right)-\left(q_h^n,\theta_{hx}^n\right)-\gamma\left(\widetilde{\psi}_h^n,\theta_{hx}^n\right)=0, \quad (4.13)$$

$$\frac{\tau}{2\triangle t}\left(\left\|q_h^n-q_h^{n-1}\right\|^2+\left\|q_h^n\right\|^2-\left\|q_h^{n-1}\right\|^2\right)+\beta\left\|q_h^n\right\|^2+\left(\theta_{hx}^n,q_h^n\right)=0. \quad (4.14)$$

Using again (4.5) and (4.6), we obtain :

$$\begin{aligned}\left(f^n,\widetilde{f}^n\right)&=\left(f^n,\tfrac{f^n-f^{n-1}}{\Delta t}\right),\\&=\tfrac{1}{2\triangle t}\left(\left\|f^n-f^{n-1}\right\|^2+\left\|f^n\right\|^2-\left\|f^{n-1}\right\|^2\right),\\&\geq\tfrac{1}{2\triangle t}\left(\left\|f^n\right\|^2-\left\|f^{n-1}\right\|^2\right),\end{aligned} \quad (4.15)$$

is result that :

$$k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\widetilde{\phi}_{hx}^n+\widetilde{\psi}_h^n+l\widetilde{w}_h^n\right)\geq\frac{k}{2\triangle t}\left(\left\|\phi_{hx}^n+\psi_h^n+lw_h^n\right\|^2-\left\|\phi_{hx}^{n-1}+\psi_h^{n-1}+lw_h^{n-1}\right\|^2\right), \quad (4.16)$$

and

$$k_0\left(w_{hx}^n-l\phi_h^n,\widetilde{w}_h^n-l\widetilde{\phi}_h^n\right)\geq\frac{k_0}{2\triangle t}\left(\left\|w_{hx}^n-l\phi_h^n\right\|^2-\left\|w_{hx}^{n-1}-l\phi_h^{n-1}\right\|^2\right), \quad (4.17)$$

summing equations (4.10)-(4.14), we have:

$$\begin{aligned}&0\geq\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}_h^{n-1}\right\|^2\right)+\tfrac{\rho_2}{2\triangle t}\left(\left\|\widetilde{\psi}_h^n\right\|^2-\left\|\widetilde{\psi}_h^{n-1}\right\|^2\right)+\\&\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{w}_h^n\right\|^2-\left\|\widetilde{w}_h^{n-1}\right\|^2\right)+\tfrac{\rho_3}{2\triangle t}\left(\left\|\theta_h^n\right\|^2-\left\|\theta_h^{n-1}\right\|^2\right)+\tfrac{\tau}{2\triangle t}\left(\left\|q_h^n\right\|^2-\left\|q_h^{n-1}\right\|^2\right)+\\&\tfrac{b}{2\triangle t}\left(\left\|\psi_{hx}^n\right\|^2-\left\|\psi_{hx}^{n-1}\right\|^2\right)+\tfrac{\rho_1}{2\triangle t}\left\|\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}\right\|^2+\tfrac{\rho_2}{2\triangle t}\left\|\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}\right\|^2+\\&\tfrac{\rho_1}{2\triangle t}\left\|\widetilde{w}_h^n-\widetilde{w}_h^{n-1}\right\|^2+\tfrac{\tau}{2\triangle t}\left\|q_h^n-q_h^{n-1}\right\|^2+\tfrac{b}{2\triangle t}\left\|\psi_{hx}^n-\psi_{hx}^{n-1}\right\|^2+\tfrac{\rho_3}{2\triangle t}\left\|\theta_h^n-\theta_h^{n-1}\right\|^2+\beta\left\|q_h^n\right\|^2+\\&k\left(\phi_{hx}^n+\psi_h^n+lw_h^n,\widetilde{\phi}_{hx}^n+\widetilde{\psi}_h^n+l\widetilde{w}_h^n\right)+lk_0\left(w_{hx}^n-l\phi_h^n,\widetilde{w}_h^n-l\widetilde{\phi}_h^n\right)\geq\\&\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}_h^{n-1}\right\|^2\right)+\tfrac{\rho_2}{2\triangle t}\left(\left\|\widetilde{\psi}_h^n\right\|^2-\left\|\widetilde{\psi}_h^{n-1}\right\|^2\right)+\tfrac{b}{2\triangle t}\left(\left\|\psi_{hx}^n\right\|^2-\left\|\psi_{hx}^{n-1}\right\|^2\right)+\\&\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{w}_h^n\right\|^2-\left\|\widetilde{w}_h^{n-1}\right\|^2\right)+\tfrac{\rho_3}{2\triangle t}\left(\left\|\theta_h^n\right\|^2-\left\|\theta_h^{n-1}\right\|^2\right)+\tfrac{\tau}{2\triangle t}\left(\left\|q_h^n\right\|^2-\left\|q_h^{n-1}\right\|^2\right)+\\&\tfrac{k}{2\triangle t}\left(\left\|\phi_{hx}^n+\psi_h^n+lw_h^n\right\|^2-\left\|\phi_{hx}^{n-1}+\psi_h^{n-1}+lw_h^{n-1}\right\|^2\right)+\tfrac{k_0}{2\triangle t}\left(\left\|w_{hx}^n-l\phi_h^n\right\|^2-\left\|w_{hx}^{n-1}-l\phi_h^{n-1}\right\|^2\right)\end{aligned} \quad (4.18)$$

$=\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\triangle t}$, which implies that:

$$\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\triangle t}\leq 0.$$

□

Remark 4.1 It is important to recognize that solving the equation (4.4) requires addressing a square system of linear algebraic equations. The preceding proof indicates that if all input data are set to zero, the resulting solution set $\left\{\widetilde{\phi}_h^n, \widetilde{\psi}_h^n, \widetilde{w}_h^n, \theta_h^n, q_h^n\right\}$ will also be zero. Consequently, this implies that the equation (4.4) has a unique solution.

4.1. Error estimate

We will now provide estimates regarding the discrepancy between the exact solution and the numerical solution.

At this point, we derive a priori error estimates concerning the numerical errors $\widetilde{\phi}^n - \widetilde{\phi}_h$, $\widetilde{\psi}^n - \widetilde{\psi}_h$, $\widetilde{w}^n - \widetilde{w}_h^n$, $\phi^n - \phi_h$, $\psi^n - \psi_h$, $w^n - w_h^n$, $\theta^n - \theta_h$, $q^n - q_h^n$. We have the following theorem :

Theorem 4.2 *For all* $\left\{\zeta_h^i,\ \chi_h^i,\ \xi_h^i,\ \eta_h^i,\ \alpha_h^i\right\}_{i=0}^n \subset S_0^h$, *there exists a constant* $C > 0$, *independent of the discretization parameters* h *and* $\triangle t$ *such that*

$$\begin{aligned}
&(\left\|\widetilde{\phi}^n - \widetilde{\phi}_h^n\right\|^2 + \left\|\widetilde{\psi}^n - \widetilde{\psi}_h^n\right\|^2 + \|\widetilde{w}^n - \widetilde{w}_h^n\|^2 + \left\|\widetilde{\psi}_x^n - \widetilde{\psi}_{hx}^n\right\|^2 + \|\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n)\|^2 \\
&+ \|w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n)\|^2 + \|\theta^n - \theta_h^n\|^2 + \|q^n - q_h^n\|^2 \leq C\triangle t \sum_{i=1}^n \left\|\widetilde{\phi}_t^i - \tfrac{\widetilde{\phi}^i - \widetilde{\phi}^{i-1}}{\triangle t}\right\|^2 + \\
&\left\|\widetilde{\psi}_t^i - \tfrac{\widetilde{\psi}^i - \widetilde{\psi}^{i-1}}{\triangle t}\right\|^2 + \left\|\widetilde{w}_t^i - \tfrac{\widetilde{w}^i - \widetilde{w}^{i-1}}{\triangle t}\right\|^2 + \left\|\theta_t^i - \tfrac{\theta^i - \theta^{i-1}}{\triangle t}\right\|^2 + \left\|q_t^i - \tfrac{q^i - q^{i-1}}{\triangle t}\right\|^2 + \left\|\widetilde{\phi}^i - \zeta_h^i\right\|^2 + \\
&\left\|\widetilde{\phi}_x^i - \zeta_{hx}^i\right\|^2 + \left\|\widetilde{\psi}^i - \chi_h^i\right\|^2 + \left\|\widetilde{\psi}_x^i - \chi_{hx}^i\right\|^2 + \|\widetilde{w}^i - \xi_h^i\|^2 + \|\widetilde{w}_x^i - \xi_{hx}^i\|^2 + \|\theta_x^i - \eta_{hx}^i\|^2 + \\
&\|q_x^i - \alpha_{hx}^i\|^2 + \|\theta^i - \eta_h^i\|^2 + \|q^i - \alpha_h^i\|^2 + \tfrac{C}{\triangle t} \sum_{i=1}^{n-1} (\left\|\widetilde{\phi}^i - \zeta_h^i - \left(\widetilde{\phi}^{i+1} - \zeta_h^{i+1}\right)\right\|^2 + \\
&\left\|\widetilde{\psi}^i - \chi_h^i - \left(\widetilde{\psi}^{i+1} - \chi_h^{i+1}\right)\right\|^2 + \|\widetilde{w}^i - \xi_h^i - (\widetilde{w}^{i+1} - \xi_h^{i+1})\|^2 + \|\theta^i - \eta_h^i - (\theta^{i+1} - \eta_h^{i+1})\|^2 + \\
&\|q^i - \alpha_h^i - (q^{i+1} - \alpha_h^{i+1})\|^2) + C(\left\|\phi^1 - \widetilde{\phi}_h^0\right\|^2 + \left\|\psi^1 - \widetilde{\psi}_h^0\right\|^2 + \|w^1 - \widetilde{w}_h^0\|^2 + \|\psi_x^0 - \psi_{hx}^0\|^2 + \\
&\|\phi_{hx}^0 + \psi_h^0 + lw_h^0 - (\phi_x^0 + \psi^0 + lw^0)\|^2 + \|w_x^0 - l\phi^0 - (w_{hx}^0 - l\phi_h^0)\|^2 + \|\theta^0 - \theta_h^0\|^2 + \|q^0 - q_h^0\|^2).
\end{aligned} \tag{4.19}$$

Proof: Step 1: For a continuous function $g(t)$, let $g^n = g(t_n)$. Subtracting the discrete equation $(4.4)_1$ and the equation $(4.2)_1$ at time t_n for $\zeta = \zeta_h \in S_0^h$, we obtain

$$\begin{aligned}
&\rho_1 \left(\widetilde{\phi}_t^n - \tfrac{\widetilde{\phi}_h^n - \widetilde{\phi}_h^{n-1}}{\triangle t}, \zeta_h\right) + k\left((\phi_x^n + \psi^n + lw^n) - (\phi_{hx}^n + \psi_h^n + lw_h^n), \zeta_{hx}\right) \\
&-lk_0\left(w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n), \zeta_h\right) = 0.
\end{aligned} \tag{4.20}$$

Thus, for all $\zeta_h \in S_0^h$, we obtain :

$$\begin{aligned}
&\rho_1 \left(\widetilde{\phi}_t^n - \tfrac{\widetilde{\phi}_h^n - \widetilde{\phi}_h^{n-1}}{\triangle t}, \widetilde{\phi}^n - \widetilde{\phi}_h^n\right) + k\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{\phi}_x^n - \widetilde{\phi}_{hx}^n\right) \\
&-lk_0\left(w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n), \widetilde{\phi}^n - \widetilde{\phi}_h^n\right) = \rho_1\left(\widetilde{\phi}_t^n - \tfrac{\widetilde{\phi}_h^n - \widetilde{\phi}_h^{n-1}}{\triangle t}, \widetilde{\phi}^n - \zeta_h\right) + \\
&+k\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{\phi}_x^n - \zeta_{hx}\right) - lk_0\left(w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n), \widetilde{\phi}^n - \zeta_h\right).
\end{aligned} \tag{4.21}$$

Similarly, from equations $(4.2)_2$–$(4.2)_5$ and $(4.4)_2$-$(4.4)_5$ we deduce, for all $\chi_h,\ \xi_h,\ \eta_h,\ \alpha_h \in S_0^h$,

$$\begin{aligned}
&\rho_2\left(\widetilde{\psi}_t^n - \tfrac{\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}}{\triangle t}, \widetilde{\psi}^n - \widetilde{\psi}_h^n\right) + b\left(\psi_x^n - \psi_{hx}^n,\ \widetilde{\psi}_x^n - \widetilde{\psi}_{hx}^n\right) + k\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{\psi}^n - \widetilde{\psi}_h^n\right) \\
&-\gamma\left(\theta^n - \theta_h^n, \widetilde{\psi}_x^n - \widetilde{\psi}_{hx}^n\right) = \rho_2\left(\widetilde{\psi}_t^n - \tfrac{\widetilde{\psi}_h^n - \widetilde{\psi}_h^{n-1}}{\triangle t}, \widetilde{\psi}^n - \chi_h\right) + b\left(\psi_x^n - \psi_{hx}^n,\ \widetilde{\psi}_x^n - \chi_{hx}\right) + \\
&k\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{\psi}^n - \chi_h\right) - \gamma\left(\theta^n - \theta_h^n, \widetilde{\psi}_x^n - \chi_{hx}\right),
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\rho_1\left(\widetilde{w}_t^n - \tfrac{\widetilde{w}_h^n - \widetilde{w}_h^{n-1}}{\triangle t}, \widetilde{w}^n - \widetilde{w}_h^n\right) + k_0\left(w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n), \widetilde{w}_x^n - \widetilde{w}_{hx}^n\right) \\
&+lk\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{w}^n - \widetilde{w}_h^n\right) = \rho_1\left(\widetilde{w}_t^n - \tfrac{\widetilde{w}_h^n - \widetilde{w}_h^{n-1}}{\triangle t}, \widetilde{w}^n - \xi_h\right) + \\
&+k_0\left(w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n), \widetilde{w}_x^n - \xi_{hx}\right) + lk\left(\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n), \widetilde{w}^n - \xi_h\right),
\end{aligned} \tag{4.23}$$

$$\begin{array}{c}\rho_3\left(\theta_t^n-\frac{\theta_h^n-\theta_h^{n-1}}{\triangle t},\theta^n-\theta_h^n\right)+(q_x^n-q_{hx}^n,\theta^n-\theta_h^n)+\gamma\left(\widetilde{\psi}_x^n-\widetilde{\psi}_{hx},\theta^n-\theta_h^n\right)=\\ \rho_3\left(\theta_t^n-\frac{\theta_h^n-\theta_h^{n-1}}{\triangle t},\theta^n-\eta_h\right)+(q_x^n-q_{hx}^n,\theta^n-\eta_h)+\gamma\left(\widetilde{\psi}_x^n-\widetilde{\psi}_{hx},\theta^n-\eta_h\right),\end{array}\tag{4.24}$$

$$\begin{array}{c}\tau(q_t^n-\frac{q_h^n-q_h^{n-1}}{\triangle t},q^n-q_h^n)+\beta\left(q^n-q_h^n,q^n-q_h^n\right)+(\theta_x^n-\theta_{hx}^n,q^n-q_h^n)=\\ \tau\left(q_t^n-\frac{q_h^n-q_h^{n-1}}{\triangle t},q^n-\alpha_h\right)+\beta\left(q^n-q_h^n,q^n-\alpha_h\right)+(\theta_x^n-\theta_{hx}^n,q^n-\alpha_h)\end{array}\tag{4.25}$$

Step2 : Using that (4.6), the first term in equation (4.21) become :

$$\begin{aligned}\left(\widetilde{\phi}_t^n-\frac{\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}}{\triangle t},\widetilde{\phi}^n-\widetilde{\phi}_h^n\right) &= \left(\widetilde{\phi}_t^n-\frac{\widetilde{\phi}^n-\widetilde{\phi}^{n-1}}{\triangle t},\widetilde{\phi}^n-\widetilde{\phi}_h^n\right)+\frac{1}{\triangle t}\left(\widetilde{\phi}^n-\widetilde{\phi}^{n-1}-\left(\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}\right),\widetilde{\phi}^n-\widetilde{\phi}_h^n\right)\\ &= \left(\widetilde{\phi}_t^n-\frac{\widetilde{\phi}^n-\widetilde{\phi}^{n-1}}{\triangle t},\widetilde{\phi}^n-\widetilde{\phi}_h^n\right)+\frac{1}{2\triangle t}\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n-\left(\widetilde{\phi}^{n-1}-\widetilde{\phi}_h^{n-1}\right)\right\|^2\\ &\quad+\frac{1}{2\triangle t}\left(\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}^{n-1}-\widetilde{\phi}_h^{n-1}\right\|^2\right).\end{aligned}\tag{4.26}$$

Then

$$\left(\widetilde{\phi}_t^n-\frac{\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}}{\triangle t},\widetilde{\phi}^n-\widetilde{\phi}_h^n\right)\geq\left(\widetilde{\phi}_t^n-\frac{\widetilde{\phi}^n-\widetilde{\phi}^{n-1}}{\triangle t},\widetilde{\phi}^n-\widetilde{\phi}_h^n\right)+\frac{1}{2\triangle t}\left(\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}^{n-1}-\widetilde{\phi}_h^{n-1}\right\|^2\right),\tag{4.27}$$

In the same way, for (4.22)-(4.25) we find

$$\left(\widetilde{\psi}_t^n-\frac{\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}}{\triangle t},\widetilde{\psi}^n-\widetilde{\psi}_h^n\right)\geq\left(\widetilde{\psi}_t^n-\frac{\widetilde{\psi}^n-\widetilde{\psi}^{n-1}}{\triangle t},\widetilde{\psi}^n-\widetilde{\psi}_h^n\right)+\frac{1}{2\triangle t}\left(\left\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\right\|^2-\left\|\widetilde{\psi}^{n-1}-\widetilde{\psi}_h^{n-1}\right\|^2\right),\tag{4.28}$$

$$\left(\widetilde{w}_t^n-\frac{\widetilde{w}_h^n-\widetilde{w}_h^{n-1}}{\triangle t},\widetilde{w}^n-\widetilde{w}_h^n\right)\geq\left(\widetilde{w}_t^n-\frac{\widetilde{w}^n-\widetilde{w}^{n-1}}{\triangle t},\widetilde{w}^n-\widetilde{w}_h^n\right)+\frac{1}{2\triangle t}\left(\|\widetilde{w}^n-\widetilde{w}_h^n\|^2-\left\|\widetilde{w}^{n-1}-\widetilde{w}_h^{n-1}\right\|^2\right),\tag{4.29}$$

$$\left(\theta_t^n-\frac{\theta_h^n-\theta_h^{n-1}}{\triangle t},\theta^n-\theta_h^n\right)\geq\left(\theta_t^n-\frac{\theta^n-\theta^{n-1}}{\triangle t},\theta^n-\theta_h^n\right)+\frac{1}{2\triangle t}\left(\|\theta^n-\theta_h^n\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right),\tag{4.30}$$

$$\left(q_t^n-\frac{q_h^n-q_h^{n-1}}{\triangle t},q^n-q_h^n\right)\geq\left(q_t^n-\frac{q^n-q^{n-1}}{\triangle t},q^n-q_h^n\right)+\frac{1}{2\triangle t}\left(\|q^n-q_h^n\|^2-\left\|q^{n-1}-q_h^{n-1}\right\|^2\right),\tag{4.31}$$

using again (4.15) for $u^n=\psi_x^n-\psi_{hx}^n$, $\phi_x^n+\psi^n+lw^n-(\phi_{hx}^n+\psi_h^n+lw_h^n)$ and $w_x^n-l\phi^n-(w_{hx}^n-l\phi_h^n)$ and adding (4.21)-(4.25) we obtain

$$\begin{aligned}
&\tfrac{\rho_1}{2\triangle t}\left(\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n\right\|^2-\left\|\widetilde{\phi}^{n-1}-\widetilde{\phi}_h^{n-1}\right\|^2+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2-\|\widetilde{w}^{n-1}-\widetilde{w}_h^{n-1}\|^2\right)+\\
&\tfrac{\rho_2}{2\triangle t}\left(\left\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\right\|^2-\left\|\widetilde{\psi}^{n-1}-\widetilde{\psi}_h^{n-1}\right\|^2\right)+\tfrac{b}{2\triangle t}\left(\|\psi_x^n-\psi_{hx}^n\|^2-\|\psi_x^{n-1}-\psi_{hx}^{n-1}\|^2\right)+\\
&\tfrac{k}{2\triangle t}\left(\|\phi_x^n+\psi^n+lw^n-(\phi_{hx}^n+\psi_h^n+lw_h^n)\|^2-\|\phi_x^{n-1}+\psi^{n-1}+lw^{n-1}-(\phi_{hx}^{n-1}+\psi_h^{n-1}+lw_h^{n-1})\|^2\right)+\\
&\tfrac{k_0}{2\triangle t}\left(\|w_x^n-l\phi^n-(w_{hx}^n-l\phi_h^n)\|^2-\|w_x^{n-1}-l\phi^{n-1}-(w_{hx}^{n-1}-l\phi_h^{n-1})\|^2\right)+\\
&\tfrac{\rho_3}{2\triangle t}\left(\|\theta^n-\theta_h^n\|^2-\|\theta^{n-1}-\theta_h^{n-1}\|^2\right)+\tfrac{\tau}{2\triangle t}\left(\|q^n-q_h^n\|^2-\|q^{n-1}-q_h^{n-1}\|^2\right)\leq C(\left\|\widetilde{\phi}_t^n-\tfrac{\widetilde{\phi}^n-\widetilde{\phi}^{n-1}}{\triangle t}\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{\phi}^n-\widetilde{\phi}^{n-1}-\left(\widetilde{\phi}_h^n-\widetilde{\phi}_h^{n-1}\right),\widetilde{\phi}^n-\zeta_h\right)+\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n\right\|^2+\left\|\widetilde{\phi}^n-\zeta_h\right\|^2+\left\|\widetilde{\phi}_x^n-\zeta_{hx}\right\|^2+\\
&\|w_x^n+l\phi^n-(w_{hx}^n+l\phi_h^n)\|^2+\left\|\widetilde{\psi}_t^n-\tfrac{\widetilde{\psi}^n-\widetilde{\psi}^{n-1}}{\triangle t}\right\|^2+\left\|\widetilde{\psi}^n-\chi_h\right\|^2+\left\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{\psi}^n-\widetilde{\psi}^{n-1}-\left(\widetilde{\psi}_h^n-\widetilde{\psi}_h^{n-1}\right),\widetilde{\psi}^n-\chi_h\right)+\|\psi_x^n-\chi_{hx}\|^2+\|\psi_x^n-\psi_{hx}^n\|^2+\left\|\widetilde{w}_t-\tfrac{\widetilde{w}^n-\widetilde{w}^{n-1}}{\triangle t}\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{w}^n-\widetilde{w}^{n-1}-\left(\widetilde{w}_h^n-\widetilde{w}_h^{n-1}\right),\widetilde{w}^n-\xi_h\right)+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2+\|\widetilde{w}^n-\xi_h\|^2+\|\widetilde{w}_x^n-\xi_{hx}\|^2+\\
&\left\|\theta_t-\tfrac{\theta^n-\theta^{n-1}}{\triangle t}\right\|^2+\tfrac{1}{\triangle t}\left(\theta^n-\theta^{n-1}-\left(\theta_h^n-\theta_h^{n-1}\right),\theta^n-\eta_h\right)+\|\theta^n-\theta_h^n\|^2+\|\theta_x^n-\eta_x^n\|^2+\\
&\|\theta^n-\eta_h^n\|^2+\|\theta_x^n-\eta_{hx}^n\|^2+\left\|q_t-\tfrac{q^n-q^{n-1}}{\triangle t}\right\|^2+\tfrac{1}{\triangle t}\left(q^n-q^{n-1}-\left(q_h^n-q_h^{n-1}\right),q^n-\alpha_h\right)+\\
&\|q^n-q_h^n\|^2+\|q^n-\alpha_h^n\|^2+\|q_x^n-q_{hx}^n\|^2+\|q_x^n-\alpha_{hx}^n\|^2).
\end{aligned} \tag{4.32}$$

Step 3. Multiplying the latter inequality by $\triangle t$ and summing over n we obtain, for all $\{\zeta_h,\ \chi_h,\ \xi_h,\ \eta_h,\ \alpha_h\}_{i=0}^n\in S_0^h$,

$$\begin{aligned}
&\left\|\widetilde{\phi}^n-\widetilde{\phi}_h^n\right\|^2+\left\|\widetilde{\psi}^n-\widetilde{\psi}_h^n\right\|^2+\|\widetilde{w}^n-\widetilde{w}_h^n\|^2+\left\|\widetilde{\psi}_x^n-\widetilde{\psi}_{hx}^n\right\|^2+\|\phi_x^n+\psi^n+lw^n-(\phi_{hx}^n+\psi_h^n+lw_h^n)\|^2+\\
&\|w_x^n-l\phi^n-(w_{hx}^n-l\phi_h^n)\|^2+\|\theta^n-\theta_h^n\|^2+\|q^n-q_h^n\|^2\leq C\triangle t\sum_{i=1}^{n}(\left\|\widetilde{\phi}^i-\widetilde{\phi}_h^i\right\|^2+\\
&\left\|\widetilde{\psi}^i-\widetilde{\psi}_h^i\right\|^2+\|\widetilde{w}^i-\widetilde{w}_h^i\|^2+\left\|\widetilde{\psi}_x^i-\widetilde{\psi}_{hx}^i\right\|^2+\left\|\phi_{hx}^i+\psi_h^i+lw_h^i-(\phi_x^i+\psi^i+lw^i)\right\|^2+\\
&\left\|w_x^i-l\phi^i-(w_{hx}^i-l\phi_h^i)\right\|^2+\left\|\theta^i-\theta_h^i\right\|^2+\left\|q^i-q_h^i\right\|^2+\left\|\widetilde{\phi}_t^i-\tfrac{\widetilde{\phi}^i-\widetilde{\phi}^{i-1}}{\triangle t}\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{\phi}^i-\widetilde{\phi}^{i-1}-\left(\widetilde{\phi}_h^i-\widetilde{\phi}_h^{i-1}\right),\widetilde{\phi}^i-\zeta_h^i\right)+\left\|\widetilde{\psi}_t^i-\tfrac{\widetilde{\psi}^i-\widetilde{\psi}^{i-1}}{\triangle t}\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{\psi}^i-\widetilde{\psi}^{i-1}-\left(\widetilde{\psi}_h^i-\widetilde{\psi}_h^{i-1}\right),\widetilde{\psi}^i-\chi_h^i\right)+\left\|\widetilde{w}_t^i-\tfrac{\widetilde{w}^i-\widetilde{w}^{i-1}}{\triangle t}\right\|^2+\\
&\tfrac{1}{\triangle t}\left(\widetilde{w}^i-\widetilde{w}^{i-1}-\left(\widetilde{w}_h^i-\widetilde{w}_h^{i-1}\right),\widetilde{w}^i-\xi_h^i\right)+\left\|\theta_t^i-\tfrac{\theta^i-\theta^{i-1}}{\triangle t}\right\|^2+\\
&+\tfrac{1}{\triangle t}\left(\theta^i-\theta^{i-1}-\left(\theta_h^i-\theta_h^{i-1}\right),\theta^i-\eta_h^i\right)+\left\|q_t^i-\tfrac{q^i-q^{i-1}}{\triangle t}\right\|^2+\tfrac{1}{\triangle t}(q^i-q^{i-1}-\left(q_h^i-q_h^{i-1}\right),q^i-\alpha_h^i)+\\
&\left\|\widetilde{\phi}^i-\zeta_h^i\right\|^2+\left\|\widetilde{\phi}_x^i-\zeta_{hx}^i\right\|^2+\left\|\widetilde{\psi}^i-\chi_h^i\right\|^2+\left\|\widetilde{\psi}_x^i-\chi_{hx}^i\right\|^2+\|\widetilde{w}^i-\xi_h^i\|^2+\|\widetilde{w}_x^i-\xi_{hx}^i\|^2+\\
&\left\|\theta_x^i-\eta_{hx}^i\right\|^2+\left\|q_x^i-\alpha_{hx}^i\right\|^2+\left\|\theta^i-\eta_h^i\right\|^2+\left\|q^i-\alpha_h^i\right\|^2)+C(\left\|\phi^1-\widetilde{\phi}_h^0\right\|^2+\left\|\psi^1-\widetilde{\psi}_h^0\right\|^2+\\
&\left\|w^1-\widetilde{w}_h^0\right\|^2+\left\|\psi_x^0-\psi_{hx}^0\right\|^2+\left\|\phi_{hx}^0+\psi_h^0+lw_h^0-(\phi_x^0+\psi^0+lw^0)\right\|^2+\left\|w_x^0-l\phi^0-(w_{hx}^0-l\phi_h^0)\right\|^2+\\
&\left\|\theta^0-\theta_n^0\right\|^2+\left\|q^0-q_n^0\right\|^2).
\end{aligned} \tag{4.33}$$

Taking into account that (as in [2] with an equivalent result for similar terms)

$$\sum_{i=1}^{n}\left(\widetilde{\phi}^{i}-\widetilde{\phi}^{i-1}-\left(\widetilde{\phi}_{h}^{i}-\widetilde{\phi}_{h}^{i-1}\right),\widetilde{\phi}^{i}-\zeta_{h}^{i}\right)=\left(\widetilde{\phi}^{n}-\widetilde{\phi}_{h}^{n},\widetilde{\phi}^{n}-\zeta_{h}^{n}\right)+\left(\widetilde{\phi}_{h}^{0}-\widetilde{\phi}^{0},\widetilde{\phi}^{1}-\zeta_{h}^{1}\right)+$$
$$\sum_{i=1}^{n-1}\left\|\widetilde{\phi}_{h}^{i}-\widetilde{\phi}_{h}^{i},\widetilde{\phi}^{i}-\zeta_{h}^{i}-\left(\widetilde{\phi}^{i+1}-\zeta_{h}^{i+1}\right)\right\|\leq C\left(\left\|\widetilde{\phi}^{n}-\widetilde{\phi}_{h}^{n}\right\|^{2}+\left\|\widetilde{\phi}^{n}-\zeta_{h}^{n}\right\|^{2}+\left\|\widetilde{\phi}_{h}^{0}-\zeta_{h}^{0}\right\|^{2}+\left\|\widetilde{\phi}^{1}-\zeta_{h}^{1}\right\|^{2}\right)$$
$$+C\triangle t\sum_{i=1}^{n-1}\left\|\widetilde{\phi}^{i}-\widetilde{\phi}_{h}^{i}\right\|^{2}+\frac{C}{\triangle t}\sum_{i=1}^{n-1}\left\|\widetilde{\phi}^{i}-\zeta_{h}^{i}-\left(\widetilde{\phi}^{i+1}-\zeta_{h}^{i+1}\right)\right\|^{2}, \tag{4.34}$$

and applying Gronwall's inequality [6] in its discrete version, the result follows. □

Corollary 4.1 *Suppose that the solution to the continuous problem is sufficiently regular, that is:*

$$\phi,\psi,\omega\in H^{3}\left(0,T;L^{2}\left(0,L\right)\right)\cap W^{1,\infty}\left(0,T;H^{1}\left(0,L\right)\right)\cap H^{2}\left(0,T;H^{1}\left(0,L\right)\right), \tag{4.35}$$

and,

$$\theta,q\in H^{2}\left(0,T;L^{2}\left(0,L\right)\right)\cap L^{\infty}\left(0,T;H^{2}\left(0,L\right)\right)\cap H^{1}\left(0,T;H^{1}\left(0,L\right)\right). \tag{4.36}$$

There exists then, a constant $C>0$ independent of h and $\triangle t$, such that :

$$\left\|\widetilde{\phi}^{n}-\widetilde{\phi}_{h}^{n}\right\|^{2}+\left\|\widetilde{\psi}^{n}-\widetilde{\psi}_{h}^{n}\right\|^{2}+\left\|\widetilde{w}^{n}-\widetilde{w}_{h}^{n}\right\|^{2}+\left\|\widetilde{\psi}_{x}^{n}-\widetilde{\psi}_{hx}^{n}\right\|^{2}+\left\|\phi_{x}^{n}+\psi^{n}+lw^{n}-\left(\phi_{hx}^{n}+\psi_{h}^{n}+lw_{h}^{n}\right)\right\|^{2}+$$
$$\left\|w_{x}^{n}-l\phi^{n}-\left(w_{hx}^{n}-l\phi_{h}^{n}\right)\right\|^{2}+\left\|\theta^{n}-\theta_{h}^{n}\right\|^{2}+\left\|q^{n}-q_{h}^{n}\right\|^{2}\leq C\left(h^{2}+\triangle t^{2}\right). \tag{4.37}$$

Proof: The result is a consequence of following estimates as in [7] and [15] :
$$\frac{1}{\triangle t}\sum_{n=1}^{N-1}\left\|\widetilde{\phi}^{n}-\zeta_{h}^{n}-\left(\widetilde{\phi}^{n+1}-\zeta_{h}^{n+1}\right)\right\|\leq Ch^{2}\left\|\widetilde{\phi}_{t}\right\|_{L^{2}(0,T;H^{1}(0,L))}^{2}.$$ □

4.2. Numerical simulation

To demonstrate the accuracy of the approximation and confirm the asymptotic behavior of the solutions, we present in this section results from some numerical examples. To solve the system (1.1), we employed an iterative method for a set of implicit equations. Refer to [23], [29] and [24] for further details.

Assuming that $\widetilde{\phi}_{h}^{n-1}$, $\widetilde{\psi}_{h}^{n-1}$, $\widetilde{w}_{h}^{n-1}$, θ_{h}^{n-1}, q_{h}^{n-1} are known and let. $\phi_{h}^{n,0}=\phi_{h}^{n-1}$, $\widetilde{\phi}_{h}^{n,0}=\widetilde{\phi}_{h}^{n-1}$, $\psi_{h}^{n,0}=\psi_{h}^{n-1}$, $\widetilde{\psi}_{h}^{n,0}=\widetilde{\psi}_{h}^{n-1}$, $\widetilde{w}_{h}^{n,0}=\widetilde{w}_{h}^{n-1}$, $w_{h}^{n,0}=w_{h}^{n-1}$,
$\theta_{h}^{n,0}=\theta_{h}^{n-1}$, $q_{h}^{n,0}=q_{h}^{n-1}$, we will solve the following system:

$$\begin{aligned}
&\tfrac{\rho_{3}}{\triangle t}(\theta_{h}^{n,j}-\theta_{h}^{n-1},\eta_{h})+(q_{hx}^{n,j-1},\eta_{h})+\gamma(\widetilde{\psi}_{x}^{n,j},\eta_{h})=0,\\
&\tfrac{\tau}{\triangle t}(q_{h}^{n,j}-q_{h}^{n-1},\alpha_{h})+\beta(q_{h}^{n,j},\alpha_{h})+(\theta_{hx}^{n,j},\alpha_{h})=0,\\
&\tfrac{\rho_{2}}{\triangle t}(\widetilde{\psi}_{h}^{n,j}-\widetilde{\psi}_{h}^{n-1},\chi_{h})+b(\psi_{hx}^{n,j},\chi_{hx})+k(\phi_{hx}^{n,j}+\psi_{h}^{n,j}+lw_{h}^{n,j-1},\chi_{h})-\gamma(\theta_{h}^{n,j-1},\chi_{hx})=0,\\
&\tfrac{\rho_{1}}{\triangle t}(\widetilde{\phi}_{h}^{n,j}-\widetilde{\phi}_{h}^{n-1},\zeta_{h})+k(\phi_{hx}^{n,j}+\psi_{h}^{n,j-1}+lw_{h}^{n,j-1},\zeta_{hx})-lk_{0}(w_{hx}^{n}-l\phi_{h}^{n},\zeta_{h})=0,\\
&\tfrac{\rho_{1}}{\triangle t}(\widetilde{w}_{h}^{n,j}-\widetilde{w}_{h}^{n-1},\ \xi_{h})+lk_{0}(w_{hx}^{n,j}-l\phi_{h}^{n,j},\xi_{hx})+lk(\phi_{hx}^{n,j}+\psi_{h}^{n,j}+lw_{h}^{n,j},\ \xi_{h})=0,
\end{aligned} \tag{4.38}$$

where, for $j=1,2,...,J$

$$\phi_{h}^{n,j}=\phi_{h}^{n-1}+\triangle t\widetilde{\phi}_{h}^{n,j},\ \psi_{h}^{n,j}=\psi_{h}^{n-1}+\triangle t\widetilde{\psi}_{h}^{n,j},\ w_{h}^{n,j}=w_{h}^{n-1}+\triangle t\widetilde{w}_{h}^{n,j}. \tag{4.39}$$

Problem (4.38) consists of five decoupled linear systems of algebraic equations, with tridiagonal matrices.

Example 1. Homogeneous problem: energy decay
We consider first the following parameters of the model:

$$\rho_1 = \rho_2 = 0.9,\ \rho_3 = 1,\ \ b = 2,\ k = 2,\ k_0 = 1, \tau = 1,\ \gamma = \sqrt{\frac{b}{\tau}(1 - \frac{\tau k \rho_3}{\rho_1})(\frac{\rho_1}{k} - \frac{\rho_2}{b})} = 0.7416,\ \beta = l = 1; \tag{4.40}$$

The discretization parameters are:

$$N = 1500,\ T = 40,\ \triangle t = \frac{T}{N},\ J = 100,\ h = \frac{1}{J},\ . \tag{4.41}$$

Along with the following initial conditions:

$$\begin{cases} \phi_0(x) = 20x(x-1)^2 = \phi_t\,(x,0) = \phi_1\,(x)\,, & \forall x \in (0,1)\,, \\ \psi_0(x) = x\cos(\frac{\pi x}{2}) = \psi_t\,(x,0)\,, & \forall x \in (0,1)\,, \\ w_0(x) = w(x,0) = x\cos(\frac{\pi x}{2}) = w_t(x,0), & \forall x \in (0,1)\,, \\ q_0(x) = q(x,0) = \sin(\pi x), & \forall x \in (0,1)\,, \\ \theta_0(x) = \theta\,(x,0) = \sin(\pi x), & \forall x \in (0,1)\,, \end{cases} \tag{4.42}$$

We present the graphs of the solutions to the system described in (1.1) (refer to Figure 2), along with the graph depicting the total discrete energy $E(t)$. This analysis reveals an exponential decay in the total discrete energy, as illustrated in Figure 1.

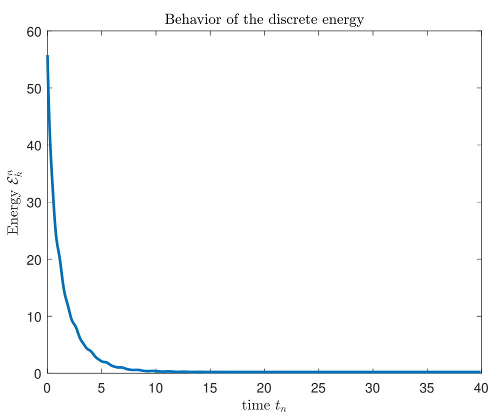

Figure 1: Behavior of the discrete energy for example 1.

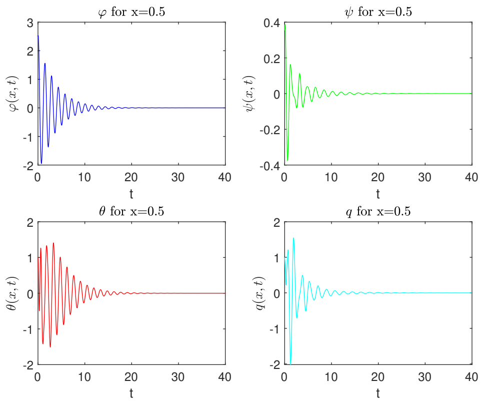

Figure 2: Numerical solution for x = 0.5 for example 1.

Example 2: Numerical convergence

The purpose of this example is to assess the numerical convergence of the numerical scheme. To achieve this, we will consider the following problem.

Subsequently, we conducted a simulation to numerically evaluate the error estimate. We addressed the modified problem (P)

Find $w : [0,1] \times [0,T] \to \mathbb{R}$, $\phi : [0,1] \times [0,T] \to \mathbb{R}$ and $\psi : [0,1] \times [0,T] \to \mathbb{R}$ respectively the longitudinal, vertical, and shear angle displacements and the temperature deviations $\theta : [0,1] \times [0,T] \to \mathbb{R}$ and $q : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$(P)\begin{cases} \rho_1\phi_{tt} - k(\phi_x + \psi + lw)_x - lk_0(w_x - l\phi) = f_1 & \text{in } (0,1)\times(0,T) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi + lw) + \gamma\theta_x = f_2 & \text{in } (0,1)\times(0,T) \\ \rho_1 w_{tt} - k_0(w_x - l\phi)_x + lk(\phi_x + \psi + lw) = f_3 & \text{in } (0,1)\times(0,T) \\ \rho_3\theta_t + q_x + \gamma\psi_{xt} = f_4 & \text{in } (0,1)\times(0,T) \\ \tau q_t + \beta q + \theta_x = f_5 & \text{in } (0,1)\times(0,T) \end{cases}$$

where $f_1,\ f_2,\ f_3,\ f_4,\ f_5$, and the initial data are calculated from the exact solution

$$\begin{cases} \phi\,(x,t) = x^5(1-x)^5 e^t, \\ \psi\,(x,t) = x^5(1-x)^5 e^t, \\ \omega\,(x,t) = x^5(1-x)^5 e^t, \\ \theta\,(x,t) = x^5(1-x)^5 e^t, \\ q\,(x,t) = x^5(1-x)^5 e^t, \end{cases}$$

Table 1: Computed errors when $T = 1$

Nel	Δt	$Error$
20	5 10^{-3}	5,6931 10^{-6}
40	2,5 10^{-3}	1,6009 10^{-6}
80	1,25 10^{-3}	5,8625 10^{-7}
160	6,25 10^{-4}	3,3313 10^{-7}
320	3,125 10^{-4}	2,6987 10^{-7}

In Table 1, we show the computed errors at $T = 1$ for different discretization parameters J and Δt , where the Error is defined as:

$$Error = (\left\|\widetilde{\phi}^n - \widetilde{\phi}_h^n\right\|^2 + \left\|\widetilde{\psi}^n - \widetilde{\psi}_h^n\right\|^2 + \|\widetilde{w}^n - \widetilde{w}_h^n\|^2 + \left\|\widetilde{\psi}_x^n - \widetilde{\psi}_{hx}^n\right\|^2 + \|\phi_x^n + \psi^n + lw^n - (\phi_{hx}^n + \psi_h^n + lw_h^n)\|^2 + \|w_x^n - l\phi^n - (w_{hx}^n - l\phi_h^n)\|^2 + \|\theta^n - \theta_h^n\|^2 + \|q^n - q_h^n\|^2)^{\frac{1}{2}}.$$

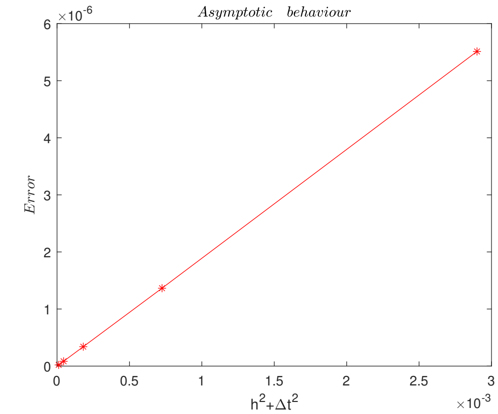

Figure 3: Asymptotic behavior of the numerical scheme.

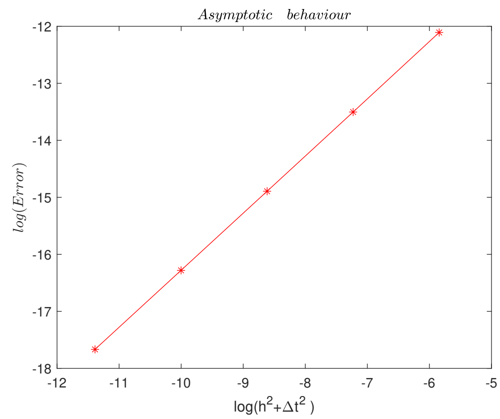

Figure 4: The evolution of log(Error).

We observe the numerical convergence in Corollary is achieved according to Figures 3-4.

The numerical scheme was implemented using MATLAB on a Intel Core i7-9750H @2.60 GHz.

5. Conclusion

In this work, we have analyzed a linear one-dimensional thermoelastic Bresse system with second sound, establishing its well-posedness and identifying conditions for exponential stability through a combination of semigroup theory, energy methods, and frequency domain analysis. To complement the theoretical study, we developed a finite element approximation in space, coupled with an implicit Euler scheme for time integration. We demonstrated that the resulting discrete energy decays over time, mirroring the continuous case, and provided a priori error estimates for both semi-discrete and fully discrete schemes. These theoretical results were supported by numerical simulations, which confirmed the expected convergence behavior and energy decay properties. Altogether, our findings provide a rigorous and comprehensive understanding of the dynamics of the system, both analytically and numerically, and offer a reliable framework for further investigation or practical implementation.

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References

1. F. Alabau Boussouira, J. Munoz Rivera, & D. Almeida Jùnior, *Stability to weak dissipative Bresse system.* Journal Of Mathematical Analysis And Applications. 374, 481-498, (2011)
2. K. Andrews, J. Fernandez, & M. Shillor, *Numerical analysis of dynamic thermoviscoelastic contact with damage of a rod.* IMA J. Appl. Math, 70, 768-795, (2005)
3. M. Alves, L. Fatori, M A. Jorge Silva & Monteiro, *Stability and optimality of decay rate for weakly dissipative Bresse system.* Math Meth. Appl. Sci, 38, 898-908, (2015)
4. M. Aouadi, M. Campo, M. Copetti, & J. Fernandez, Existence, *stability and numerical results for a Timoshenko beam with thermodiffusion effects.* Math Phys., 70, (2019)
5. C. Benchimol, *A note on weak stabilizability of contraction semigroups.* SIAM J. Control Optim. 16, 373-379, (1978)
6. M. Campo, J. Fernandez, K. Kuttler, M. Shillor, & J. Viano, *Numerical analysis and simulations of a dynamic frictionless contact problem with damage.* Comput. Methods Appl. Mech. Eng. 196, 476-488, (2006)
7. P. Ciarlet, *The Finite Element Method for Elliptic Problems.* (Eds: P. G. Ciarlet, J. L. Lions), Handbook Of Numerical Analysis. vol. II, North Holland, 17-352, (1991)
8. P. Ciarlet, *Basic error estimates for elliptic problems.* Handbook Of Numerical Analysis, Vol 2., 17-351, (1993)
9. W. Charles, J. Soriano, F. Nascimentoc, & J. Rodrigues, *Decay rates for Bresse system with arbitrary nonlinear localized damping.* J. Diff. Equa. 255, 2267-2290, (2013)
10. L. Fatori, & R. Monteiro, *The optimal decay rate for a weak dissipative Bresse system.* Appl. Math Lett. 25, 600-604, (2012)
11. L. Fatori, & J. Rivera, *Rates of decay to weak thermoelastic Bresse system.* J. Appl. Math IMA. 75, 881-904, (2010)
12. A. Guesmia, & M. Kafini, *Bresse system with infinite memories.* Math. Meth. Appl. Sci. 38, 2389-2402, (2015)
13. A. Guesmia, & M. Kafini, *Uniform and weak stability of Bresse system with two infinite memories.* Math Phys. 67, 1-39, (2016)
14. A. Guesmia, *Asymptotic stability of Bresse system with one infinite memory in the longitudinal displacements.* Medi. J. Math. 14, 1-19, (2017)
15. W. Han, M. Shillor, & M. Sofonea, *Variational and numerical analysis of a quasi-static viscoelastic problem with normal compliance, friction and damage,* J. Comp. Appl. Math. 137, 377-398, (2001)
16. F. Huang, *Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces.* Ann. Of Diff. Eqs. 1, 43-56, (1985)
17. A. Keddi, T. Apalara, & S. Messaoudi, *Exponential and polynomial decay in a thermoelastic-Bresse system with second sound.* Appl. Math Optim. 77, 315-341, (2018)
18. J. Lagnese, G. Leugering, & J. Schmidt, *Modelling Analysis and Control of Dynamic Elastic Multi-Link Structures.* (Foundations & Applications, (1994)
19. Z. Liu, & B. Rao, *Characterization of polynomial decay rate for the solution of linear evolution equation.* Math Phys. 56, 630-644, (2005)
20. Z. Liu, & B. Rao, *Energy decay rate of the thermoelastic Bresse system.* Z. Angew. Math Phys. 60, 54-69, (2009)
21. J. Pruss, *On the spectrum of C0 semigroups.* Trans. Amer Math. Soc. 284, 847-857, (1984)
22. Santos & Almeida Jùnior, D. *Asymptotic behavior to Bresse system with past history.* Quart. Appl. Math. 73, 23-54, (2015)
23. A. Smouk, & A. Radid, *Discrete energy behavior of Timoshenko system with Cattaneo's law.* Computational And Applied Mathematics. 43(197), 1-22, (2024)
24. A. Smouk, & A. Radid, *Numerical studies of a Timoshenko system with the second sound.* Mathematical Modeling And Computing. 11(4), 911-922, (2024)
25. J. Soriano, W. Charles, & R. Schulz, *Asymptotic stability for Bresse systems.* J. Math Anal. Appl. 412, 369-380, (2014)
26. J. Soriano, J. Rivera, & L. Fatori, *Bresse system with indefinite damping.* J. Math Anal. Appl. 387, 284-290, (2012)
27. A. Soufyane, & B. Said-Houari, *The effect of the wave speeds and the frictional damping terms on the decay rate of the Bresse system.* Evol. Equa. Cont. Theory. 3, 713-738, (2014)
28. W. Youssef, & A. Wehbe, *Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedbacks.* J. Math Phys. 51, 1-17, (2010)
29. A. Radid, A. Smouk, and K. Rhofir. *Numerical stability of discrete energy for a thermoelastic-Bresse system with second sound.*, Journal of Mathematical Modeling, 12, 4, 671-686, (2024)

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