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The Laplacian Minimum Boundary Pendant Dominating Energy of a Graph

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Abstract: Let G be a finite, simple, and undirected graph with vertex set $V(G)$. A subset $S \subseteq V(G)$ is called a boundary pendant dominating set if the induced subgraph $< S >$ is boundary dominated by atleast one pendant vertex. The boundary pendant dominating number, denoted $\gamma^B_{pe}(G)$, is the minimum cardinality among all boundary pendant dominating set of G. In this research article, we compute the Laplacian minimum boundary pendant dominating energy $LE^B_{pe}(G)$ for several standard graphs, and establish corresponding upper and lower bounds for $LE^B_{pe}(G)$.

Keywords: Laplacian minimum boundary pendant dominating set, Laplacian minimum boundary pendant dominating matrix, Laplacian minimum boundary pendant dominating eigenvalues, Laplacian minimum boundary pendant dominating energy.

Contents

1. Introduction

The concept of boundary domination in graphs was first introduced by K. M. Kathiresan, G. Marimuthu, and M. Sivanandha Saraswathy in 2010 [11]. This idea extends traditional domination parameters by emphasizing the effect that chosen vertices have on the boundary of the remaining graph. Consider a graph G with n vertices and m edges, which is a simple, finite, undirected, and non-empty graph without loops or multiple edges between the same pair of vertices.

The distance between two vertices u and v in a graph is defined as the length of the shortest path connecting them. A vertex v is called a boundary neighbor of u if v is the nearest boundary of u.

For a vertex $u \in V$, the boundary neighborhood of u, denoted by $N^B(u)$ vertex to u is defined as $N^B(u) = \{u \in V : d(u, w) \leq d(u, v) \; for \; all \; w \in N^B(u)\}$. The cardinality of $N^B(u)$ denoted by $deg^B(G)$ in G. The maximum and minimum boundary degrees of vertices in a graph G are denoted by $\Delta^B(G)$ and $\delta^B(G)$ respectively. That is, $\Delta^B(G) = max_{u \in V} |N^B(u)|$ and $\delta^B(G) = min_{u \in V} |N^B(u)|$.

A vertex u boundary dominates a vertex v if v is a boundary neighbor of u. A subset $B \subseteq V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by atleast one vertex in B. The boundary domination number of G, denoted by $\gamma^B(G)$, is the minimum cardinality among all boundary dominating sets of G.

The concept of graph energy was introduced by I. Gutman [8] in year 1978. Let $A = (a_{ij})$ be the adjacency matrix of a graph G, and let its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arranged in non-increasing order.

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Since A is a real symmetric matrix, all the eigenvalues of G and their sum equal to boundary domination number. The energy of a graph G, denoted by $E(G)$ is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

For an in depth treatment of the mathematical theory of graph energy, refer to [7]. I. Gutman and B. Zhou [6,9] defined the Laplacian energy of a graph G in 2006. Recently, minimum covering energy [1] has become an interesting way to connect traditional graph coverings with the graph's spectral (eigenvalue) properties. The Laplacian matrix of a graph G is denoted by $L = (l_{ij})$ is a square matrix of order $n \times n$. The elements of the Laplacian matrix are

$$l_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i, & \text{i = j} \end{cases}$$

where d_i is the vertex of v_i. M. R. Rajesh Kanna and G. Sridhara [22] was computed the Laplacian minimum dominating energy of some standard graphs. Let $D(G)$ be the diagonal matrix of vertex degrees of a graph G and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the Laplacian eigenvalues. The Laplacian energy is denoted by $LE(G)$ and is defined as follows:

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|$$

Mohammed Alatif introduced the concept of the Laplacian minimum boundary dominating energy of a graph [14]. Motivated by these papers the present authors defined the Laplacian minimum boundary pendant dominating energy of a graph. The concept of pendant dominating energy [17,18,19,20] and the Laplacian energy [10,25,27] can be analyzed through various bounds and inequalities. The Laplacian energy of a graph computed from the eigenvalues [24,26] of its Laplacian matrix, finds wide ranging and important applications such as in chemistry, high resolution satellite image classification and segmentation and in uncovering semantic structures within image hierarchies.

1.1. The Minimum Boundary Dominating Energy of Graphs

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. A subset B of $V(G)$ is called a boundary dominating set if every vertex in $V - B$ is boundary dominated by atleast one vertex in B. The boundary domination number $\gamma_b(G)$ of G is the minimum cardinality of a boundary dominating set. Any boundary dominating set with minimum cardinality is called a minimum boundary dominating set. Let B be a minimum boundary dominating set of the graph G. The minimum boundary dominating matrix of G is the $n \times n$ matrix defined by $A_B(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_j \in N_b(v_i) \\ 1, & \text{if } i = j \text{ and } v_i \in B \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_B(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_B(G))$. The minimum boundary dominating eigenvalues of a graph G are the eigenvalues of $A_B(G)$. Since $A_B(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum boundary dominating energy of a graph G is defined as

$$E_B(G) = \sum_{i=1}^{n} |\lambda_i|$$

1.2. The Laplacian Minimum Boundary Dominating Energy of Graphs

Let $D_B(G)$ be a diagonal matrix of boundary vertex degrees of a graph G. Then $L_B(G) = D_B(G) - A_B(G)$ is called the Laplacian minimum boundary dominating matrix of G. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $L_B(G)$, arranged in non-increasing order and are called Laplacian minimum boundary dominating eigenvalues of G. The Laplacian minimum boundary dominating energy of a graph G is defined as

$$LE_B(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{\Omega}{n} \right|$$

where $\Omega = \sum_{i=1}^{n} deg_B(v_i)$. Here v_i is the boundary of v, and $\frac{\Omega}{n}$ is the average boundary degree G.

2. The Laplacian Minimum Boundary Pendant Dominating Energy of a Graph

Let G be a graph of order p and degree q, with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and edge set $E(G)$. A subset S of $V(G)$ is called a boundary pendant dominating set if the induced subgraph $< S >$ contains atleast one boundary pendant vertex. The boundary pendant domination number, denoted by $\gamma_{pe}^{B}(G)$, is the minimum cardinality of such a boundary pendant dominating set. Let S be a Laplacian minimum boundary pendant dominating set of G. The Laplacian minimum boundary pendant dominating matrix of order $p \times p$ is the adjacency matrix $A_{pe}^{B}(G) = a_{ij}^{B}$ where

$$a_{ij}^{B} = \begin{cases} 1, & \text{if } v_j \in N^B(v_i) \\ 1, & \text{if } v_i = v_j \text{ for } i = j \\ 1, & \text{if } v_i \sim v_j \text{ and } v_i \in S \\ 0, & \text{otherwise} \end{cases}$$

Let $D_{pe}^{B}(G)$ be a diagonal matrix of the boundary vertex degree of G. The Laplacian minimum boundary pendant dominating matrix of G is given by $L_{pe}^{B}(G) = D_{pe}^{B}(G) - A_{pe}^{B}(G)$ where $D_{pe}^{B}(G)$ and $A_{pe}^{B}(G)$ are the diagonal matrix and adjacency matrix of G respectively. Let $\beta_1, \beta_2, \ldots, \beta_p$ be the Laplacian minimum pendant dominating boundary eigenvalues of G. The Laplacian minimum pendant dominating boundary energy, denoted by $LE_{pe}^{D}(G)$ and is defined as

$$LE_{pe}^{B}(G) = \sum_{i=1}^{p} \left| \beta_i - \frac{\Phi}{p} \right|$$

where $\Phi = \sum_{i=1}^{n} deg^B(v_i)$. Here $deg^B(v_i)$ is the pendant boundary degree of v_i and $\left| \beta_i - \frac{\Phi}{p} \right|$ represents absolute value of the difference between each eigenvalue β_i and the average pendant boundary of the graph G. The average pendant boundary of G is computed as $\frac{\Phi}{p}$ where Φ is the sum of total number of boundary edges and p is the number of vertices in G. As $L_{pe}^{B}(G)$ is real and symmetric matrix, the characteristic polynomial of $L_{pe}^{B}(G)$ is given by $f_p(G, \beta) = (L_{pe}^{B}(G) - \beta I)$ and the set of eigenvalues $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p|$ with their algebraic multiplicities $r_1, r_2, \ldots, r_p$ of $L_{pe}^{B}(G)$ is called the Laplacian minimum pendant dominating boundary spectra of G, denoted by $Spec(L_{pe}^{B}(G))$ and is as follows:

$$Spec(L_{pe}^{B}(G)) = \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_p \\ r_1 & r_2 & \ldots & r_p \end{pmatrix}$$

The Laplacian minimum pendant dominating boundary energy with algebraic multiplicities r_i is

$$LE_{pe}^{B}(G) = \sum_{i=1}^{p} \left| \beta_i - \frac{\Phi}{p} \right| r_i$$

2.1. Example

Let G be a graph with the vertex set $\{v_1, v_2, v_3, v_4, v_4\}$ as shown in the FIGURE-1

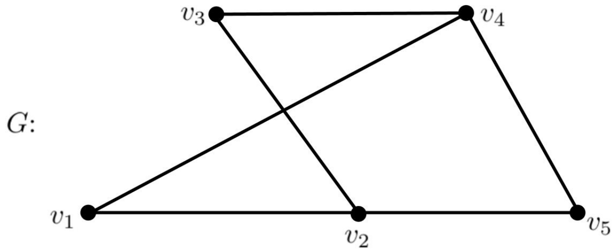

FIGURE - 1

. (i) Let $S_1 = \{v_2, v_3\}$ is the boundary pendant dominating set. The adjacency matrix, diagonal matrix and Laplacian matrix is as follows:

$$A^B_{pe}(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } D^B_{pe}(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L^B_{pe}(G) = D^B_{pe}(G) - A^B_{pe}(G) = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

The characteristic polynomial of the Laplacian matrix $L^B_{pe}(G)$ is $(\beta-3)(\beta^2-2\beta-1)(\beta^2-\beta-1) = 0$. The Laplacian minimum boundary pendant dominating eigenvalues are $\beta_1 = -0.61803, \beta_2 = -0.41421, \beta_3 = 1.61803, \beta_4 = 2.41421, \beta_5 = 3$. The average boundary degree is $\frac{\Phi}{p} = \frac{8}{5} \approx 1.6$. Therefore, the Laplacian minimum boundary pendant dominating energy is $LE^B_{pe}(G) \approx 6.46448$.
(ii) Let $S_2 = \{v_1, v_4\}$ is the boundary pendant dominating set. The adjacency matrix, diagonal matrix and Laplacian matrix is as follows:

$$A^B_{pe}(G) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } D^B_{pe}(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L^B_{pe}(G) = D^B_{pe}(G) - A^B_{pe}(G) = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

The characteristic polynomial of the Laplacian matrix $L^B_{pe}(G)$ is $(\beta - 3)(\beta^2 - 2\beta - 1)(\beta^2 - \beta - 1) = 0$. This polynomial is same as (i) and hence we conclude that the Laplacian minimum boundary pendant dominating energy is $LE^B_{pe}(G) \approx 6.46448$.

Theorem 2.1 *Let G be a bull graph, then* $LE^B_{pe}(G) \approx 9.59524$

Proof: Let G be a simple, undirected bull graph with 5 vertices and 5 edges. The graph is in the form of a triangle with two disjoint pendant edges, which resembles a bull's head with horns as shown in the Figure 2.

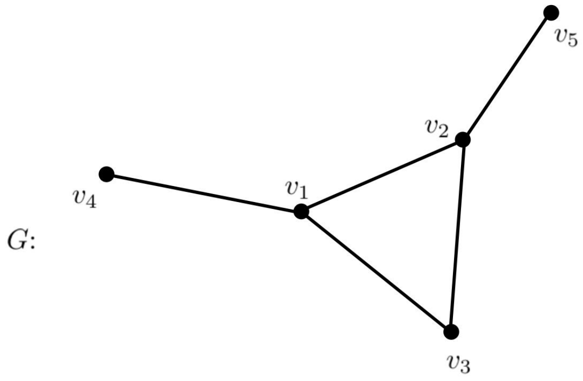

Figure 2: Bull graph with 5 vertices and 5 edges

Let $S = \{v_1, v_2\}$ is the boundary pendant dominating set. The adjacency matrix, diagonal matrix and Laplacian matrix is as follows:

$$A^B_{pe}(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } D^B_{pe}(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$L^B_{pe}(G) = D^B_{pe}(G) - A^B_{pe}(G) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}$$

The characteristic polynomial of the Laplacian matrix is $(\beta - 1)(\beta^2 - 4\beta - 1)(\beta^2 - 3\beta - 2) = 0$. The Laplacian minimum boundary pendant dominating eigenvalues are $\beta_1 = -0.56155, \beta_2 = -0.23607, \beta_3 = 1, \beta_4 = 3.56155, \beta_5 = 4.23607$. The average boundary degree is $\frac{\Phi}{p} = \frac{10}{5} = 2$. Therefore, the Laplacian minimum boundary pendant dominating energy is $LE^B_{pe}(G) \approx 9.59524$. □

3. Fundamental Properties on Eigenvalues of $LE^B_{pe}(G)$

Theorem 3.1 *Let G be a simple graph having the vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$, the edge set $E(G)$ and the pendant dominating set $S = \{u_1, u_2, u_3, \ldots, u_p\}$. If $\beta_1, \beta_2, \ldots, \beta_p$ are the eigenvalues of the Laplacian minimum boundary pendant dominating matrix $L^B_{pe}(G)$ and $\Phi = \sum_{i=1}^{n} deg^B(v_i)$ then*

(i) $\sum_{i=1}^{p} \beta_i = \Phi - |S|$

(ii) $\sum_{i=1}^{p} \beta_i^2 = \Phi + \sum_{i=1}^{p} (deg^B(v_i) - c_i)^2$ *where* $c_i = \begin{cases} 1, & \text{if } v_i \in S \\ 0, & \text{if } v_i \notin S \end{cases}$

Proof: (i) By definition, the sum of the principal diagonal elements of $L^B_{pe}(G)$ is equal to $\sum_{i=1}^{p}(deg^B(v_i) - |S| = \Phi - |S|$. Also the sum of eigenvalues of $L^B_{pe}(G)$ is the trace of $L^B_{pe}(G)$, it follows that

$$\sum_{i=1}^{p} \beta_i = \sum_{i=1}^{p} a_{ii} = \sum_{i=1}^{p} deg^B(v_i) - |S| = \Phi - |S|$$

(ii) Similarly, the sum of squares of the eigenvalues of $L^B_{pe}(G)$ is the trace of $(L^B_{pe}(G))^2$. Therefore,

$$\begin{aligned}\sum_{i=1}^{p}\beta_i^2 &= \sum_{i=1}^{p}\beta_i\sum_{j=1}^{p}\beta_j = \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_i\beta_j = \sum_{i=1}^{p}\sum_{j=1}^{p}l_{ij}l_{ji}\\ &= \sum_{i=1}^{p}(l_{ii})^2 + \sum_{i\neq j}l_{ij}l_{ji} = \sum_{i=1}^{p}(l_{ii})^2 + 2\sum_{i<j}(l_{ij})^2\\ \therefore \sum_{i=1}^{p}\beta_i^2 &= \Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2 \;\; where \;\; c_i = \begin{cases} 1, & \text{if } v_i \in S\\ 0, & \text{if } v_i \notin S\end{cases}\end{aligned}$$

□

Theorem 3.2 *If the sum of the absolute eigenvalues of the Laplacian minimum boundary pendant dominating matrix $L^B_{pe}(G)$ is a rational number, then*

$$\sum_{i=1}^{p}|\beta_i| \equiv \gamma^B_{pe}(G)(mod\ 2)$$

Proof: Let $\beta_1, \beta_2, \ldots, \beta_p$ be the eigenvalues of the Laplacian minimum boundary pendant dominating matrix $L^B_{pe}(G)$ of a graph G, of which $\beta_1, \beta_2, \ldots, \beta_r$ are positive and the rest of them are non-positive, then

$$\begin{aligned}\sum_{i=1}^{p}|\beta_i| &= \big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(\beta_{p+1} + \cdots + \beta_p\big)\\ &= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(\beta_1 + \beta_2 + \cdots + \beta_p\big)\\ &= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \sum_{i=1}^{p}\beta_i\\ &= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r\big) - \big(2|E| - |S|\big)\\ &= 2\big(\beta_1 + \beta_2 + \cdots + \beta_r - |E|\big) - |S|\\ \therefore \sum_{i=1}^{p}|\beta_i| &\equiv \gamma^B_{pe}(G)(mod\ 2)\end{aligned}$$

□

Theorem 3.3 *Let G be a graph of order p and size q. Let $\beta_1(G)$ be the largest eigenvalue of $A^B_{pe}(G)$ and $\gamma^B_{pe}(G)$ is the minimum boundary pendant domination number, then*

$$\beta_1(G) \geq \frac{\Phi + \gamma^B_{pe}(G)}{p}$$

Proof: Let G be a graph of order p and let $\beta_1(G)$ be the largest minimum boundary pendant dominating eigenvalue of $A^B_{pe}(G)$. Then form [3], we have $\beta_1 = max_{X\neq 0}\Big(\frac{X^T L^B_{pe} X}{X^T X}\Big)$ where X is any non-zero vector, X^T is its transpose and L^B_{pe} is a matrix. If we choose $X = I = (1, 1, \ldots, 1)^T$. Then, we have

$$\beta_1(G) \geq \frac{I^T L^B_{pe} I}{I^T I} = \frac{\Phi + \gamma^B_{pe}(G)}{p}$$

□

4. Bounds on $LE_{pe}^B(G)$

McLelland's [13] gave upper and lower bounds for ordinary energy of a graph. AleksiÂtc [2] gave upper and lower bounds for Laplacian energy of a graph. Similar bounds for $LE_{pe}^B(G)$ are given in the following theorem.

Theorem 4.1 *(Upper Bound) Let G be a simple connected graph of order p, $\Phi = \sum_{i=1}^{p} deg^B(v_i)$ and S is a minimum boundary pendant dominating set. If $\sum_{i=1}^{p} |\beta_i|$ is a rational number, then*

$$\left(\gamma_{pe}^B(G) + 2c_i - \Phi\right) \leq LE_{pe}^B(G) \leq \left(\gamma_{pe}^B(G) + 2c_i + \Phi\right)$$

Proof: By definition, we have

$$LE_{pe}^B(G) = \sum_{i=1}^{p} \left|\beta_i - \frac{\Phi}{p}\right| \leq \sum_{i=1}^{p} |\beta_i| + \Phi$$

Now

$$\begin{aligned} LE_{pe}^B(G) \leq & \sum_{i=1}^{p} |\beta_i| + \Phi \\ = & |S| + 2c_i + \Phi \\ \implies LE_{pe}^B(G) \leq & \gamma_{pe}^B(G) + 2c_i + \Phi \end{aligned}$$

Also,

$$\begin{aligned} LE_{pe}^B(G) \geq & \sum_{i=1}^{p} |\beta_i| - \Phi \\ = & |S| + 2c_i - \Phi \\ \implies LE_{pe}^B(G) \geq & \gamma_{pe}^B(G) + 2c_i - \Phi \end{aligned}$$

From both we conclude that

$$\left(\gamma_{pe}^B(G) + 2c_i - \Phi\right) \leq LE_{pe}^B(G) \leq \left(\gamma_{pe}^B(G) + 2c_i + \Phi\right)$$

and

$$LE_{pe}^B \in (\gamma_{pe}^B(G) + 2c_i - \Phi, \gamma_{pe}^B + 2c_i + \Phi)$$

□

Theorem 4.2 *(Lower Bound) Let G be a simple connected graph having p vertices and $\Phi = \sum_{i=1}^{p} deg^B(v_i)$. If S is the minimum boundary pendant dominating set and $\Delta = |det(L_{pe}^B(G))|$, then*

$$LE_{pe}^B(G) \geq \sqrt{\Phi + \sum_{i=1}^{p} (deg^B(v_i) - c_i)^2 + p(p-1)\Delta^{\frac{2}{p}}} - \Phi$$

Proof: Consider

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 = \left(\sum_{i=1}^{p}|\beta_i|\right)\left(\sum_{i=1}^{p}|\beta_i|\right)$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 = \sum_{i=1}^{p}|\beta_i|^2 + \sum_{i\neq j}|\beta_i||\beta_j|$$

$$\Longrightarrow \sum_{i\neq j}|\beta_i||\beta_j| = \left(\sum_{i=1}^{p}|\beta_i|\right)^2 - \sum_{i=1}^{p}|\beta_i|^2$$

Applying inequality between the Arithmetic and Geometric means for $p(p-1)$ terms

$$\frac{\sum\limits_{i\neq j}|\beta_i||\beta_j|}{p(p-1)} \geq \left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\sum_{i\neq j}|\beta_i||\beta_j| \geq p(p-1)\left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 - \sum_{i=1}^{p}|\beta_i|^2 \geq p(p-1)\left(\prod_{i\neq j}|\beta_i||\beta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 - \left(\Phi + \sum_{i=1}^{p}(deg^B(v_i)-c_i)^2\right) \geq p(p-1)\left(\prod_{i\neq j}|\beta_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}}$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 - \left(\Phi + \sum_{i=1}^{p}(deg^B(v_i)-c_i)^2\right) \geq p(p-1)\left(\prod_{i\neq j}|\beta_i|\right)^{\frac{2}{p}}$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 \geq \Phi + \sum_{i=1}^{p}(deg^B(v_i)-c_i)^2 + p(p-1)|det(L_{pe}^B(G))|^{\frac{2}{p}}$$

$$\Longrightarrow \left(\sum_{i=1}^{p}|\beta_i|\right) \geq \sqrt{\Phi + \sum_{i=1}^{p}(deg^B(v_i)-c_i)^2 + p(p-1)|det(L_{pe}^B(G))|^{\frac{2}{p}}}$$

By Triangular inequality, we have

$$|\beta_i| - \left|\frac{\Phi}{p}\right| \leq \left|\beta_i - \frac{\Phi}{p}\right| \quad \forall i$$

$$|\beta_i| - \frac{\Phi}{p} \leq \left|\beta_i - \frac{\Phi}{p}\right|$$

$$\sum_{i=1}^{p}|\beta_i| - \sum_{i=1}^{p}\frac{\Phi}{p} \leq \sum_{i=1}^{p}\left|\beta_i - \frac{\Phi}{p}\right|$$

$$\sum_{i=1}^{p}|\beta_i| - \Phi \leq \sum_{i=1}^{p}|\beta_i| - \Phi$$

$$\Longrightarrow \sum_{i=1}^{p}|\beta_i| - \Phi \leq LE_{pe}^B(G)$$

Therefore, from the above we conclude that

$$LE_{pe}^{B}(G) \geq \sqrt{\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2 + p(p-1)\Delta^{\frac{2}{p}} - \Phi}$$

□

Theorem 4.3 *Let G be a simple connected graph of order p, $\Phi = \sum_{i=1}^{p} deg^{B}(v_i)$ and S is a minimum boundary pendant dominating set of G, then*

$$LE_{pe}^{B}(G) \leq \sqrt{p\Big(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2\Big) + \Phi\Big(\Phi - \gamma_{pe}^{B}(G)\Big)}$$

Proof: The Cauchy's-Schwartz inequality is

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

Put $a_i = 1$ and $b_i = \left|\beta_i - \frac{\Phi}{p}\right|$ in the above inequality, then

$$\begin{aligned}
\left(\sum_{i=1}^{p}\left|\beta_i - \frac{\Phi}{p}\right|\right)^2 &\leq \Big(\sum_{i=1}^{p} 1\Big)\left(\sum_{i=1}^{p}\left|\beta_i - \frac{\Phi}{p}\right|^2\right) \\
\big(LE_{pe}^{B}(G)\big)^2 &\leq p\left(\sum_{i=1}^{p}\left(|\beta_i|^2 + \frac{\Phi^2}{p^2} - 2|\beta_i|\left|\frac{\Phi}{p}\right|\right)\right) \\
&= p\left(\sum_{i=1}^{p}|\beta_i|^2 + \sum_{i=1}^{p}\frac{\Phi^2}{p^2} - \frac{2\Phi}{p}\sum_{i=1}^{p}|\beta_i|\right) \\
&= p\left(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2 + \frac{\Phi^2}{p} - \frac{\Phi}{p}\big(\Phi - |S|\big)\right) \\
&= p\left(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2 + \frac{\Phi^2}{p} - \frac{\Phi\gamma_{pe}^{B}(G)}{p}\right) \\
&= p\Big(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2\Big) + \Phi^2 - \Phi\gamma_{pe}^{B}(G) \\
&= p\Big(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2\Big) + \Phi\big(\Phi - \gamma_{pe}^{B}(G)\big) \\
\big(LE_{pe}^{B}(G)\big)^2 &\leq p\Big(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2\Big) + \Phi\big(\Phi - \gamma_{pe}^{B}(G)\big) \\
\therefore LE_{pe}^{B}(G) &\leq \sqrt{p\Big(\Phi + \sum_{i=1}^{p}(deg^{B}(v_i) - c_i)^2\Big) + \Phi\big(\Phi - \gamma_{pe}^{B}(G)\big)}
\end{aligned}$$

□

Theorem 4.4 *(Upper Bound) Let G be a simple connected graph having p vertices and $\Phi = \sum_{i=1}^{p} deg^{B}(v_i)$, then*

$$LE^B_{pe}(G) \leq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right)} + \Phi$$

Proof: The Cauchy's-Schwartz inequality is

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\beta_i|$ in the above inequality

$$\begin{aligned}\left(\sum_{i=1}^{p} |\beta_i|\right)^2 &\leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} |\beta_i|^2\right)\\ &= p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right)\\ \Longrightarrow \sum_{i=1}^{p} |\beta_i| &\leq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right)}\end{aligned}$$

By Triangular inequality, we have

$$\begin{aligned}\left|\beta_i - \frac{\Phi}{p}\right| &\leq |\beta_i| + \left|\frac{\Phi}{p}\right| = |\beta_i| + \frac{\Phi}{p}\\ \sum_{i=1}^{p}\left|\beta_i - \frac{\Phi}{p}\right| &\leq \sum_{i=1}^{p} |\beta_i| + \sum_{i=1}^{p} \frac{\Phi}{p}\\ \Longrightarrow LE^B_{pe}(G) &\leq \sum_{i=1}^{p} |\beta_i| + \Phi\end{aligned}$$

From the above result we conclude that

$$LE^B_{pe}(G) \leq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right)} + \Phi$$

□

Theorem 4.5 *Let G be a simple connected graph having p vertices and $\Phi = \sum_{i=1}^{p} deg^B(v_i)$. Let $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p|$ be a non-increasing order of the Laplacian minimum boundary pendant dominating eigenvalues of $L^B_{pe}(G)$, then*

$$LE^B_{pe}(G) \geq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2 - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2\right)} - 2q$$

where $\alpha(p) = p\lceil \frac{p}{2} \rceil \left(1 - \frac{1}{p}\lceil \frac{p}{2} \rceil\right)$ and $[x]$ denotes the integral part of a real number.

Proof: Let $a, a_1, a_2, \ldots, a_p, A$ and $b, b_1, b_2, \ldots, b_p, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for all $i = 1, 2, \ldots, p$. Then, the following inequality holds

$$\left|p\sum_{i=1}^{p} a_i b_i - \sum_{i=1}^{p} a_i \sum_{i=1}^{p} b_i\right| \leq \alpha(p)(A - a)(B - b)$$

Put $a_i = b_i = |\beta_i|$, $a = b = |\beta_p|$ and $A = B = |\beta_1|$ in the above inequality

$$\left| p\left(\sum_{i=1}^{p}|\beta_i|^2 - \left(\sum_{i=1}^{p}|\beta_i|\right)^2\right)\right| \leq \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2$$

$$\left| p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right) - \left(\sum_{i=1}^{p}|\beta_i|\right)^2\right| \leq \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2$$

$$p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right) - \left(\sum_{i=1}^{p}|\beta_i|\right)^2 \leq \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2$$

$$p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right) - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2 \leq \left(\sum_{i=1}^{p}|\beta_i|\right)^2$$

$$\left(\sum_{i=1}^{p}|\beta_i|\right)^2 \geq p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right) - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2$$

$$\Longrightarrow \left(\sum_{i=1}^{p}|\beta_i|\right) \geq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2\right) - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2}$$

By definition, we have $LE^B_{pe}(G) = \sum_{i=1}^{p}\left|\beta_i - \frac{\Phi}{p}\right|$ and by Triangular inequality, we have

$$LE^B_{pe}(G) \geq \sum_{i=1}^{p}\Big|\beta_i\Big| - \sum_{i=1}^{p}\frac{\Phi}{p}$$

$$LE^B_{pe}(G) \geq \sum_{i=1}^{p}\Big|\beta_i\Big| - \Phi$$

From the above, we obtain result

$$LE^D_{pe}(G) \geq \sqrt{p\left(\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2 - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2\right)} - \Phi$$

□

Theorem 4.6 *Let G be a simple connected graph having p vertices and $\Phi = \sum_{i=1}^{p} deg^B(v_i)$. Let $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_p| > 0$ be a non-increasing order of the Laplacian minimum boundary pendant dominating eigenvalues of $L^B_{pe}(G)$, then*

$$LE^B_{pe}(G) \geq \frac{\Phi + \sum_{i=1}^{p}(deg^B(v_i) - c_i)^2 - \alpha(p)\Big(|\beta_1| - |\beta_p|\Big)^2}{\Big(|\beta_1| + |\beta_p|\Big)} - \Phi$$

Proof: Let $a_i \neq 0, b_i, r$ and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds

$$\sum_{i=1}^{p} b_i^2 + rR\sum_{i=1}^{p} a_i \leq \big(r + R\big)\sum_{i=1}^{p} a_i b_i$$

Put $b_i = |\beta_i|$, $a_i = 1$, $r = |\beta_p|$ and $R = |\beta_1|$ in the above inequality, then

$$\sum_{i=1}^{p} |\beta_i|^2 + |\beta_1||\beta_p| \sum_{i=1}^{p} 1 \leq \big(|\beta_1| + |\beta_p|\big) \sum_{i=1}^{p} |\beta_i|$$

$$\Phi + \sum_{i=1}^{p} (deg^B(v_i) - c_i)^2 + p|\beta_1||\beta_p| \leq \big(|\beta_1| + |\beta_p|\big) \sum_{i=1}^{p} |\beta_i|$$

$$\frac{\Phi + \sum_{i=1}^{p} (deg^B(v_i) - c_i)^2 + p|\beta_1||\beta_p|}{\big(|\beta_1| + |\beta_p|\big)} \leq \sum_{i=1}^{p} |\beta_i|$$

By definition, we have

$$LE^B_{pe}(G) = \sum_{i=1}^{p} \Big|\beta_i - \frac{\Phi}{p}\Big|$$

$$LE^B_{pe}(G) \geq \sum_{i=1}^{p} \big|\beta_i\big| - \sum_{i=1}^{p} \frac{\Phi}{p}$$

$$LE^B_{pe}(G) \geq \sum_{i=1}^{p} \big|\beta_i\big| - \Phi$$

From the above, we conclude

$$LE^B_{pe}(G) \geq \frac{\Phi + \sum_{i=1}^{p} (deg^B(v_i) - c_i)^2 + p|\beta_1||\beta_p|}{\big(|\beta_1| + |\beta_p|\big)} - \Phi$$

$\square$

5. $LE^B_{pe}(G)$ for Various Standard Graphs

Theorem 5.1 *For a complete bipartite graph $K_{p,p}$, we have $LE^B_{pe}(K_{p,p}) = (2p-4) + 2\sqrt{p^2 - 2p + 5}$ where $p \geq 2$*

Proof: Let $K_{p,p}$ be a complete bipartite graph having the vertex set $V(K_{p,p}) = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$. The minimum boundary pendant dominating set is $S = \{u_1, v_1\}$. The associated adjacency matrix and diagonal matrix are as follows:

$$A^B_{pe}(K_{p,p}) = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 0 \end{pmatrix}_{2p \times 2p} \quad \text{and}$$

$$D^B_{pe}(K_{p,p}) = \begin{pmatrix} (p-1) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & (p-1) & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (p-1) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (p-1) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & (p-1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & (p-1) \end{pmatrix}_{2p\times 2p}$$

The Laplacian matrix is $L^B_{pe}(K_{p,p}) = D^B_{pe}(K_{p,p}) - A^B_{pe}(K_{p,p})$

$$L^B_{pe}(K_{p,p}) = \begin{pmatrix} (p-2) & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ -1 & (p-1) & \dots & -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & (p-1) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (p-2) & -1 & \dots & -1 \\ 0 & 0 & \dots & 0 & -1 & (p-1) & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -1 & -1 & \dots & (p-1) \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the Laplacian matrix $L^B_{pe}(K_{p,p})$ is

$$f_p(K_{p,p},\beta) = \big(\beta - p\big)^{2p-4}\big(\beta^2 - (p-1)\beta - 1\big)^2 = 0$$

The eigenvalues are $\beta = p$ with multiplicity $(2p-4)$, $\beta = \frac{(p-1)\pm\sqrt{p^2-2p+5}}{2}$ with multiplicity 2 each. The average boundary degree of $K_{p,p}$ is

$$\tfrac{\Phi}{p} = \tfrac{2p(p-1)}{2p} = p - 1$$

Therefore, the Laplacian minimum boundary pendant dominating energy of $K_{p,p}$ is

$$LE^B_{pe}(K_{p,p}) = (2p-4) + 2\sqrt{p^2-2p+5}$$

□

Theorem 5.2 *For a double star graph* $S_{p,p}$, *we have* $LE^B_{pe}(S_{p,p}) = \frac{4p^2+2p\sqrt{p+3}-12}{p}$ *where* $p \geq 2$

Proof: Let $S_{p,p}$ be a double star graph having the vertex set $V(S_{p,p}) = \{v_0, v_1, v_2, \dots, v_{p-1}, u_0, u_1, \dots, u_{p-1}\}$, where v_0 and u_0 are the two central vertices. The minimum boundary pendant dominating set is $S = \{v_0, u_0\}$. The associated adjacency matrix and diagonal matrix are as follows:

$$A^B_{pe}(S_{p,p}) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 1 & \dots & 0 \end{pmatrix}_{2p\times 2p} \quad \text{and}$$

$$D^B_{pe}(S_{p,p}) = \begin{pmatrix} (p-1) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & (p+1) & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (p+1) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (p-1) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & (p+1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & (p+1) \end{pmatrix}_{2p\times 2p}$$

The Laplacian matrix is $L^B_{pe}(S_{p,p}) = D^B_{pe}(S_{p,p}) - A^B_{pe}(S_{p,p})$

$$L^B_{pe}(S_{p,p}) = \begin{pmatrix} (p-2) & 0 & \dots & 0 & 0 & -1 & \dots & -1 \\ 0 & (p+1) & \dots & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \dots & (p+1) & -1 & -1 & \dots & -1 \\ 0 & -1 & \dots & -1 & (p-2) & 0 & \dots & 0 \\ -1 & -1 & \dots & -1 & 0 & (p+1) & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 & 0 & -1 & \dots & (p+1) \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the Laplacian matrix $L^B_{pe}(S_{p,p})$ is

$$f_p(S_{p,p},\beta) = \big(\beta-(p+2)\big)^{2p-4}\big(\beta^2-2p\beta+(p^2-p-3)\big)\big(\beta^2-2\beta-(p^2-5p+7)\big) = 0$$

The eigenvalues are $\beta = (p+2)$ with multiplicity $(2p-4)$, $\beta = p \pm \sqrt{p+3}$ and $\beta = 1 \pm \sqrt{p^2-5p+8}$ with multiplicity 1 each. The average boundary degree of $S_{p,p}$ is

$$\tfrac{\Phi}{p} = \tfrac{2p^2+2p-4}{2p} = \tfrac{p^2+p-2}{p}$$

Therefore, the Laplacian minimum boundary pendant dominating energy of $S_{p,p}$ is

$$LE^B_{pe}(S_{p,p}) = \tfrac{4p^2+2p\sqrt{p+3}-12}{p}$$

□

Theorem 5.3 *For a dumbbell graph $D_{p,p}$, we have* $LE^B_{pe}(D_{p,p}) = \frac{p^2+3p-6}{p} + \sqrt{p^2+6p-3}$

Proof: Let $D_{p,p}$ be a dumbbell graph having the vertex set $V(D_{p,p}) = \{v_0, v_1, v_2, \dots, v_{p-1}, u_0, u_1, \dots, u_{p-1}\}$. The minimum pendant dominating degree set is $S = \{v_0, u_0\}$. The associated adjacency matrix and diagonal matrix are as follows:

$$A^B_{pe}(D_{p,p}) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \end{pmatrix}_{2p\times 2p} \quad \text{and}$$

$$D^B_{pe}(D_{p,p}) = \begin{pmatrix} (p-1) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & p & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (p-1) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & p \end{pmatrix}_{2p\times 2p}$$

The Laplacian matrix is $L^B_{pe}(D_{p,p}) = D^B_{pe}(D_{p,p}) - A^B_{pe}(D_{p,p})$

$$L^B_{pe}(D_{p,p}) = \begin{pmatrix} (p-2) & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & p & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p & -1 & -1 & \dots & -1 \\ 0 & -1 & \dots & -1 & (p-2) & 0 & \dots & 0 \\ -1 & -1 & \dots & -1 & 0 & p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 & 0 & 0 & \dots & p \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of the Laplacian matrix $L^B_{pe}(D_{p,p})$ is

$$f_p(D_{p,p},\beta) = \big(\beta - p\big)^{2p-4}\big(\beta^2 - (3p-3)\beta + (2p^2-6p+3)\big)\big(\beta^2 - (p-1)\beta - 1\big) = 0$$

The eigenvalues are $\beta = p$ with multiplicity $(2p-4)$, $\beta = \frac{(3p-3)\pm\sqrt{p^2+6p-3}}{2}$ and $\beta = \frac{(p-1)\pm\sqrt{p^2-2p+5}}{2}$ with multiplicity 1 each. The average boundary degree of $D_{p,p}$ is

$$\tfrac{2q}{p} = \tfrac{2p^2-2}{2p} = \tfrac{p^2-1}{p}$$

Therefore, the Laplacian minimum boundary pendant dominating energy of $D_{p,p}$ is

$$LE^B_{pe}(D_{p,p}) = \tfrac{p^2+3p-6}{p} + \sqrt{p^2+6p-3}$$

□

6. Conclusion

The Laplacian dominating boundary distribution plays a key role in spectral analysis of complex networks. This study investigates its properties using linear algebra, introducing the boundary pendant dominating matrix a novel degree based representation of a simple graph G. This matrix captures domination related structure while enabling more efficient spectral analysis. Our approach reduces matrix dimensions in eigenvalue computations, often allowing explicit eigenvalues determination. We also introduce degree energy, a new spectral invariant inspired by classical graph energy, defined via the boundary pendant dominating matrix. Theoretical bounds are established, and exact values are computed for standard graph families, deepening the algebraic understanding of domination in networks.

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