Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.79623

Energy Estimates and Existence of Non-trivial Solutions for Robin Problems Involving p-Laplacian Operator

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ABSTRACT: This paper studies the existence of non-trivial solutions and energy estimates for a nonlinear elliptic problem driven by the p-Laplacian under Robin boundary conditions, which model various physical phenomena such as heat transfer and fluid flow with boundary interactions. Using a recent local minimum theorem, we establish existence results under suitable growth and Ambrosetti-Rabinowitz (AR) conditions. We identify intervals of the parameter λ where solutions exist and extend the result to all $\lambda > 0$ under $(p-1)$-sublinear growth at zero and infinity. An illustrative example is also provided.

Key Words: Robin problem, p-Laplacian, non-trivial solutions, energy estimates.

Contents

1. Introduction

In this work, we investigate the existence of non-trivial weak solutions and energy estimates for the following nonlinear elliptic problem with homogeneous Robin boundary conditions:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \ = \lambda f(x,u), & in \quad \Omega, \\ \frac{\partial u}{\partial v} + \alpha(x)|u|^{p-2}u = 0, & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω, is a non-empty bounded open set in $\mathbb{R}^N$ (for $N \geq 3$), that has a smooth boundary $\partial\Omega$ and λ is a positive real parameter. The differential operator p-Laplacian is defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, for $1 < p < N$. Moreover, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\alpha \in L^\infty(\partial\Omega), \alpha(x) \geq 0$ a.e. on $\partial\Omega$. The generalized normal derivative $\frac{\partial u}{\partial v}$ is defined by $\frac{\partial u}{\partial v} = |\nabla u|^{p-2}\nabla u \cdot v(x)$, where $v(x)$ be the unit normal vector pointing outward from the boundary of Ω at $x \in \partial\Omega$.

Partial differential equations involving the p-Laplacian have been widely studied due to their applications in nonlinear elasticity, fluid dynamics, and other fields involving non-Newtonian flows and processes with non-standard diffusion. As shown in [8,12], and the references therein, a rich literature exists addressing various boundary value problems involving the p-Laplacian. Among these, problems with Robin-type boundary conditions are of particular interest, as they encompass both Dirichlet and Neumann cases and model energy transfer across boundaries.

Numerous analytical techniques have been employed to study such problems, including fixed point theory, the method of sub- and supersolutions, and variational methods. For more details, see [1,10,11, 19,21,22,23,24].

In [1], a parametric Robin problem involving the p-Laplacian with an indefinite potential and a superlinear term not satisfying the Ambrosetti-Rabinowitz (AR) condition is considered. In [10], the existence of two nontrivial solutions is established using variational methods under the AR condition for the problem (1.1). The study in [24] investigates a (p, q)-equation under Robin conditions and proves a bifurcation-type result using variational and comparison techniques. Additional works such as [4,6,13] address related questions involving Neumann, singular, or periodic boundary conditions.

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2010 *Mathematics Subject Classification:* 35A15, 35J60.
Submitted October 17, 2025. Published December 19, 2025

In this paper, we establish the existence of nontrivial solutions and energy estimates for Robin problems driven by the p-Laplacian without requiring asymptotic conditions on the nonlinear term at zero or infinity (see Theorem 3.1); an explicit example is given in Corollary 3.1. Moreover, Theorem 3.2 provides an existence result under a sublinear growth condition at the origin. When the nonlinearity also exhibits sublinear growth at infinity, the associated energy functional is coercive, and the existence of (possibly trivial) solutions follows from direct minimization methods (see Remark 3.4).

The main novelty of our work lies in the application of a recent local minimum theorem to nonlinear Robin problems involving the p-Laplacian. To demonstrate the effectiveness of our approach, we conclude with a concrete example.

For a thorough treatment of the subject, we refer the reader to [7,17,18]

The structure of the paper is as follows: In Section 2, we present the main definitions and mathematical tools required for the proofs of our main results. Section 3 is devoted to the statement and proof of the main results, along with an illustrative example.

2. Starting Points and Foundational Notation

Let $A : X \to X^*$ be a functional, where X is a real Banach space and X^* is its dual. We say that A has the s_+-property, iff for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

1. $u_n \rightharpoonup u$ (weakly in X), and
2. $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$,

implies that $u_n \to u$ (strongly in X).

Here, $\langle .,. \rangle$ denotes the duality pairing between X and X^*.

In this paper, X denotes the Sobolev space $W^{1,p}(\Omega)$ with the following norm

$$\|u\| = \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}},$$

for all $u \in X$. For $1 < p < N, p^* = \frac{Np}{N-p}$ and $u \in W^{1,p}(\Omega)$, one has there exists a positive constant T such that

$$\|u\|_{L^{p^*}(\Omega)} \leq T\|u\|,$$

where T has been defined by Talenti (see [27]) and

$$T \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-P}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}\right)^{\frac{1}{N}},$$

where Γ is the Euler function.

Moreover, by Hölder's inequality for every $u \in W^{1,p}(\Omega)$ and $s \in [1, p^*]$, one has

$$\|u\|_{L^s(\Omega)} \leq \kappa_s\|u\|, \tag{2.1}$$

where

$$\kappa_s = T|\Omega|^{\frac{p^*-s}{p^*s}}, \tag{2.2}$$

and $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbb{R}$. On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure, denoted by $\sigma(.)$. We will also use the boundary Lebesgue spaces $L^p(\partial\Omega)$, for $1 \leq p \leq \infty$ in the standard case. The theory of Sobolev spaces guarantees the existence of a unique continuous linear operator $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called the trace map, such that

$$\gamma_0(u) = u|_{\partial\Omega}, \quad \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

Therefore, the trace map extends the notion of boundary values to all Sobolev functions. The operator $\gamma_0 : W^{1,p}(\Omega) \to L^\eta(\partial\Omega)$ is compact for $\eta \in \left[1, \frac{(N-1)p}{N-p}\right)$, if $N > p$ and for all $\eta \geq 1$ if $N < p$, with

$$im\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega), \left(p' = \frac{p}{p-1}\right), \quad ker\gamma_0 = W_0^{1,p}(\Omega).$$

As usual, we drop the notation of the trace map γ writing simply u in place of $\gamma(u)$.

In our study of the problem (1.1), we work with the negative p-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$. Evidently, $-\Delta_p$ is continuous, bounded, pseudomonotone and satisfies the s_+-property (see [5], [20]). Moreover, we suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, in other word $f(.,t)$ is measurable for every $t \in \mathbb{R}$ and $f(x,.)$ is continuous for almost every $x \in \Omega$. Moreover, f satisfies in the following conditions:

(l_1) the subcritical growth condition: There exist two non-negative constants a_1, a_2 and a constant $s \in]p, p^*[$, such that

$$|f(x,t)| \leq a_1 + a_2|t|^{s-1}, \forall (x,t) \in \Omega \times \mathbb{R}.$$

(l_2) Ambrosetti-Rabinowitz (AR)-condition: there exist two constants $\mu > p$ and $M > 0$, such that

$$0 < \mu F(x,t) < tf(x,t), \forall x \in \Omega, |t| \geq M,$$

where $F(x,t) = \int_0^t f(x,\zeta)d\zeta, \forall (x,t) \in \Omega \times \mathbb{R}$.

We define the functional $I_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_{\partial\Omega} \alpha(x)|u(x)|^p d\sigma \tag{2.3}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx,$$

for every $u \in W^{1,p}(\Omega)$.

Lemma 2.1 *The functionals Φ, Ψ and I_λ are Gâteaux differentiable functionals. More precisely, we have*

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega |u(x)|^{p-2}u(x)v(x)dx + \int_{\partial\Omega} \alpha(x)|u(x)|^{p-2}uvd\sigma$$

and

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx,$$

for every $u, v \in W^{1,p}(\Omega)$.

We say that $u \in W^{1,p}(\Omega)$ is a weak solution of the problem (1.1), if

$$I_\lambda'(u)(v) = 0, \forall v \in W^{1,p}(\Omega),$$

which is equivalent to

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega |u(x)|^{p-2}u(x)v(x)dx + \int_{\partial\Omega} \alpha(x)|u(x)|^{p-2}uvd\sigma = \lambda\int_\Omega f(x,u(x))v(x)dx.$$

Definition 2.1 *Let S be a real reflexive Banach space. The functional I satisfies in the Palais-Smale condition (in short (PS)-condition), if any sequence $\{u_k\} \subset S$ such that*

(i_1) $\{I(u_k)\}$ *is bounded,*

(i_2) $\lim_{k\to\infty} \|I'(u_k)\|_{S^*} = 0$,

has a convergent subsequence.

Fix two real numbers $r_1 < r_2$ and put $I = \Phi - \lambda\Psi$ where $\Phi, \Psi : S \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals. We say I satisfies in the Palais-Smale condition cut off lower at r_1 and upper at r_2(${}^{[r_1]}(PS)^{[r_2]}$-condition), if any sequence $\{u_k\}$ such that (i_1), (i_2) and

(i_3) $r_1 < \Phi(u_k) < r_2$,

possesses a convergent subsequence. Obviously, if $r_1 = -\infty$ and $r_2 = \infty$ it coincides with the classical (PS)-condition. Moreover, we denote the condition by $(PS)^{[r_2]}$ if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ and by ${}^{[r_1]}(PS)$ if $r_1 \in \mathbb{R}$ and $r_2 = \infty$.

Now, we recall the basic tools that will be utilized in the following section. For all $r_1, r_2, r \in \mathbb{R}$ with $r_1 < r_2$, set

$$\vartheta(r_1, r_2) = \inf_{u\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{v\in\Phi^{-1}(r_1,r_2)} \Psi(v) - \Psi(u)}{r_2 - \Phi(u)},$$

$$\beta(r_1, r_2) = \sup_{u\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(u) - \sup_{v\in\Phi^{-1}(-\infty,r_1]} \Psi(v)}{\Phi(u) - r_1},$$

and

$$\beta(r) = \sup_{u\in\Phi^{-1}(r,+\infty)} \frac{\Psi(u) - \sup_{v\in\Phi^{-1}(-\infty,r]} \Psi(v)}{\Phi(u) - r}.$$

In order to prove the existence of at least one non-zero solution for the problem (1.1), we use the following version of Theorem 5.1 and Theorem 5.3 in [2], that was derived from Ricceri variational principle as presented in [26, Theorem 2.5].

Theorem 2.1 *Let, X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that, there are $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$ and $\vartheta(r_1, r_2) < \beta(r_1, r_2)$ and for each*

$$\lambda \in \left(\frac{1}{\beta(r_1, r_2)}, \frac{1}{\vartheta(r_1, r_2)}\right),$$

the function $I_\lambda = \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition. Then for all

$$\lambda \in \left(\frac{1}{\beta(r_1, r_2)}, \frac{1}{\vartheta(r_1, r_2)}\right),$$

there exists $u_{0\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0\lambda}) \le I_\lambda(u)$, for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0\lambda}) = 0$.

Theorem 2.2 *Let X be a real finite dimensional Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differetiable function whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differetiable function. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that $\beta(r) > 0$ and for each $\lambda > \frac{1}{\beta(r)}$, the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Then for each*

$$\lambda \in \left(\frac{1}{\beta(r)}, +\infty\right),$$

there is $u_{0\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0\lambda}) \le I_\lambda(u)$, for all $u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0\lambda}) = 0$.

We refer interested readers to [9], [13], [14], and [16], where Theorems 2.1 and 2.2 have been applied to establish the existence of solutions for various boundary value problems

3. Main Results

Put

$$k = \frac{|\Omega| + \alpha_\infty |\partial\Omega|}{|\Omega|^{\frac{p}{p^*}}} T^p,$$

that $|\partial\Omega| = \int_{\partial\Omega} d\sigma = \sigma(\partial\Omega)$ and $\alpha_\infty = ess\sup_\Omega \alpha(x)$.

For two non-negative constants c and d with

$$c \neq k^{\frac{1}{p}} d,$$

put

$$a_c(d) = \frac{a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s - \int_\Omega F(x,d)dx}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}(c^p - kd^p)},$$

where κ_s is defined in (2.2).

We now state and prove the main result of this paper, which establishes a qualitative property of solutions to problem (1.1).

Theorem 3.1 *Assume that f be a Carathéodory function that satisfies both the subcritical growth condition and AR-condition and there exist three positive constants c_1, c_2 and d with*

$$\frac{c_1}{|\Omega|^{\frac{1}{N}} T} < d < \frac{c_2}{k^{\frac{1}{p}}}, \tag{3.1}$$

such that

$$a_{c_2}(d) < a_{c_1}(d). \tag{3.2}$$

Then, for each $\lambda \in \left(\frac{1}{a_{c_1}(d)}, \frac{1}{a_{c_2}(d)}\right)$, the problem (1.1) admits at least two non-trivial solutions u_1 and u_2 in X, such that

$$\frac{|\Omega|^{\frac{1}{p^*}} c_1}{T\left(1 + a_\infty |\partial\Omega| T^p |\Omega|^{p - \frac{p}{p^*}}\right)^{\frac{1}{p}}} < \|u_i\| < \frac{|\Omega|^{\frac{1}{p^*}} c_2}{T}, \quad i = 1, 2.$$

Proof: Let us apply Theorem 2.1 to the functionals Φ and Ψ. By (2.3), we have Φ is coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$. We claim that, Φ has a continuous inverse on X^*. By direct calculation, one has

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u - v \rangle &= \int_\Omega |\nabla(u-v)(x)|^{p-2} \nabla(u-v)(x) \cdot \nabla(u-v)(x) dx \\
&+ \int_\Omega |(u-v)(x)|^{p-2} (u-v)(x)(u-v)(x) dx \\
&+ \int_{\partial\Omega} \alpha(x) |(u-v)(x)|^{p-2} (u-v)(u-v) d\sigma \\
&\geq \|u-v\|^p,
\end{aligned}$$

from this we deduce that Φ is uniformly monotone in X. Put $v = 0$, by $p > 1$ we have

$$\lim_{\|u\|\to\infty} \frac{\Phi'(u)[u]}{\|u\|} = +\infty,$$

so, Φ' is coercive. By [28, Theorem 26. A], we can see that Φ admits a continuous inverse on X^*. By condition (l_1), we have

$$F(x,t) \leq a_1|t| + \frac{a_2}{s}|t|^s, \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.3}$$

Then, from (3.3) it is easy to see that

$$\Psi(u) = \int_\Omega F(x,u(x))dx \leq a_1\|u\|_{L^1} + \frac{a_2}{s}\|u\|_{L^s}^s.$$

For all $u \in W^{1,p}(\Omega)$ with $\Phi(u) \leq r$, inequality (2.3) implies $\|u\| \leq (pr)^{\frac{1}{p}}$. Thus,

$$\Phi^{-1}(]-\infty,r]) \subseteq \{u \in W^{1,p}(\Omega) : \|u\| \leq (pr)^{\frac{1}{p}}\}.$$

Then, by (2.1) and (2.3) for every $u \in X$ such that $\Phi(u) \leq r$, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) \leq \sup_{u\in\Phi^{-1}(-\infty,r]} a_1\kappa_1\|u\| + \frac{a_2}{s}\kappa_s^s\|u\|^s \leq a_1\kappa_1(rp)^{\frac{1}{p}} + \frac{a_2}{s}\kappa_s^s(rp)^{\frac{s}{p}}.$$

Define $\omega(x) = d$ as a constant function in $\omega \in X$. Then,

$$\frac{d^p|\Omega|}{p} \leq \Phi(\omega) = \frac{d^p}{p}\Big(\int_\Omega dx + \int_{\partial\Omega}\alpha(x)d\sigma\Big) \leq \frac{d^p}{p}(|\Omega| + \alpha_\infty|\partial\Omega|) = \frac{d^p|\Omega|^{\frac{p}{p^*}}}{pT^p}k,$$

and

$$\Psi(\omega) = \int_\Omega F(x,d)dx.$$

Now, we set

$$r_1 = \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}c_1^p, \quad r_2 = \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}c_2^p.$$

By (3.1), a direct computation yields $r_1 < \Phi(\omega) < r_2$. Therefore, we have

$$\begin{aligned}
\vartheta(r_1,r_2) &= \inf_{u\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{v\in\Phi^{-1}(r_1,r_2)}\Psi(v) - \Psi(u)}{r_2 - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_2)}\Psi(v) - \Psi(\omega)}{r_2 - \Phi(\omega)} \\
&\leq \frac{a_1(r_2p)^{\frac{1}{p}}\kappa_1 + \frac{a_2}{s}(r_2p)^{\frac{s}{p}}\kappa_s^s - \int_\Omega F(x,d)dx}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}c_2^p - \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}kd^p} \\
&= a_{c_2}(d),
\end{aligned}$$

on the other hand, arguing as before, we have

$$\begin{aligned}
\beta(r_1,r_2) &= \sup_{u\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(u) - \sup_{v\in\Phi^{-1}(-\infty,r_1]}\Psi(v)}{\Phi(u) - r_1} \\
&\geq \frac{\Psi(\omega) - \sup_{v\in\Phi^{-1}(-\infty,r_1]}\Psi(v)}{\Phi(\omega) - r_1} \\
&\geq \frac{\int_\Omega F(x,d)dx - a_1(r_1p)^{\frac{1}{p}}\kappa_1 - \frac{a_2}{s}(r_1p)^{\frac{s}{p}}\kappa_s^s}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}kd^p - \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}c_1^p} \\
&= a_{c_1}(d),
\end{aligned}$$

from (3.2), one has $\vartheta(r_1, r_2) < \beta(r_1, r_2)$. By conditions $(l_1), (l_2)$ and [10, Lemma 1], the functional I_λ satisfies (PS)-condition. Hence, from Theorem 2.1 for each $\lambda \in \left(\frac{1}{a_{c_1}(d)}, \frac{1}{a_{c_2}(d)}\right)$, the functional I_λ admits at least one non-trivial critical point u_1 such that $r_1 < \Phi(u_1) < r_2$. This condition by the following inequality

$$\frac{1}{p}\|u_1\|^p \leq \Phi(u_1) \leq \frac{1}{p}\|u_1\|^p + \frac{1}{p} a_\infty |\partial\Omega| \kappa_1^p \|u_1\|^p$$
$$\leq \frac{1}{p}\|u_1\|^p \left(1 + a_\infty |\partial\Omega| T^p |\Omega|^{\frac{p(p^*-1)}{p^*}}\right),$$

is equivalent to

$$\frac{|\Omega|^{\frac{1}{p^*}} c_1}{T\left(1 + a_\infty |\partial\Omega| T^p |\Omega|^{p-\frac{p}{p^*}}\right)^{\frac{1}{p}}} < \|u_1\| < \frac{|\Omega|^{\frac{1}{p^*}} c_2}{T}.$$

Then, $u_1 \in X$ is a non-trivial local minimum for I_λ in X. Without loss of generality, we can assume that u_1 is a strict local minimum of I_λ in the X. So, there is $\tau > 0$ such that $\inf_{\|u-u_1\|=\tau} I_\lambda(u) > I_\lambda(u_1)$. Furthermore, (l_2) implies through standard arguments that I_λ is unbounded from below. Consequently, there exists $u_2 \in X$ satisfying $I_\lambda(u_2) < I_\lambda(u_1)$, for which I_λ verifies the mountain pass geometry. Moreover, by using (AR) again, we deduce that I_λ satisfies the Palais-Smale condition. Applying the classical Theorem of Ambrosetti and Rabinowitz, we obtain a second critical point $\tilde{u} \in X$ of I_λ with $I_\lambda(\tilde{u}) > I_\lambda(u_1)$. Thus, u_1 and $\tilde{u}$ are two distinct weak solutions to the problem (1.1). □

Remark 3.1 *Owing to the (AR)-condition in Theorem 3.1, we infer that the energy functional I_λ is unbounded from below and fulfills the classical (PS)-condition, so the classical Mountain Pass Theorem can be used. Hence, (AR)-condition could be replaced by a Cerami-type condition (C_e) in Theorem 3.1 to attain the second solution. The reader is referred to the papers [3,15,25], for more information on the subject.*

The functional $I \in C^1(S, \mathbb{R})$ verifies the (C_e)-condition, if every sequence $\{u_k\}$ such that

$$I(u_k) \to c, \quad \|I'(u_k)\|(1 + \|u_k\|) \to 0,$$

has a convergent subsequence.

Remark 3.2 *Theorem 3.1 ensures that the problem (1.1) admits at least two non-trivial solutions. One of these solutions has been achieved based on the classical Ambrosetti-Rabinowitz condition on the data by $f(x,0) \neq 0$ for every $x \in \Omega$. If the condition $f(x,0) \neq 0$ for every $x \in \Omega$ does not hold, the second solution u_2 to the problem (1.1) may indeed be zero.*

Below we give a particular case of Theorem 3.1.

Corollary 3.1 *Assume that, f be a Carathéodory function that satisfies both the subcritical growth condition and AR-condition and there exist two positive constants c and d with*

$$kd^p < c^p,$$

such that

$$\frac{\int_\Omega F(x,d)dx}{kd^p} > \frac{a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s}{c^p}. \tag{3.4}$$

Then, for each

$$\lambda \in \left(\frac{|\Omega|^{\frac{p}{p^*}} kd^p}{pT^p \int_\Omega F(x,d)dx}, \frac{|\Omega|^{\frac{p}{p^*}} c^p}{pT^p \left(a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s\right)} \right),$$

the problem (1.1) *admits at least two non-zero weak solution* u_1, u_2 *in* X*, such that*

$$\|u_i\| < \frac{|\Omega|^{\frac{1}{p^*}} c_2}{T}, \quad i = 1, 2.$$

Proof: In Theorem 3.1, we set $c_1 = 0$ and $c_2 = c$. Indeed, by (3.4) we have

$$\begin{aligned} a_c(d) &= \frac{a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s - \int_\Omega F(x,d)dx}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p}(c^p - kd^p)} \\ &< \frac{a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p} c^p} \\ &< \frac{\int_\Omega F(x,d)dx}{\frac{|\Omega|^{\frac{p}{p^*}}}{pT^p} kd^p} = a_0(d). \end{aligned}$$

So, Theorem 3.1 ensures the result. □

By applying Corollary 3.1, we obtain the following result.

Theorem 3.2 *Assume that* f *is a Carathéodory function that satisfies both the subcritical growth condition and AR-condition and*

$$\lim_{\xi \to 0^+} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} = +\infty. \tag{3.5}$$

Moreover, let $c > 0$ *and put*

$$\lambda_c^* = \frac{|\Omega|^{\frac{p}{p^*}} c^p}{pT^p \left(a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s \right)}.$$

Then, for every $\lambda \in (0, \lambda_c^*)$*, the problem* (1.1) *admits at least one non-zero weak solution* $u_{0,\lambda} \in X$ *such that* $\|u_{0,\lambda}\| \leq \frac{|\Omega|^{\frac{1}{p^*}} c}{T}$ *and* $\lim_{\lambda \to 0^+} \|u_{0,\lambda}\| = 0$.

Proof: Fix $\lambda \in (0, \lambda_c^*)$. By (3.5), there is a positive constant d with $kd^p < c^p$, such that

$$\frac{|\Omega|^{\frac{p}{p^*}} kd^p}{pT^p \int_\Omega F(x,d)dx} < \lambda < \frac{|\Omega|^{\frac{p}{p^*}} c^p}{pT^p \left(a_1 \frac{|\Omega|^{\frac{1}{p^*}} c}{T} \kappa_1 + \frac{a_2}{s} \frac{|\Omega|^{\frac{s}{p^*}} c^s}{T^s} \kappa_s^s \right)}.$$

Applying Corollary 3.1, the problem (1.1) admits at least one non-zero solution $u_{0,\lambda}$ such that $\|u_{0,\lambda}\| \leq \frac{|\Omega|^{\frac{1}{p^*}} c}{T}$. Then, for every $\lambda \in (0, \lambda_c^*)$, there exists at least one non-zero weak solution $u_{0,\lambda} \in \Phi^{-1}(0, r_2)$ for the problem (1.1) and we get

$$\frac{1}{p}\|u_{0,\lambda}\|^p \leq \Phi(u_{0,\lambda}) \leq r_2 = \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p} c_2^p,$$

so,

$$\|u_{0,\lambda}\| \leq \frac{|\Omega|^{\frac{1}{p^*}} c_2}{T}. \tag{3.6}$$

Hence, from (3.6), (l_1) and (2.2) one has

$$\Big|\int_{\Omega} f(x,u_{0,\lambda}(x))u_{0,\lambda}(x)dx\Big| \leq a_1 \frac{|\Omega|^{\frac{1}{p^*}}c_2}{T}\kappa_1 + \frac{a_2}{s}\frac{|\Omega|^{\frac{s}{p^*}}c_2^s}{T^s}\kappa_s^s, \tag{3.7}$$

for every $\lambda \in (0,\lambda_c^*)$. Since, $I'_\lambda(u_{0,\lambda})(v) = 0$ for all $\lambda \in (0,\lambda_c^*)$ and every $v \in X$, in particular $I'_\lambda(u_{0,\lambda})(u_{0,\lambda}) = 0$, we have

$$\Phi^{'}(u_{0,\lambda})(u_{0,\lambda}) = \lambda \int_{\Omega} f(x,u_{0,\lambda}(x))u_{0,\lambda}(x)dx,$$

for every $\lambda \in (0,\lambda_c^*)$. Then, from

$$0 \leq \|u_{0,\lambda}\|^p \leq \Phi^{'}(u_{0,\lambda})(u_{0,\lambda}),$$

and bearing (3.7) in mind, it follows

$$\lim_{\lambda\to 0^+} \|u_{0,\lambda}\|^p \leq \lim_{\lambda\to 0^+} \lambda\Psi^{'}(u_{0,\lambda})(u_{0,\lambda}) = 0,$$

consequently, $\lim_{\lambda\to 0^+}\|u_{0,\lambda}\|^p = 0$. □

Example 3.1 *Assume that* $N = 3, p = 2, a_1 = 1, a_2 = 3, s = 3, c = 1, \Omega = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3, |x_1^2 + x_2^2 + x_3^2 \leq 1\}, \alpha(x) = 1$ *and so* $p^* = 6$. *Set* $f(x,t) = 1 + 2t^2$, *for* $t \in \mathbb{R}$. *We have*

$$|f(x,t)| \leq 1 + 3t^2,$$

then, (l_1) *holds. On the other hand*

$$F(x,t) = \int_0^t f(x,\zeta)d\zeta = t + \frac{2}{3}t^3,$$

and

$$\lim_{|t|\to\infty} \frac{tf(x,t)}{F(x,t)} < \infty,$$

so, (l_2) *holds. A simple computation shows that*

$$\limsup_{t\to 0^+} \frac{F(t)}{t^p} = +\infty.$$

Also by definition of T, *one has*

$$T \leq (3\pi)^{\frac{-1}{2}} \left(\frac{\Gamma(3)}{\Gamma(\frac{3}{2})}\right)^{\frac{1}{3}},$$

and

$$\lambda^* = \frac{|\Omega|^{\frac{-2}{3}}}{4T^2} \geq \frac{(\frac{4\pi}{3})^{\frac{-2}{3}} \times 3\pi}{4\left(\frac{\Gamma(3)}{\Gamma(\frac{3}{2})}\right)^{\frac{2}{3}}}.$$

Applying Theorem 3.2 for every $\lambda \in \left(0, \frac{(\frac{4\pi}{3})^{\frac{-2}{3}} \times 3\pi}{4\left(\frac{\Gamma(3)}{\Gamma(\frac{3}{2})}\right)^{\frac{2}{3}}}\right)$, *the problem (1.1) has at least one non-zero weak solution.*

Remark 3.3 *Bearing in mind the above assumptions, we claim that the mapping* $\lambda \to I_\lambda(u_{0,\lambda})$ *is negative and strictly decreasing on the interval* $(0, \lambda_c^*)$. *Let* $u_{0,\lambda}$ *be the global minimum of the functional* I_λ *restricted to* $\Phi^{-1}(0, r_2)$, *where* r_2 *is given by* $r_2 = \frac{|\Omega|^{\frac{p}{p^*}}}{pT^p} c_2^p$, *that is a critical point (local minimum) of* I_λ *in* X. *Furthermore, since* $\omega \in \Phi^{-1}(0, r_2)$ *and*

$$\frac{\Phi(\omega)}{\Psi(\omega)} = \frac{\frac{d^p |\Omega|^{\frac{p}{p^*}}}{pT^p} k}{\int_\Omega F(x, d)dx} < \lambda,$$

it follows that

$$I_\lambda(u_{0,\lambda}) \leq I_\lambda(\omega) = \Phi(\omega) - \lambda\Psi(\omega) < 0.$$

We see that

$$I_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right),$$

for any $u \in X$ *and fix* $0 < \lambda_1 < \lambda_2 < \lambda_c^*$. *Define*

$$n_{\lambda_1} = \left(\frac{\Phi(u_{0,\lambda_1})}{\lambda_1} - \Psi(u_{0,\lambda_1})\right) = \inf_{u \in \Phi^{-1}(0,r_2)} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u)\right)$$

and

$$n_{\lambda_2} = \left(\frac{\Phi(u_{0,\lambda_2})}{\lambda_2} - \Psi(u_{0,\lambda_2})\right) = \inf_{u \in \Phi^{-1}(0,r_2)} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u)\right).$$

Arguing as before, we have $n_{\lambda_i} < 0$, *for* $i = 1, 2$ *and* $n_{\lambda_2} \leq n_{\lambda_1}$, *by* $\lambda_1 < \lambda_2$. *Owing to*

$$I_{\lambda_2}(u_{0,\lambda_2}) = \lambda_2 n_{\lambda_2} \leq \lambda_2 n_{\lambda_1} < \lambda_1 n_{\lambda_1} = I_{\lambda_1}(u_{0,\lambda_1}),$$

we observe that the mapping $\lambda \to I_\lambda(u_{0,\lambda})$ *is strictly decreasing in* $(0, \lambda_c^*)$. *So, the proof is complete.*

Remark 3.4 *We note that, if* f *is* $(p-1)$*-sublinear at infinity, Theorem 3.2 guarantees the existence of at least one non-zero weak solution for the problem (1.1), for every positive parameter* λ. *This ensured solution is non-trivial, whereas the classical direct method only ensures the existence of a solution that may be zero.*

Remark 3.5 *If* f *be a non-negative function, we deduce that the attained weak solution is also non-negative. Indeed, let* u_0 *be a weak solution of the problem (1.1). Arguing by contradiction, assume the set* $\mathcal{A} = \{x \in \Omega | u_0(x) < 0\}$ *has positive measure. Set* $v_0(x) = \min\{0, u_0(x)\}$ *for every* $x \in \Omega$. *Evidently,* $v_0 \in X$ *and*

$$\int_\Omega |\nabla u_0(x)|^{p-2} \nabla u_0(x) \cdot \nabla v_0(x) dx + \int_\Omega |u_0(x)|^{p-2} u_0(x) v_0(x) dx$$
$$+ \int_{\partial\Omega} \alpha(x) |u_0(x)|^{p-2} u_0 v_0 d\sigma - \lambda \int_\Omega f(x, u_0(x)) v_0(x) dx = 0.$$

So, we observe that

$$0 \leq \|u_0\|^p \leq \Phi^{'}(u_0)(u_{0,\lambda}) = \lambda \int_\Omega f(x, u_0(x)) u_0(x) dx \leq 0.$$

Thus, $u_0 = 0$, *which contradicts the definition of* $\mathcal{A}$. *Consequently,* u_0 *must be non-negative.*

Remark 3.6 *By a careful analysis of the proof of Theorem 3.2, we can see that the result remains true, when condition (3.5) is replaced by the more general assumption*

$$\limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} = +\infty.$$

Furthermore, in the autonomous case, the previous asymptotic condition at zero, can be expressed as:

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = +\infty.$$

We now present an application of Theorem 2.2, which will be used to establish the existence of multiple solutions to problem (1.1).

Theorem 3.3 *Suppose that f be a Carathéodory function and there exist two positive constants $\overline{c}$ and $\overline{d}$ that*

$$\frac{\overline{c}}{|\Omega|^{\frac{1}{N}}T} < \overline{d},$$

$$\int_\Omega F(x,\overline{d})dx > a_1 \frac{|\Omega|^{\frac{1}{p^*}}\overline{c}}{T}\kappa_1 + \frac{a_2}{s}\frac{|\Omega|^{\frac{s}{p^*}}\overline{c}^s}{T^s}\kappa_s^s,$$

and

$$\limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)dx}{|\xi|^p} \le 0. \tag{3.8}$$

Then, for any $\lambda > \bar{\lambda}$, where

$$\bar{\lambda} = \frac{\frac{|\Omega|^{\frac{p}{p^*}}k\overline{d}^p}{pT^p} - \frac{|\Omega|^{\frac{p}{p^*}}\overline{c}^p}{pT^p}}{\int_\Omega F(x,\overline{d})dx - a_1\frac{|\Omega|^{\frac{1}{p^*}}\overline{c}}{T}\kappa_1 - \frac{a_2}{s}\frac{|\Omega|^{\frac{s}{p^*}}\overline{c}^s}{T^s}\kappa_s^s},$$

the problem (1.1) admits at least one non-trivial solution $u_{0,\lambda} \in X$, such that

$$r = \frac{|\Omega|^{\frac{p}{p^*}}\overline{c}^p}{pT^p} < \Phi(u_{0,\lambda}) < \frac{1}{p}\|u_{0,\lambda}\|^p\Big(1 + a_\infty|\partial\Omega|T^p|\Omega|^{p-\frac{p}{p^*}}\Big),$$

so,

$$\frac{|\Omega|^{\frac{1}{p^*}}\overline{c}}{T\Big(1 + a_\infty|\partial\Omega|T^p|\Omega|^{p-\frac{p}{p^*}}\Big)^{\frac{1}{p}}} \le \|u_{0,\lambda}\|.$$

Proof: Our aim is to apply Theorem 2.2. Take into account the real Banach space X and the functionals Φ and Ψ, as in the proof of Theorem 3.1. The functionals Φ and Ψ satisfy all the assumptions requested in Theorem 2.2. We show that functional I_λ for each $\lambda > 0$, is coercive. Fix $0 < \delta < \frac{1}{\lambda p\kappa_p^p}$. By (3.8), there exist a function $\rho_\delta : \Omega \to \mathbb{R}$ such that $\int_\Omega \rho_\delta(x) < \infty$ and $F(x,t) \le \delta|t|^p + \rho_\delta(x)$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Then for any $u \in X$, we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\ge \frac{\|u\|^p}{p} - \lambda\int_\Omega F(x,u(x))dx \\
&\ge \frac{\|u\|^p}{p} - \lambda\delta\|u\|_{L^p}^p - \lambda\int_\Omega \rho_\delta(x) \\
&\ge \frac{\|u\|^p}{p} - \lambda\delta\kappa_p^p\|u\|^p - \lambda\int_\Omega \rho_\delta(x) \\
&= \Big(\frac{1}{p} - \lambda\delta\kappa_p^p\Big)\|u\|^p - \lambda\int_\Omega \rho_\delta(x),
\end{aligned}$$

so,

$$\lim_{\|u\|\to\infty}(\Phi(u)-\lambda\Psi(u))=\infty,$$

this implies that the functional I_λ is coercive. Let $\overline{r}=\frac{|\Omega|^{\frac{p}{p^*}}\overline{c}^p}{pT^p}$, arguing as in the proof of Theorem 3.1, we can see that

$$\beta_2(\overline{r})\geq\frac{\int_\Omega F(x,\overline{d})dx-a_1\frac{|\Omega|^{\frac{1}{p^*}}\overline{c}}{T}\kappa_1+\frac{a_2}{s}\frac{|\Omega|^{\frac{s}{p^*}}\overline{c}^s}{T^s}\kappa_s^s}{\frac{|\Omega|^{\frac{p}{p^*}}k\overline{d}^p}{pT^p}-\frac{|\Omega|^{\frac{p}{p^*}}\overline{c}^p}{pT^p}}.$$

From our assumptions, it can be concluded that $\beta_2(\overline{r})>0$. Therefore, Theorem 2.2 guarantees that the functional $\Phi-\lambda\Psi$ has at least one local minimum $u_{0,\lambda}$ such that

$$\Phi(u_{0,\lambda})>\frac{|\Omega|^{\frac{p}{p^*}}\overline{c}^p}{pT^p}.$$

□

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