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Positive Solutions for Fractional Boundary Value Problem with Laplacian Operator: Existence and Asymptotic Behavior

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ABSTRACT: This paper deals with existence and uniqueness of a positive solution for the fractional boundary value problem $D^{\beta}(\rho(x)D^{\alpha}u) = a(x)u^{\sigma}$ in $(0,1)$ with the condition

$$\lim_{x\to 0} D^{\beta-1}(\rho(x)D^{\alpha}u(x)) = \lim_{x\to 1}\rho(x)D^{\alpha}u(x) = 0 \text{ and } \lim_{x\to 0} D^{\alpha-1}u(x) = u(1) = 0,$$

where $\beta, \alpha \in (1,2]$, $\sigma \in (-1,1)$, the differential operator is taken in the Riemann-Liouville sense and $\rho, a\ : (0,1) \longrightarrow \mathbb{R}$ are nonnegative and continuous functions that may are singular at $x = 0$ or $x = 1$ and satisfies some appropriate conditions. We also give the global behavior of a such solution.

Key Words: Fractional differential equation, p-Laplacian operator, Dirichlet problem, positive solution, Schauder fixed point theorem.

Contents

1. Introduction

Many papers on fractional differential equations have been recently received much attention. It is caused by the intensive development of the theory of fractional calculus itself and by the applications of such construction in various fields of sciences and engineering, such as control, porus media, chemistry, physics, etc. For example and details, see [12] and [13].

In this paper, we consider the following nonlinear fractional problem

$$\begin{cases} D^{\beta}(\rho(x)D^{\alpha}u) = a(x)u^{\sigma}, \quad x \in (0,1), \\ \lim\limits_{x\to 0} D^{\beta-1}(\rho(x)D^{\alpha}u(x)) = \lim\limits_{x\to 1}\rho(x)D^{\alpha}u(x) = 0, \\ \lim\limits_{x\to 0} D^{\alpha-1}u(x) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in (1,2]$ and for $t \in \mathbb{R}$ and $\sigma \in (-1,1)$. The functions ρ and a are positive and continuous in $(0,1)$ that may are singular at $x = 0$ or $x = 1$ and satisfying some conditions detailed bellow and D^{α} is the Riemann-Liouville fractional derivative. Then, we will adress the question of existence, uniqueness and exact asymptotic behavior of positive solutions to problem (1.1).
For readers convenience, we recall that for a measurable function v, the Riemann-Liouville derivative $D^{\beta}v$ of order $\beta > 0$ is defined by

$$D^{\beta}v(x) = \frac{1}{\Gamma(n-\beta)}\Big(\frac{d}{dx}\Big)^{n}\int_0^x (x-t)^{n-\beta-1}v(t)dt,$$

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provided that the right-hand sides is pointwise defined on $(0,1]$. Here $n=[\beta]+1$, where $[\beta]$ means the integer part of the number β and Γ is the Euler Gamma function.

This work is motivated by recent advances in the study of fractional differential equations involving singular or sublinear nonlinearities with different boundary conditions (see $[7,9,11,15-18,22,24,27-28]$ and the references therein). Namely, in [28], the authors investigate the existence of positive solutions of the following problem

$$\begin{cases} D^{\beta}(\Phi_p(D^{\alpha}u)) = f(x,u), \quad x\in(0,1),\\ u(0)=0,\ u(1)=au(\xi),\\ D^{\alpha}u(0)=0,\ D^{\alpha}u(1)=bD^{\alpha}u(\varsigma), \end{cases}$$

where $\alpha,\beta\in[1,2]$, for $t\in\mathbb{R}$, $\Phi_p:=t|t|^{p-2}$, $(p>2)$, $a,b\in[0,1]$, $\xi,\varsigma\in(0,1)$ and f is a nonnegative continuous function on $(0,1)\times[0,\infty)$. Under some appropriate conditions, the authors proved by the Schauder's fixed-point theorem the existence and asymptotic behavior of positive continuous solution. Liu in [17], considered the fractional differential equation

$$D^{\beta}(\rho(x)\Phi_p(D^{\alpha}u(x))) = f(x,u(x)), \quad x\in(0,1),$$

where $\alpha,\beta\in(0,1]$, $\rho\in C((0,1))$ and f is a nonnegative function on $(0,1]\times\mathbb{R}$ allowed to be singular at $t=0$. The author proved the existence of positive solution with fractional nonlocal integral boundary conditions.

Recently, in [19], **Mâagli et al**. considered the following problem

$$\begin{cases} D^{\alpha}u(x) = -a(x)u^{\sigma}, \quad x\in(0,1),\\ \lim\limits_{x\to0^+} D^{\alpha-1}u(x)=0, \quad u(1)=0, \end{cases} \tag{1.2}$$

where $1<\alpha\le 2$, $-1<\sigma<1$ and the function a is required to satisfy some assumptions related to $\mathcal{K}$, the set of all Karamata functions L defined on $(0,\eta]$, by

$$L(t) := c\exp\left(\int_t^{\eta}\frac{z(s)}{s}ds\right),$$

for some $\eta>1$, where $c>0$ and z is a continuous function on $[0,\eta]$, with $z(0)=0$.
To describe the result of [19] in more details, we need some notations.

- For two nonnegative functions f and g defined on a set S, the notation $f(x)\approx g(x)$, $x\in S$ means that there exists $c>0$ such that $\frac{1}{c}f(x)\le g(x)\le cf(x)$, for all $x\in S$.
- For $\alpha\in(1,2]$, we denote by $\mathcal{C}_{2-\alpha}([0,1])$ the set of all measurable functions f such that $t\to t^{2-\alpha}f(t)$ is continuous on $[0,1]$.

In [19], **Mâagli et al**. studied problem (1.2) where a verifies
$(\mathbf{H}_0)$ $a\in C((0,1))$ satisfying for each $x\in(0,1)$,

$$a(x)\approx x^{-\lambda}(1-x)^{-\mu}L(x)\tilde{L}(1-x),$$

where $\lambda+(2-\alpha)\sigma\le 1$, $\mu\le\alpha$ and $L,\tilde{L}\in\mathcal{K}$ such that

$$\int_0^{\eta}\frac{L(t)}{t^{\lambda+(2-\alpha)\sigma}}dt<\infty \quad\text{and}\quad \int_0^{\eta}\frac{\tilde{L}(t)}{t^{\mu-\alpha+1}}dt<\infty.$$

Based on the Schauder fixed-point theorem, the authors showed in [19] the following result.

Theorem 1.1 *Assume that a satisfies (H_0). Then problem (1.2) has a unique positive solution $u\in C_{2-\alpha}([0,1])$ satisfying for $x\in(0,1)$,*

$$u(x)\approx x^{\alpha-2}(1-x)^{\min(1,\frac{\alpha-\mu}{1-\sigma})}\Psi_{\tilde{L},\mu,\sigma,\alpha}(1-x), \tag{1.3}$$

where $\Psi_{\tilde{L},\mu,\sigma,\alpha}$ *defined on* $(0,1)$ *by*

$$\Psi_{\tilde{L},\mu,\sigma,\alpha}(t) := \begin{cases} 1, & if\ \mu < \sigma+\alpha-1, \\ \left(\int_t^\eta \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ \mu = \sigma+\alpha-1, \\ \left(\tilde{L}(t)\right)^{\frac{1}{1-\sigma}}, & if\ \ \sigma+\alpha-1 < \mu < \alpha, \\ \left(\int_0^t \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ \mu = \alpha. \end{cases}$$

2. Main Results

The main goal of this paper is to improve and extend the above results on the boundary behavior of solutions to problem (1.1). More precisely, we consider the following hypotheses.
$(\mathbf{H}_1)$ a and ρ are nonnegative functions in $C((0,1))$ satisfying for each $x \in (0,1)$,

$$a(x) \approx x^{-\lambda}(1-x)^{-\mu} L_1(x) L_2(1-x),$$

where $\lambda + (2-\alpha)\sigma \leq 1$, $\mu \in \mathbb{R}$ and $L_1, L_2 \in \mathcal{K}$ such that

$$\int_0^\eta \frac{L_1(t)}{t^{\lambda+(2-\alpha)\sigma}} dt < \infty$$

and

$$\rho(x) \approx x^\gamma (1-x)^r,$$

where $\gamma < \beta - 1$ and $r < \alpha + 1$.

We introduce the quantities

$$\delta_1 = \min(1, \alpha - r + 1), \delta_2 = \frac{\beta - \mu - r + \alpha}{1-\sigma} \text{ and } \delta_3 = \min(1, \alpha - r).$$

The above values of δ_1, δ_2 and δ_3 are related to the boundary behavior of solutions to problem (1.1), as it will be explained in our main results stated in the following theorems.

Theorem 2.1 *Assume* $(\mathbf{H}_1)$ *and suppose that*

$$\mu < \beta - 1 + \sigma\delta_1 \text{ and } r < \alpha + 1.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_1} N(1-x),$$

where

$$N(x) = \begin{cases} 1, & if\ r \neq \alpha, \\ \log(\frac{\eta}{x}), & if\ r = \alpha. \end{cases}$$

Theorem 2.2 *Assume* $(\mathbf{H}_1)$ *and suppose that*

$$\mu = \beta - 1 + \sigma\delta_1,$$

$$r < \alpha + 1 \text{ and } r \neq \alpha.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_1} N(1-x),$$

where

$$N(x) = \begin{cases} 1, & if\ r < \alpha, \\ \left(\int_x^\eta \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha < r < \alpha+1. \end{cases}$$

Theorem 2.3 *Assume* $(\mathbf{H}_1)$ *and suppose that*

$$\beta - 1 + \sigma < \mu < \beta + \sigma \text{ and } r \le \alpha - 1 + \beta + \sigma - \mu.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)N(1-x),$$

where

$$N(x) = \begin{cases} 1, & if\ r < \alpha - 1 + \beta + \sigma - \mu, \\ \left(\int_x^\eta \frac{(L_2(s))}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ r = \alpha - 1 + \beta + \sigma - \mu. \end{cases}$$

Theorem 2.4 *Assume* $(\mathbf{H}_1)$ *and suppose that*

$$\beta - 1 < \mu - \delta_2\sigma < \beta$$

and

$$\alpha - 1 < r + \mu - \delta_2\sigma - \beta < \alpha.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_2}(L_2(1-x))^{\frac{1}{1-\sigma}}.$$

Theorem 2.5 *Assume* $(\mathbf{H}_1)$ *and suppose that* $\beta - 1 < \mu < \beta$, $r = \alpha + \beta - \mu$ *and* L_2 *satisfy*

$$\int_0^\eta \frac{(L_2(t))}{t} dt < \infty.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}\left(\int_0^{1-x} \frac{(L_2(t))}{t} dt\right)^{\frac{1}{1-\sigma}}.$$

Theorem 2.6 *Assume* $(\mathbf{H}_1)$ *and suppose that*

$$\mu = \beta + \sigma\delta_3,$$

$$r < \alpha,\ r \neq \alpha - 1$$

and L_2 *satisfy*

$$\int_0^\eta \frac{L_2(s)}{s} ds < \infty.$$

Then problem (1.1) *has a unique positive solution* $u \in C_{2-\alpha}([0,1])$ *satisfying for* $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_3}N(1-x),$$

where

$$N(x) = \begin{cases} 1, & if\ r < \alpha - 1, \\ \left(\int_0^x \frac{L_2(t)}{t} dt\right)^{\frac{1}{1-\sigma}}, & if\ \alpha - 1 < r < \alpha. \end{cases}$$

The rest of the paper is as follows. In Section 3, we state some already known results on Karamata functions. Also, we give some necessary definitions and lemmas from fractional calculus theory. In section 4, we present some necessary conditions to existence result and we prove our main results stated in Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6 which illustrated by an example. The last Section is reserved to a conclusion.

3. Preliminary Results

In what follows, we are quoting without proof some fundamental properties of functions belonging to the class $\mathcal{K}$ collected from [8] and [26]. We recall that a function L defined on $(0,\eta]$ belongs to the class $\mathcal{K}$, if

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}ds\right),$$

for some $\eta > 1$, $c > 0$ and z is a continuous function on $[0,\eta]$ with $z(0) = 0$.

Proposition 3.1 *(i) A function L is in $\mathcal{K}$ if and only if L is a positive function in $\mathcal{C}^1((0,\eta])$ such that*

$$\lim_{t\to 0^+} \frac{tL^{'}(t)}{L(t)} = 0. \tag{3.1}$$

(ii) Let $L_1,\ L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then we have

$$L_1 + L_2 \in \mathcal{K}, L_1L_2 \in \mathcal{K} \text{ and } L_1^p \in \mathcal{K}.$$

(iii) Let $L \in \mathcal{K}$ and $\varepsilon > 0$. Then we have

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0.$$

As a standard example of functions belonging to the class $\mathcal{K}$ (see [26]), we give

Example 3.1 Let $m \in \mathbb{N}^*$ and $\eta > 0$. Let $(\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$ and w be a sufficiently large positif real number such that the function

$$L(t) = \prod_{1\le i\le m} \left(\log_i(\frac{w}{t})\right)^{\mu_i}$$

is defined and positive on $(0,\eta]$, where $\log_i t = \log\circ....\circ\log t$ (i *times*). Then we have $L \in \mathcal{K}$.

Lemma 3.1 *(i) Let L be a function in $\mathcal{K}$. Then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\displaystyle\int_t^\eta \frac{L(s)}{s}ds} = 0.$$

In particular,

$$t \longmapsto \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}.$$

(ii) If $\displaystyle\int_0^\eta \frac{L(s)}{s}ds$ converges, then

$$\lim_{t\to 0^+} \frac{L(t)}{\displaystyle\int_0^t \frac{L(s)}{s}ds} = 0.$$

In particular,

$$t \longmapsto \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}.$$

Applying Karamata's theorem, we get the following.

Lemma 3.2 *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$ for some $\eta > 1$. We have*
(i) If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s)ds$ converges and

$$\int_0^t s^\gamma L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

(ii) If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)ds$ diverges and

$$\int_t^\eta s^\gamma L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

Since our approach is based on potential theory, we recall in the following some basic tools. For $\gamma \in (1,2]$, we denote by $G_\gamma(x,t)$ the Green's function for the boundary value problem (1.2). From [19], we have

$$G_\gamma(x,t) = \frac{1}{\Gamma(\gamma)}[x^{\gamma-2}(1-t)^{\gamma-1} - ((x-t)^+)^{\gamma-1}],$$

where $x^+ = \max(x,0)$.

Proposition 3.2 *(see [19]) Let $1 < \gamma \leq 2$ and f be a nonnegative measurable function on $(0,1)$. Then we have*
(i) For $x, t \in (0,1)$,

$$G_\gamma(x,t) \approx x^{\gamma-2}(1-t)^{\gamma-2}(1-\max(x,t)).$$

(ii) For $x \in (0,1)$, $G_\gamma f(x) := \int_0^1 G_\gamma(x,t)f(t)dt < \infty$ if and only if $\int_0^1 (1-t)^{\gamma-1}f(t)dt < \infty$.
(iii) If the map $t \to (1-t)^{\gamma-1}f(t)$ is continuous and integrable on $(0,1)$, then $G_\gamma f$ is the unique solution in $C_{2-\gamma}([0,1])$ of the boundary value problem

$$\begin{cases} D^\gamma u(x) = -f(x), \quad x \in (0,1), \\ \lim\limits_{x\to 0^+} D^{\gamma-1}u(x) = 0, \; u(1) = 0. \end{cases}$$

Here, below we provide a crucial property concerning continuity.

Lemma 3.3 *Let φ and ρ be tow nonnegative functions in $\mathcal{B}((0,1))$ such that*

$$\int_0^1 (1-t)^{\alpha-1}\frac{1}{\rho(t)}G_\beta\varphi(t)dt < \infty.$$

Then the family

$$\mathcal{F} = \{Sf : x \longmapsto x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta f)(x);\; |f| \leq \varphi\}$$

is relatively compact in $C([0,1])$.

Proof: Let $f \in \mathcal{B}((0,1))$ such that $|f(x)| \leq \varphi(x)$ for all $x \in (0,1)$.
Let $x \in (0,1)$, we have

$$\begin{aligned} Sf(x) &= x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta f)(x) \\ &= x^{2-\alpha}\int_0^1 G_\alpha(x,t)(\frac{1}{\rho(t)}G_\beta f(t)dt. \end{aligned}$$

Using Proposition 3.2 (i), we obtain that

$$|Sf(x)| \leq \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{(1-t)^{\alpha-1}}{\rho(t)}G_\beta\varphi(t)dt < \infty.$$

Thus $\mathcal{F}$ is uniformly bounded. Now, let us prove that $\mathcal{F}$ is equicontinuous in $[0,1]$.
Let $x, y \in (0,1)$, then we have

$$\begin{aligned}
|Sf(x) - Sf(y)| &= \left|x^{2-\alpha} G_\alpha(\frac{1}{\rho} G_\beta f))(x) - y^{2-\alpha} G_\alpha(\frac{1}{\rho} G_\beta f)(y)\right| \\
&= \frac{1}{\Gamma(\alpha)} \Big| \int_0^1 \left(y^{2-\alpha}((y-t)^+)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}\right) (\frac{1}{\rho(t)} G_\beta f(t)) dt \Big| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 \left|y^{2-\alpha}((y-t)^+)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}\right| (\frac{1}{\rho(t)} G_\beta \varphi(t)) dt.
\end{aligned}$$

For every $t \in (0,1)$, we have

$$\left|y^{2-\alpha}((y-t)^+)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}\right| \longrightarrow 0 \text{ as } \left|x-y\right| \longrightarrow 0$$

and

$$\left|y^{2-\alpha}((y-t)^+)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}\right| \leq 2(1-t)^{\alpha-1}.$$

Then we obtain by Lesbegue's theorem that

$$|Sf(x) - Sf(y)| \longrightarrow 0 \quad \text{as } |x-y| \longrightarrow 0.$$

Now, let $x \in (0,1)$, we have

$$\begin{aligned}
\left|Sf(x) - \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-t)^{\alpha-1}}{\rho(t)} G_\beta f(t) dt\right| &= \frac{1}{\Gamma(\alpha)} \Big| \int_0^1 x^{2-\alpha}((x-t)^+)^{\alpha-1} (\frac{1}{\rho(t)} G_\beta f(t)) dt \Big| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 x^{2-\alpha}((x-t)^+)^{\alpha-1} (\frac{1}{\rho(t)} G_\beta \varphi(t)) dt.
\end{aligned}$$

Using the fact that for $t \in (0,1)$,

$$x^{2-\alpha}((x-t)^+)^{\alpha-1} \longrightarrow 0 \text{ as } x \longrightarrow 0$$

and

$$0 \leq x^{2-\alpha}((x-t)^+)^{\alpha-1} \leq (1-t)^{\alpha-1},$$

we get again by Lesbegue's theorem that

$$\left|Sf(x) - \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-t)^{\alpha-1}}{\rho(t)} G_\beta f(t) dt\right| \longrightarrow 0 \quad \text{as } x \longrightarrow 0.$$

Furthermore, for $x \in (0,1)$, we have

$$\left|Sf(x)\right| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 ((1-t)^{\alpha-1} - x^{2-\alpha}((x-t)^+)^{\alpha-1}) (\frac{1}{\rho(t)} G_\beta \varphi(t)) dt.$$

By similar arguments as above, we deduce that

$$\left|Sf(x)\right| \longrightarrow 0 \quad \text{as } x \longrightarrow 1.$$

Finally, we conclude that the family $\mathcal{F}$ is equicontinuous in $[0,1]$. Hence, by Ascoli's theorem, we deduce that $\mathcal{F}$ is relatively compact in $\mathcal{C}([0,1])$. □

The following Lemma is due to [19].

Lemma 3.4 *Let $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$ and let for $x \in (0,1)$*

$$b(t) = t^{-\lambda_1}(1-t)^{-\mu_1}\tilde{L}_1(t)\tilde{L}_2(1-t),$$

with $\lambda_1 \leq 1$ and $\mu_1 \leq \beta$. Assume that

$$\int_0^\eta t^{-\lambda_1}\tilde{L}_1(t)dt < \infty \text{ and } \int_0^\eta t^{\beta-1-\mu_1}\tilde{L}_2(t)dt < \infty.$$

Then we have for $x \in (0,1)$,

$$x^{2-\beta}G_\beta b(x) \approx (1-x)^{\min(1,\beta-\mu_1)}\Psi_{\tilde{L}_2,\mu_1,0,\beta}(1-x).$$

Remark 3.1 We need to verify condition $\int_0^\eta t^{-\lambda_1}\tilde{L}_1(t)dt < \infty$ and $\int_0^\eta t^{\beta-1-\mu_1}\tilde{L}_2(t)dt < \infty$ in Lemma 3.4, only if $\lambda_1 = 1$ and $\mu_1 = \beta$. This is due to Lemma 3.2.

4. Proof of Main Results

We begin this section by stating the following proposition, that will play a crucial role in the proofs of our main results.

Proposition 4.1 *Let for $t \in (0,1)$,*

$$w(t) = a(t)t^{(\alpha-2)\sigma}\theta^\sigma(1-t).$$

Assume $(\mathbf{H}_1)$ and suppose that there exist a nonnegative function θ in $C([0,1])$ such that

$$\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty \tag{4.1}$$

and

$$x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \approx \theta(1-x). \tag{4.2}$$

Then problem (1.1) has a unique solution $u \in C_{2-\alpha}([0,1])$ satisfying for each $x \in (0,1)$

$$u(x) \approx x^{\alpha-2}\theta(1-x). \tag{4.3}$$

Proof: Let $m \geq 1$ and θ be a nonnegative function satisfying for each $x \in (0,1)$

$$\frac{1}{m}\theta(1-x) \leq x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \leq m\theta(1-x). \tag{4.4}$$

Put $c_0 := m^{\frac{1}{1-|\sigma|}}$. We consider the closed convex set given by

$$Y := \{v \in \mathcal{C}([0,1]); \frac{1}{c_0}\theta(1-x) \leq v(x) \leq c_0\theta(1-x)\}.$$

Using Lemma 3.3 and Proposition 3.2 (ii), we easily see that the function $x \mapsto x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x)$ belongs to $\mathcal{C}([0,1])$ and satisfies (4.4). So Y is not empty. In order to use the Schauder's fixed point theorem, we denote $\tilde{a}(x) = x^{(\alpha-2)\sigma}a(x)$ and we define the operator T on Y by

$$Tv(x) = x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta(\tilde{a}v^\sigma))(x).$$

We need to check that the operator T has a fixed point v in Y. For this choice of c_0, we will prove that T map Y into itself. Indeed, let $v \in Y$, by using (4.4), we have

$$\begin{aligned} Tv(x) &\leq c_0^{|\sigma|}x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \\ &\leq c_0^{|\sigma|}m\theta(1-x) \\ &= c_0\theta(1-x) \end{aligned}$$

and

$$\begin{aligned} Tv(x) &\geq c_0^{-|\sigma|}x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \\ &\geq \frac{1}{c_0^{|\sigma|}m}\theta(1-x) \\ &= \frac{1}{c_0}\theta(1-x). \end{aligned}$$

Furthermore, using (4.4), we have for each $x \in (0,1)$

$$G_\alpha(\frac{1}{\rho}G_\beta w)(x) < \infty.$$

This implies by Proposition 3.2 (ii) that

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{\rho(t)}G_\beta w(t)dt < \infty.$$

Hence, it follows from Lemma 3.3 that the family TY is relatively compact in $\mathcal{C}([0,1])$. So Y is invariant under T.
Next, we shall prove the continuity of T. Let $(v_k)_k$ be a sequence in Y which converges uniformly to v in Y.
For $x \in (0,1)$, we have

$$\begin{aligned} |Tv_k(x) - Tv(x)| &= x^{2-\alpha}\big|G_\alpha(\frac{1}{\rho}G_\beta(\tilde{a}v_k^\sigma))(x) - G_\alpha(\frac{1}{\rho}G_\beta(\tilde{a}v^\sigma))(x)\big| \\ &\leq x^{2-\alpha}\int_0^1 G_\alpha(x,t)\big|\frac{1}{\rho(t)}(G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t))\big|dt. \end{aligned}$$

For $t \in (0,1)$, we have

$$\big|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\big| \leq \int_0^1 G_\beta(t,s)\big|(\tilde{a}v_k^\sigma)(s) - (\tilde{a}v^\sigma)(s)\big|ds$$

and for every $s \in (0,1)$,

$$\big|(\tilde{a}v_k^\sigma)(s) - (\tilde{a}v^\sigma)(s)\big| \leq 2c_0^{|\sigma|}w(s).$$

Using Proposition 3.2 (ii) and (4.1), we obtain by Lesbegue's theorem that

$$\big|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\big| \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

We deduce that

$$\big|\frac{1}{\rho(t)}(G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t))\big| \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

We have

$$\big|\frac{1}{\rho(t)}(G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t))\big| \leq 2c_0^{|\sigma|}(\frac{1}{\rho(t)}G_\beta w(t)).$$

Using (4.4), we obtain by Lesbegue's theorem that for $x \in [0,1]$

$$Tv_k(x) \longrightarrow Tv(x), \text{ as } k \longrightarrow \infty.$$

Since TY is a relatively compact in $\mathcal{C}([0,1])$, we have the uniform convergence, namely

$$\|Tv_k - Tv\|_\infty \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

Thus, we have proved that T is a compact mapping from Y into itself. It follows by the Schauder fixed point theorem that there exists $v \in Y$ such that $Tv = v$. Put

$$u(x) = x^{\alpha-2}v(x),$$

then $u \in \mathcal{C}_{2-\alpha}([0,1])$ and u satisfies the equation

$$u(x) = G_\alpha(\frac{1}{\rho}G_\beta(au^\sigma))(x).$$

Then due to Lemma 3.4, u is a positive continuous solution of problem (1.1).
Finally, let us prove that u is the unique positive continuous solution satisfying (4.3). To this aim, we assume that (1.1) has two positive solutions u and v satisfying (4.3). Then, there exists a constant $m > 1$ such that

$$\frac{1}{m}v \le u \le mv.$$

This implies that the set

$$J := \left\{t \in (1,\infty) : \frac{1}{t}v \le u \le tv\right\}$$

is not empty. Now, put $c := \inf J$, then we aim to show that $c = 1$. Suppose that $c > 1$. Then by simple calculus, we obtain that

$$\begin{cases} D^\beta\big(\rho(D^\alpha(c^{|\sigma|}v) - D^\alpha u)\big) = a(c^{|\sigma|}v^\sigma - u^\sigma) \ge 0, \\ \lim\limits_{x\to 0} D^{\beta-1}\big(\rho(D^\alpha(c^{|\sigma|}v) - D^\alpha u)\big)(x) = 0, \\ \lim\limits_{x\to 1} \rho(x)\big(D^\alpha(c^{|\sigma|}v)(x) - D^\alpha u(x)\big) = 0. \end{cases}$$

We conclude by Proposition 3.2 (iii) that

$$\rho\big(D^\alpha(c^{|\sigma|}v) - D^\alpha u\big) = -G_\beta(a(c^{|\sigma|}v^\sigma - u^\sigma)) \le 0.$$

Then we have

$$D^\alpha(c^{|\sigma|}v) \le D^\alpha u.$$

Which implies that

$$\begin{cases} D^\alpha(c^{|\sigma|}v - u) \le 0, \\ \lim\limits_{x\to 0} D^{\alpha-1}(c^{|\sigma|}v - u)(x) = 0, \\ c^{|\sigma|}v(1) - u(1) = 0. \end{cases}$$

Using again Proposition 3.2 (iii), we conclude that

$$c^{|\sigma|}v - u \ge 0.$$

By symmetry, we obtain that $v \le c^{|\sigma|}u$. So $c^{|\sigma|} \in J$. Since $|\sigma| < 1$ and $c > 1$, we have $c^{|\sigma|} < c$. This yields to a contradiction with the fact that $c = \inf J$. Hence $c = 1$ and consequently $u = v$.

□

Proof of Theorem 2.1:

Suppose that

$$\mu < \beta - 1 + \sigma\delta_1 \text{ and } r < \alpha + 1.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = x^{\delta_1}N(x),$$

where

$$N(x) = \begin{cases} 1, & if\ r \neq \alpha, \\ \log(\frac{\eta}{x}), & if\ r = \alpha. \end{cases}$$

Put

$$w(x) = a(x)x^{(\alpha-2)\sigma}\theta^{\sigma}(1-x).$$

Using $(\mathbf{H}_1)$, we deduce that

$$w(x) \approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{\delta_1\sigma-\mu}L_1(x)L_2(1-x)N^{\sigma}(1-x).$$

Since $\mu - \delta_1\sigma < \beta - 1$, we conclude by Lemma 3.4 that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}(1-x).$$

This implies by Proposition 3.2 (ii) that $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$ and for $x \in (0,1)$, we have

$$\frac{1}{\rho}G_\beta w(x) \approx x^{-\gamma-2+\beta}(1-x)^{-r+1}.$$

Using again Lemma 3.4, we deduce that for $x \in (0,1)$

$$\begin{aligned} x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) &\approx \begin{cases} (1-x), & if\ r<\alpha, \\ (1-x)\log(\frac{\eta}{1-x}), & if\ r=\alpha, \\ (1-x)^{\alpha-r+1}, & if\ \alpha<r \end{cases} \\ &= \theta(1-x). \end{aligned}$$

It follows by Proposition 4.1 that problem (1.1) has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_1}N(1-x) = x^{\alpha-2}\theta(1-x).$$

Proof of Theorem 2.2:

Suppose that

$$\mu = \beta - 1 + \sigma\delta_1,$$

$$r < \alpha + 1 \text{ and } r \neq \alpha.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = x^{\delta_1}N(x),$$

where

$$N(x) = \begin{cases} 1, & if\ r<\alpha+1, \\ \left(\int_x^\eta \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha<r<\alpha+1. \end{cases}$$

Put

$$w(x) = a(x)x^{(\alpha-2)\sigma}\theta^{\sigma}(1-x),\ x \in (0,1).$$

Thus

$$w(x) \approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{\delta_1\sigma-\mu}L_1(x)L_2(1-x)N^{\sigma}(1-x).$$

Since $\mu - \delta_1\sigma = \beta - 1$, we conclude by Lemma 3.4 that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}(1-x)\begin{cases} \int_{1-x}^\eta \frac{L_2(s)}{s}ds, & if\ r<\alpha, \\ \left(\int_{1-x}^\eta \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha<r<\alpha+1. \end{cases}$$

This implies by Proposition 3.2 (ii) that $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$ and for $x \in (0,1)$, we have

$$\begin{aligned} \frac{1}{\rho}G_\beta w(x) &\approx x^{-\gamma-2+\beta}(1-x)^{-r+1} \\ &\times \begin{cases} \left(\int_{1-x}^{\eta} \frac{L_2(s)}{s}ds\right), & if\ r < \alpha, \\ \left(\int_{1-x}^{\eta} \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha < r < \alpha+1. \end{cases} \end{aligned}$$

Using again $(\mathbf{H}_1)$ and by Lemma 3.4, we deduce that for $x \in (0,1)$

$$\begin{aligned} x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) &\approx (1-x)^{\delta_1}N(1-x) \\ &= \theta(1-x). \end{aligned}$$

Hence it follows from Proposition 4.1 that problem (1.1) has a unique positive solution u in $C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}\theta(1-x).$$

Proof of Theorem 2.3:

Suppose that

$$\beta - 1 + \sigma < \mu < \beta + \sigma \text{ and } r \leq \alpha - 1 + \beta + \sigma - \mu.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = xN(x),$$

where

$$N(x) = \begin{cases} 1, & if\ r < \alpha - 1 + \beta + \sigma - \mu, \\ \left(\int_x^{\eta} \frac{(L_2(s))}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ r = \alpha - 1 + \beta + \sigma - \mu. \end{cases}$$

From hypothesis $(\mathbf{H}_1)$, we have

$$\begin{aligned} w(x) &:= a(x)x^{(\alpha-2)\sigma}\theta^\sigma(1-x) \\ &\approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{\sigma-\mu}L_1(x)L_2(1-x)N^\sigma(1-x). \end{aligned}$$

Using the fact that $\beta - 1 < \mu - \sigma < \beta$, we deduce by Lemma 3.4, that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}(1-x)^{\beta-\mu+\sigma}L_2(1-x)N^\sigma(1-x).$$

Then by Proposition 3.2 (ii), we have $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$.
Moreover, for $x \in (0,1)$, we have

$$\frac{1}{\rho}G_\beta w(x) \approx x^{-\gamma-2+\beta}(1-x)^{-\mu+\beta+\sigma-r}L_2(1-x)N^\sigma(1-x).$$

By Lemma 3.4, we obtain that for $x \in (0,1)$

$$\begin{aligned} x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) &\approx (1-x)N(1-x) \\ &= \theta(1-x). \end{aligned}$$

The proof of our Theorem 2.3 ends by using Proposition 4.1.

Proof of Theorem 2.4:

Suppose that

$$\beta - 1 < \mu - \delta_2\sigma < \beta \text{ and } \alpha - 1 < r + \mu - \delta_2\sigma - \beta < \alpha.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = x^{\delta_2}(L_2(x))^{\frac{1}{1-\sigma}}.$$

From hypothesis $(\mathbf{H}_1)$, we have for $x \in (0,1)$

$$\begin{aligned} w(x) &:= a(x)x^{(\alpha-2)\sigma}\theta^{\sigma}(1-x) \\ &\approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{\delta_2\sigma-\mu}L_1(x)(L_2(1-x))^{\frac{1}{1-\sigma}}. \end{aligned}$$

Using the fact that $\beta - 1 < \mu - \delta_2\sigma < \beta$, we deduce by Lemma 3.4, that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}(1-x)^{\beta-\mu+\delta_2\sigma}(L_2(1-x))^{\frac{1}{1-\sigma}}.$$

Then by Proposition 3.2 (ii), we have $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$.
Moreover, for $x \in (0,1)$, we have

$$\frac{1}{\rho}G_\beta w(x) \approx x^{-\gamma-2+\beta}(1-x)^{-\mu+\beta+\delta_2\sigma-r}(L_2(1-x))^{\frac{1}{1-\sigma}}.$$

By Lemma 3.4, we obtain that for $x \in (0,1)$

$$x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \approx (1-x)^{\delta_2}(L_2(1-x))^{\frac{1}{1-\sigma}} = \theta(1-x).$$

The proof of our Theorem 2.4 ends by using Proposition 4.1.

Proof of Theorem 2.5:

Suppose that $\beta - 1 < \mu < \beta$, $r = \alpha + \beta - \mu$ and L_2 satisfy

$$\int_0^\eta \frac{(L_2(t)}{t}dt < \infty.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = \left(\int_0^x \frac{(L_2(t))}{t}dt\right)^{\frac{1}{1-\sigma}}.$$

Then, for $x \in (0,1)$

$$\begin{aligned} w(x) &:= a(x)x^{(\alpha-2)\sigma}\theta^{\sigma}(1-x) \\ &\approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{-\mu}L_1(x)L_2(1-x)\left(\int_0^{1-x} \frac{(L_2(t))}{t}dt\right)^{\frac{\sigma}{1-\sigma}}. \end{aligned}$$

We conclude by Lemma 3.4 that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}(1-x)^{\beta-\mu}L_2(1-x)\left(\int_0^{1-x} \frac{(L_2(t))}{t}dt\right)^{\frac{\sigma}{1-\sigma}}.$$

This implies by Proposition 3.2 (ii) that $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$ and for $x \in (0,1)$, we have

$$\frac{1}{\rho}G_\beta w(x) \approx x^{-\gamma-2+\beta}(1-x)^{-\mu+\beta-r}L_2(1-x)\left(\int_0^{1-x}\frac{(L_2(t))}{t}dt\right)^{\frac{\sigma}{1-\sigma}}.$$

Using again $(\mathbf{H}_1)$ and by Lemma 3.4, we deduce that for $x \in (0,1)$

$$x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \approx \theta(1-x).$$

Hence it follows from Proposition 4.1 that problem (1.1) has a unique positive solution u in $C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,

$$u(x) \approx x^{\alpha-2}\theta(1-x).$$

Proof of Theorem 2.6:

Suppose that $\mu = \beta + \sigma\delta_3$, $r < \alpha$, $r \neq \alpha - 1$ and L_2 satisfy

$$\int_0^\eta \frac{L_2(s)}{s}ds < \infty.$$

Let θ be the function defined on $[0,1]$ by

$$\theta(x) = x^{\delta_3}N(x),$$

where

$$N(x) = \begin{cases} 1, & if\ r < \alpha - 1, \\ \left(\int_x^\eta \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha - 1 < r < \alpha. \end{cases}$$

From hypothesis $(\mathbf{H}_1)$, we have for $x \in (0,1)$

$$\begin{aligned} w(x) &:= a(x)x^{(\alpha-2)\sigma}\theta^\sigma(1-x) \\ &\approx x^{-\lambda+(\alpha-2)\sigma}(1-x)^{\sigma\delta_3-\mu}L_1(x)L_2(1-x)N^\sigma(1-x). \end{aligned}$$

Since $\mu - \sigma\delta_3 = \beta$, we deduce by Lemma 3.4, that for $x \in (0,1)$

$$G_\beta w(x) \approx x^{\beta-2}\begin{cases} \int_{1-x}^\eta \frac{L_2(s)}{s}ds, & if\ r < \alpha - 1, \\ \left(\int_{1-x}^\eta \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha - 1 < r < \alpha. \end{cases}$$

This implies by Proposition 3.2 (ii) that $\int_0^1 (1-t)^{\beta-1}w(t)dt < \infty$ and for $x \in (0,1)$, we have

$$\frac{1}{\rho}G_\beta w(x) \approx x^{-\gamma-2+\beta}(1-x)^{-r}\begin{cases} \left(\int_{1-x}^\eta \frac{L_2(s)}{s}ds\right), & if\ r < \alpha - 1, \\ \left(\int_{1-x}^\eta \frac{L_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}}, & if\ \alpha - 1 < r < \alpha. \end{cases}$$

Using again the same arguments, we obtain for $x \in (0,1)$

$$x^{2-\alpha}G_\alpha(\frac{1}{\rho}G_\beta w)(x) \approx \theta(1-x).$$

The proof ends by applying Proposition 4.1.
As an application of our main results, we give the following example.

Example

Let $\beta, \alpha \in (1,2]$ and $\sigma \in (-1,1)$. Let a and ρ be two positive continuous functions on $(0,1)$ such that

$$a_1(x) \approx (1-x)^{-\mu} \left(\log(\frac{3}{1-x})\right)^{-2}, \ \mu \in \mathbb{R}$$

and

$$\rho(x) \approx x^{\gamma}(1-x)^{r},$$

where $\gamma < \beta - 1$ and $r < \alpha + 1$.
Then problem (1.1) has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for each $x \in (0,1)$:

- $\mu < \beta - 1 + \sigma\delta_1$ and $r \leq \alpha + 1$, then by Theorem 2.1

$$u(x) \approx x^{\alpha-2}(1-x) \begin{cases} 1, & if\ r \neq \alpha+1, \\ \log(\frac{\eta}{1-x}), & if\ r = \alpha+1. \end{cases}$$

- $\mu = \beta - 1 + \sigma\delta_1$, $r < \alpha + 1$ and $r \neq \alpha$, then by Theorem 2.2

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_1} \begin{cases} 1, & if\ r < \alpha, \\ \left(\log(\frac{3}{1-x})\right)^{-\frac{1}{1-\sigma}}, & if\ \alpha < r < \alpha+1. \end{cases}$$

- $\beta - 1 + \sigma < \mu < \beta + \sigma$ and $r \leq \alpha - 1 + \beta + \sigma - \mu$, then by Theorem 2.3

$$u(x) \approx x^{\alpha-2}(1-x) \begin{cases} 1, & if\ r < \alpha - 1 + \beta + \sigma - \mu, \\ \left(\log(\frac{3}{1-x})\right)^{-\frac{1}{1-\sigma}}, & if\ r = \alpha - 1 + \beta + \sigma - \mu. \end{cases}$$

- $\beta - 1 + \sigma\delta_2 < \mu < \beta + \sigma\delta_2$ and $\alpha - 1 < r - \beta - \sigma\delta_2 + \mu < \alpha$, then by Theorem 2.4

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_2} \left(\log(\frac{3}{1-x})\right)^{-\frac{2}{1-\sigma}}.$$

- $\beta - 1 < \mu < \beta$ and $r = \alpha + \beta - \mu$, then by Theorem 2.5

$$u(x) \approx x^{\alpha-2} \left(\log(\frac{3}{1-x})\right)^{-\frac{1}{1-\sigma}}.$$

- $\mu = \beta + \sigma\delta_3$, $r < \alpha$ and $r \neq \alpha - 1$, then by Theorem 2.6

$$u(x) \approx x^{\alpha-2}(1-x)^{\delta_3} \begin{cases} 1, & if\ r < \alpha - 1, \\ \left(\log(\frac{3}{1-x})\right)^{\frac{-1}{1-\sigma}}, & if\ \alpha - 1 < r < \alpha. \end{cases}$$

5. Conclusion

In this paper, we have shown the existence, uniqueness and asymptotic behavior of continuous positive solution for a nonlinear boundary value problem of fractional differential equation with p-Laplacian. Our existence results are based on Schauder's fixed point theorem, while the asymptotic behavior is obtained by the behavior of the Green's function and Karamata regular variation theory. Also, we have given an example illustrating our main results.

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