Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.79639

Novel Extensions of Hermite–Hadamard Type Inequalities via the Riemann–Liouville Fractional Integral Operator

Aftab Ahmed, Arslan Munir, Ather Qayyum, Muhammad Muawwaz* and Sara Ghareeb

ABSTRACT: This study presents new Hermite–Hadamard type inequalities formulated via the Riemann–Liouville fractional integral operator for functions whose derivatives, in absolute value, satisfy convexity conditions. By exploiting fundamental properties of convex functions, the proposed results extend classical Hermite–Hadamard inequalities into the fractional calculus setting. This generalized framework highlights the interplay between convexity and modern fractional integral operators, thereby enriching and unifying several recent developments in the literature. In addition, numerical examples and graphical illustrations are provided to support the theoretical findings, demonstrating the sharpness, effectiveness, and applicability of the obtained inequalities in advanced fractional calculus research.

Keywords: Convex functions, Riemann–Liouville fractional integrals, Hermite–Hadamard type inequalities.

Contents

1. Introduction

The Hermite–Hadamard inequality is a basic result in convex analysis that provides both lower and upper bounds for the integral mean of a convex function over a compact interval. Due to its fundamental role in analysis, the inequality independently developed by Hermite and Hadamard has seen widespread investigation, optimization, and applied sciences. It characterizes the relationship between convexity and integral estimation, offering a powerful tool for deriving various functional and fractional integral inequalities. The Hermite–Hadamard inequality concerning convex functions is a strong tool across numerous branches of Mathematics. We can define it as:

$$Q\left(\frac{c_a+c_b}{2}\right) \leq \frac{1}{c_b-c_a}\int_{c_a}^{c_b} Q\left(\rho_1\right) d\rho_1 \leq \frac{Q\left(c_a\right)+Q\left(c_b\right)}{2}. \tag{1.1}$$

Integral inequalities are essential to many theoretical and mathematical analysis. Significant contributions have been made in this area, particularly concerning Riemann–Liouville and Atangana–Baleanu fractional integral operators [1,2,3,4,5]. Dragomir et al. [6] pioneered the development of Hermite–Hadamard-type results within the framework of Riemann–Liouville fractional integrals [18]. Building on this foundation, researchers have employed generalized fractional calculus to formulate refined inequalities of Simpson–type [7,8] and trapezoidal–type [9,10]. On the other hand, Ostrowski–type inequalities have been extensively investigated in the classical setting [17]. Moreover, recent work has paid particular attention to midpoint–type fractional inequalities tailored for convex functions [11,12], demonstrating the ongoing development of this field. The theory of convex functions and convex sets has received increasing attention in recent

* Corresponding author.
2020 *Mathematics Subject Classification:* Primary 26A33, 26D07; Secondary 26D10, 26D15.
Submitted October 19, 2025. Published March 28, 2026

years, which has resulted in creative generalizations across several mathematical contexts. Inspired by these developments, this paper serves as a basis for additional research on fractional integral inequalities by reviewing key definitions pertaining to convex functions and the left–right Riemann–Liouville fractional integral operators.

Definition 1.1 *The function $Q:[c_a,c_b]\subset\Re\to\Re$, convexity of a function is confirmed when the following inequality is satisfied:*

$$Q\left(\Phi\tilde{\sigma}_1+(1-\Phi)\,\tilde{\sigma}_2\right)\leq\Phi Q\left(\tilde{\sigma}_1\right)+(1-\Phi)\,Q\left(\tilde{\sigma}_2\right),$$

for all $\tilde{\sigma}_1,\tilde{\sigma}_2\in[c_a,c_b]$ and $\Phi\in[0,1]$. The function Q is termed concave when its negation $-Q$, exhibits convex behavior.

Definition 1.2 *[13] Suppose $Q\in L_1[c_a,c_b]$. The fractional integrals $J^{\beta}_{c_a+}Q$ and $J^{\beta}_{c_b-}Q$ corresponding to the Riemann–Liouville definition and of order $\beta>0$, are expressed as follows:*

$$J^{\beta}_{c_a+}Q=\frac{1}{\Gamma(\beta)}\int_{c_a}^{\tilde{\sigma}_1}\left(\tilde{\sigma}_1-y\right)^{\beta-1}Q(y)\,dy,\quad\tilde{\sigma}_1>c_a,$$

and

$$J^{\beta}_{c_b-}Q=\frac{1}{\Gamma(\beta)}\int_{\tilde{\sigma}_1}^{c_b}\left(y-\tilde{\sigma}_1\right)^{\beta-1}Q(y)\,dy,\quad\tilde{\sigma}_1<c_b,$$

Here, $\Gamma(\beta)$ is the function of Gamma and $J^{0}_{c_a+}Q\left(\tilde{\sigma}_1\right)=J^{0}_{c_b-}Q\left(\tilde{\sigma}_1\right)=Q\left(\tilde{\sigma}_1\right).$

In [14], H. Wang et al. introduced a set of new inequalities derived through the use of the Riemann–Liouville fractional integral operator.

Theorem 1.1 *Consider a differentiable function $Q:[c_a,c_b]\to\Re$ defined on the interval (c_a,c_b) with $0<\beta\leq1$. If the absolute value of its derivative, $|Q'|$ is convex on $[c_a,c_b]$, then the following result holds:*

$$\begin{aligned}&\left|\frac{Q(c_a)+Q(c_b)}{2}-\frac{\Gamma(\beta+1)}{2(c_b-c_a)^{\beta}}\left[J^{\beta}_{(c_b)^-}Q(c_a)+J^{\beta}_{(c_a)^+}Q(c_b)\right]\right|\\ \leq&\ \frac{(c_b-c_a)\left(2^{\beta}-1\right)}{2^{\beta+1}(\beta+1)}\left[|Q'(c_a)|+|Q'(c_b)|\right].\end{aligned}$$

Proposition 1.1 *If the Assumptions in Theorem 1.1 are satisfied and we take $\beta=1$, the resulting expression reduces to a trapezoidal inequality.*

$$\left|\frac{Q(c_a)+Q(c_b)}{2}-\frac{1}{c_b-c_a}\int_{c_a}^{c_b}Q\left(\tilde{\sigma}_1\right)d\tilde{\sigma}_1\right|\leq\frac{(c_b-c_a)}{8}\left(|Q'(c_a)|+|Q'(c_a)|\right),$$

which is obtained by Dragomir in [15].

Theorem 1.2 *Consider a differentiable function $Q:[c_a,c_b]\to\Re$ defined on the interval (c_a,c_b) with $0<\beta\leq1$. If the absolute value of its derivative, $|Q'|$ is convex on $[c_a,c_b]$, then the following result holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2(c_b-c_a)^{\beta}}\left[J^{\beta}_{(c_b)^-}Q(c_a)+J^{\beta}_{(c_a)^+}Q(c_b)\right]-Q\left(\frac{c_a+c_b}{2}\right)\right|\\ \leq&\ \frac{c_b-c_a}{4(\beta+1)}\left(\frac{2^{\beta-1}(\beta-1)+1}{2^{\beta-1}}\right)\left[|Q'(c_a)|+|Q'(c_b)|\right].\end{aligned}$$

Proposition 1.2 *If the Assumptions in Theorem 1.2 are satisfied and we take $\beta = 1$, the resulting expression reduces to a midpoint inequality.*

$$\left|\frac{1}{c_b - c_a}\int_{c_a}^{c_b} Q(\tilde{\sigma}_1)\, d\tilde{\sigma}_1 - Q\left(\frac{c_a + c_b}{2}\right)\right| \leq \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{8}\left[|Q'(\tilde{\alpha}_1)| + |Q'(\tilde{\alpha}_2)|\right],$$

which was derived by Kirmaci in [16].

Significant contributions related to the right-hand side of equation (1.1) were made by Dragomir and Agarwal in [15].

Lemma 1.1 *Consider a differentiable function $Q : I^0 \subset \Re \to \Re$ where $I^0, c_a, c_b \in I^0$ with $c_a < c_b$. If the derivative $Q' \in L[c_a, c_b]$, then the equality below holds:*

$$\begin{aligned} &\frac{Q(c_a) + Q(c_b)}{2} - \frac{1}{c_b - c_a}\int_{c_a}^{c_b} Q(\tilde{\sigma}_1)\, d\tilde{\sigma}_1 \\ =\ & \frac{c_b - c_a}{2}\int_0^1 (1 - 2y)\, Q'(yc_a + (1 - y)c_b)\, dy. \end{aligned} \tag{1.2}$$

Theorem 1.3 *Consider a differentiable function $Q : I^0 \subset \Re \to \Re$ where $I^0, c_a, c_b \in I^0$ with $c_a < c_b$. If the derivative $Q' \in L[c_a, c_b]$ is convex, then the inequality below holds:*

$$\begin{aligned} &\left|\frac{Q(c_a) + Q(c_b)}{2} - \frac{1}{c_b - c_a}\int_{c_a}^{c_b} Q(\tilde{\sigma}_1)\, d\tilde{\sigma}_1\right| \\ \leq\ & \frac{(c_b - c_a)}{8}\left[|Q'(c_a)| + |Q'(c_b)|\right]. \end{aligned}$$

There are two primary components to the current paper. The main ideas and theories that underpin the investigation are introduced in the first section. It describes the reasons for the research and establishes the foundation for the next advancements. The results obtained for each of the inequalities under examination are presented in the second section, which is broken up into smaller portions. Making use of methods from fractional calculus, and places these findings in the framework of recent research. It is hoped that the information offered here may spur new developments and study in this area.

2. Main Findings

Lemma 2.1 *Consider a differentiable function $Q : I^0 \subset \Re \to \Re$ where $I^0, c_a, c_b \in I^0$ with $c_a < c_b$. If the derivative $Q' \in L[c_a, c_b]$, then the equality below holds:*

$$\begin{aligned} &\frac{1}{c_b - c_a}\int_{\Phi c_b + (1-\Phi)c_a}^{\Phi c_b + (1-\Phi)c_a} Q(\tilde{\sigma}_1)\, d\tilde{\sigma}_1 - \frac{(1 - 2\Phi)}{2}\left[Q(\Phi c_a + (1 - \Phi)c_b) + Q(\Phi c_b + (1 - \Phi)c_a)\right] \\ =\ & \frac{(1 - 2\Phi)^2 (c_b - c_a)}{2}\int_0^1 (1 - 2y)\, Q'\left[y(\Phi c_a + (1 - \Phi)c_b) + (1 - y)(\Phi c_b + (1 - \Phi)c_a)\right] dy. \end{aligned}$$

Proof: Let

$$\begin{aligned} I &= \int_0^1 (1 - 2y)\, Q'\left[y(\Phi c_a + (1 - \Phi)c_b) + (1 - y)(\Phi c_b + (1 - \Phi)c_a)\right] dy \\ &= \int_0^1 Q'\left[y(\Phi c_a + (1 - \Phi)c_b) + (1 - y)(\Phi c_b + (1 - \Phi)c_a)\right] dy \\ &\quad -2\int_0^1 yQ'\left[y(\Phi c_a + (1 - \Phi)c_b) + (1 - y)(\Phi c_b + (1 - \Phi)c_a)\right] dy \\ I &= I_1 + I_2. \end{aligned}$$

The desired outcome is obtained by systematically applying integration by parts to each integral.

$$\begin{aligned} I_1 &= \int_0^1 Q'\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]dy \\ &= \frac{1}{(1-2\Phi)(c_b-c_a)}\left|Q\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]\right|_0^1 \\ &= \frac{Q\left(\Phi c_a+(1-\Phi)c_b\right)}{(1-2\Phi)(c_b-c_a)}-\frac{Q\left(\Phi c_b+(1-\Phi)c_a\right)}{(1-2\Phi)(c_b-c_a)}. \end{aligned} \tag{2.1}$$

Similarly, we have

$$\begin{aligned} I_2 &= -2\int_0^1 yQ'\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]dy \\ &= -\frac{2}{(1-2\Phi)(c_b-c_a)}\left|yQ\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]\right|_0^1 \\ &\quad +\frac{2}{(1-2\Phi)(c_b-c_a)}\int_0^1 Q\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]dy \\ &= -\frac{2Q\left(\Phi c_a+(1-\Phi)c_b\right)}{(1-2\Phi)(c_b-c_a)}+\frac{2}{(1-2\Phi)^2(c_b-c_a)^2}\int_{(\Phi c_b+(1-\Phi)c_a)}^{(\Phi c_a+(1-\Phi)c_b)} Q(\tilde{\sigma}_1)\,d\tilde{\sigma}_1. \end{aligned} \tag{2.2}$$

Adding the equalities (2.1) and (2.2), we obtained

$$\begin{aligned} I &= \frac{2}{(1-2\Phi)^2(c_b-c_a)^2}\int_{(\Phi c_b+(1-\Phi)c_a)}^{(\Phi c_a+(1-\Phi)c_b)} Q(\tilde{\sigma}_1)\,d\tilde{\sigma}_1-\frac{Q\left(\Phi c_a+(1-\Phi)c_b\right)}{(1-2\Phi)(c_b-c_a)} \\ &\quad -\frac{Q\left(\Phi c_b+(1-\Phi)c_a\right)}{(1-2\Phi)(c_b-c_a)}. \end{aligned} \tag{2.3}$$

Multiplying both sides equality (2.3), with $\frac{(1-2\Phi)^2(c_b-c_a)}{2}$, we have

$$\begin{aligned} &\frac{(1-2\Phi)^2(c_b-c_a)}{2}I \\ &= \frac{1}{(c_b-c_a)}\int_{(\Phi c_b+(1-\Phi)c_a)}^{(\Phi c_a+(1-\Phi)c_b)} Q(\tilde{\sigma}_1)\,d\tilde{\sigma}_1-\frac{(1-2\Phi)}{2}\left[Q\left(\Phi c_a+(1-\Phi)c_b\right)+Q\left(\Phi c_b+(1-\Phi)c_a\right)\right]. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &\frac{1}{c_b-c_a}\int_{\Phi c_b+(1-\Phi)c_a}^{\Phi c_b+(1-\Phi)c_a} Q(\tilde{\sigma}_1)\,d\tilde{\sigma}_1-\frac{(1-2\Phi)}{2}\left[Q\left(\Phi c_a+(1-\Phi)c_b\right)+Q\left(\Phi c_b+(1-\Phi)c_a\right)\right] \\ &= \frac{(1-2\Phi)^2(c_b-c_a)}{2}\int_0^1 (1-2y)\,Q'\left[y\left(\Phi c_a+(1-\Phi)c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi)c_a\right)\right]dy. \end{aligned}$$

□

Remark 2.1 In Lemma 2.1, if we take $\Phi=0$ then we will get the equation (1.2)

$$\begin{aligned} &\frac{Q(c_a)+Q(c_b)}{2}-\frac{1}{c_b-c_a}\int_{c_a}^{c_b} Q(\tilde{\sigma}_1)\,d\tilde{\sigma}_1 \\ &= \frac{(c_b-c_a)}{2}\int_0^1 (1-2y)\,Q'\left[yc_a+(1-y)c_b\right]dy. \end{aligned}$$

3. Hermite–Hadamard Inequalities in Fractional Calculus

As shown below, Hermite–Hadamard inequalities can be reformulated through fractional integration.

Theorem 3.1 *Suppose $Q : [c_a, c_b] \to \Re$ be a positive function with $0 \leq c_a < c_b$ and $Q \in L_1 [c_a, c_b]$. If Q is a convex on $[c_a, c_b]$, then the following inequality involving fractional integrals holds:*

$$Q\left(\frac{c_a + c_b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(1-2\Phi)^{\beta}(c_b - c_a)^{\beta}}\left[J^{\beta}_{c_a+} Q(c_b) + J^{\beta}_{c_b-} Q(c_a)\right] \leq \frac{Q(c_a) + Q(c_b)}{2}$$

Proof: Since Q is convex on $[c_a, c_b]$, we have for $\tilde{\sigma}_1, \tilde{\sigma}_2 \in [c_a, c_b]$ with $\Phi = \frac{1}{2}$,

$$Q\left(\frac{\tilde{\sigma}_1 + \tilde{\sigma}_2}{2}\right) \leq \frac{Q(\tilde{\sigma}_1) + Q(\tilde{\sigma}_2)}{2} \tag{3.1}$$

$$\tilde{\sigma}_1 = y(\Phi c_a + (1-\Phi) c_b) + (1-y)(\Phi c_b + (1-\Phi) c_a),$$

and

$$\tilde{\sigma}_2 = (1-y)(\Phi c_a + (1-\Phi) c_b) + y(\Phi c_b + (1-\Phi) c_a)$$

$$\begin{aligned} 2Q\left(\frac{c_a + c_b}{2}\right) \leq\ & Q[y(\Phi c_a + (1-\Phi) c_b) + (1-y)(\Phi c_b + (1-\Phi) c_a)] \\ & + Q[(1-y)(\Phi c_a + (1-\Phi) c_b) + y(\Phi c_b + (1-\Phi) c_a)]. \end{aligned}$$

Applying multiplying by $y^{\beta-1}$ followed by integration on the interval $[0, 1]$, we derive the next expression:

$$\begin{aligned} \frac{2}{\beta} Q\left(\frac{c_a + c_b}{2}\right) \leq\ & \int_0^1 y^{\beta-1} Q[y(\Phi c_a + (1-\Phi) c_b) + (1-y)(\Phi c_b + (1-\Phi) c_a)]\, dy \\ & + \int_0^1 y^{\beta-1} Q[(1-y)(\Phi c_a + (1-\Phi) c_b) + y(\Phi c_b + (1-\Phi) c_a)]\, dy \\ =\ & \int_{\Phi c_b + (1-\Phi) c_a}^{\Phi c_a + (1-\Phi) c_b} \left(\frac{c_a - u + (c_b - c_a)\Phi}{(1-2\Phi)(c_b - c_a)}\right)^{\beta-1} Q(u) \frac{du}{(1-2\Phi)(c_b - c_a)} \\ & + \int_{\Phi c_a + (1-\Phi) c_b}^{\Phi c_b + (1-\Phi) c_a} \left(\frac{v - c_b + (c_b - c_a)\Phi}{(1-2\Phi)(c_b - c_a)}\right)^{\beta-1} Q(v) \frac{dv}{(1-2\Phi)(c_b - c_a)} \\ =\ & \frac{\Gamma(\beta)}{2(1-2\Phi)^{\beta}(c_b - c_a)^{\beta}}\left[J^{\beta}_{c_a+} Q(c_b + (c_b - c_a)\Phi) + J^{\beta}_{c_b-} Q(c_a + (c_b - c_a)\Phi)\right]. \end{aligned}$$

i.e.

$$Q\left(\frac{c_a + c_b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(1-2\Phi)^{\beta}(c_b - c_a)^{\beta}}\left[J^{\beta}_{c_a+} Q(c_b) + J^{\beta}_{c_b-} Q(c_a)\right].$$

The 1st inequality is proved of this theorem.

For proving the 2nd inequality of this theorem, we first observe that if Q is convex and $\Phi \in [0, 1]$, it yields.

$$\begin{aligned} & Q[y(\Phi c_a + (1-\Phi) c_b) + (1-y)(\Phi c_b + (1-\Phi) c_a)] \\ \leq\ & yQ(\Phi c_a + (1-\Phi) c_b) + (1-y) Q(\Phi c_b + (1-\Phi) c_a), \end{aligned}$$

and

$$\begin{aligned} & Q[(1-y)(\Phi c_a + (1-\Phi) c_b) + y(\Phi c_b + (1-\Phi) c_a)] \\ \leq\ & (1-y) Q(\Phi c_a + (1-\Phi) c_b) + yQ(\Phi c_b + (1-\Phi) c_a). \end{aligned}$$

Adding both inequalities, we have.

$$\begin{aligned} & Q\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] \\ & +Q\left[(1-y)\left(\Phi c_a+(1-\Phi) c_b\right)+y\left(\Phi c_b+(1-\Phi) c_a\right)\right] \\ \leq\ & y Q\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y) Q \Phi c_b+(1-\Phi) c_a \\ & +(1-y) Q\left(\Phi c_a+(1-\Phi) c_b\right)+y Q\left(\Phi c_b+(1-\Phi) c_a\right). \end{aligned}$$

By multiplying both side's by $y^{\beta-1}$, and subsequently integrating the resulting inequality w.r.t y over the interval $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 y^{\beta-1} Q\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y \\ & +\int_0^1 y^{\beta-1} Q\left[(1-y)\left(\Phi c_a+(1-\Phi) c_b\right)+y\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y \\ \leq\ & y\left[Q\left(c_a\right)+Q\left(c_b\right)\right]+(1-y)\left[Q\left(c_a\right)+Q\left(c_b\right)\right] \int_0^1 y^{\beta-1} d y . \end{aligned}$$

$$\begin{aligned} \frac{\Gamma(\beta)}{(1-2 \Phi)^\beta\left(c_b-c_a\right)^\beta}\left[J_{c_a+}^\beta Q\left(c_b\right)+J_{c_b-}^\beta Q\left(c_a\right)\right] & \leq \frac{Q\left(c_a\right)+Q\left(c_b\right)}{\beta} \\ \frac{\Gamma(\beta+1)}{2(1-2 \Phi)^\beta\left(c_b-c_a\right)^\beta}\left[J_{c_a+}^\beta Q\left(c_b\right)+J_{c_b-}^\beta Q\left(c_a\right)\right] & \leq \frac{Q\left(c_a\right)+Q\left(c_b\right)}{2} . \end{aligned}$$

Therefore, we get

$$Q\left(\frac{c_a+c_b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(1-2 \Phi)^\beta\left(c_b-c_a\right)^\beta}\left[J_{c_a+}^\beta Q\left(c_b\right)+J_{c_b-}^\beta Q\left(c_a\right)\right] \leq \frac{Q\left(c_a\right)+Q\left(c_b\right)}{2} .$$

□

Corollary 3.1 *In Theorem 3.1, if we put $\Phi=0$, then we get*

$$Q\left(\frac{c_a+c_b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2\left(c_b-c_a\right)^\beta}\left[J_{c_a+}^\beta Q\left(c_b\right)+J_{c_b-}^\beta Q\left(c_a\right)\right] \leq \frac{Q\left(c_a\right)+Q\left(c_b\right)}{2} .$$

Remark 3.1 In Theorem 3.1, if we put $\Phi=0$ and $\beta=1$ then it gave equation (1.1).

Lemma 3.1 *Consider a function which is differentiable $Q:\left[c_a, c_b\right] \rightarrow \Re$ on $\left(c_a, c_b\right)$ with $c_a<c_b$. If $Q^{\prime} \in\left[c_a, c_b\right]$, then the given inequalities verified:*

$$\begin{aligned} & \frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)+Q\left(\Phi c_a+(1-\Phi) c_b\right)}{2}-\frac{\Gamma(\beta+1)}{2(1-2 \Phi)^\beta\left(c_b-c_a\right)^\beta}\left[J_{c_a+}^\beta Q\left(c_b\right)+J_{c_b-}^\beta Q\left(c_a\right)\right] \\ = & \frac{(1-2 \Phi)\left(c_b-c_a\right)}{2} \int_0^1\left[(1-y)^\beta-y^\beta\right] Q^{\prime}\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y . \end{aligned} \tag{3.2}$$

Proof: It is sufficient to examine that

$$\begin{aligned} I & =\int_0^1\left[(1-y)^\beta-y^\beta\right] Q^{\prime}\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y \\ & =\int_0^1(1-y)^\beta Q^{\prime}\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y \\ & -\int_0^1 y^\beta Q^{\prime}\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] d y \\ I & =I_1+I_2 . \end{aligned} \tag{3.3}$$

Taking integration by parts

$$
\begin{aligned}
I_1 &= \int_0^1 (1-y)^{\beta} Q'\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] dy \\
&= \frac{(1-y)^{\beta}}{(1-2\Phi)(c_b-c_a)} \left|Q\ \left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right]\right|_0^1 \\
&\quad -\frac{1}{(1-2\Phi)(c_b-c_a)} \int_0^1 \Gamma(1-y)^{\beta-1} Q\ \left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] dy \\
&= \frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\beta}{(1-2\Phi)(c_b-c_a)} \int_{c_a}^{c_b}\left(\frac{c_a-\tilde{\sigma}_1}{c_a-c_b}\right)^{\beta-1} \frac{Q(\tilde{\sigma}_1)}{(1-2\Phi)(c_a-c_b)} d\tilde{\sigma}_1 \\
I_1 &= \frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\Gamma(\beta+1)}{(1-2\Phi)^2(c_b-c_a)^{\beta+1}} J_{c_b-}^{\beta} Q(c_a), \qquad (3.4)
\end{aligned}
$$

and similarly, we get

$$
\begin{aligned}
I_2 &= -\int_0^1 y^{\beta} Q'\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] dy \\
&= \frac{y^{\beta}}{(1-2\Phi)(c_b-c_a)} \left|Q\ \left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right]\right|_0^1 \\
&\quad -\frac{\beta}{(1-2\Phi)(c_b-c_a)} \int_0^1 y^{\beta-1} Q\ \left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] dy \\
&= \frac{Q\left(\Phi c_a+(1-\Phi) c_b\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\beta}{(1-2\Phi)(c_b-c_a)} \int_{c_a}^{c_b}\left(\frac{\tilde{\sigma}_1-c_b}{c_b-c_a}\right)^{\beta-1} \frac{Q(\tilde{\sigma}_1)}{(1-2\Phi)(c_a-c_b)} d\tilde{\sigma}_1 \\
I_2 &= \frac{Q\left(\Phi c_a+(1-\Phi) c_b\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\Gamma(\beta+1)}{(1-2\Phi)^2(c_b-c_a)^{\beta+1}} J_{c_a+}^{\beta} Q(c_b). \qquad (3.5)
\end{aligned}
$$

Using (3.4) and (3.6) in (3.3), it follows that

$$
\begin{aligned}
I &= \frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\Gamma(\beta+1)}{(1-2\Phi)^2(c_b-c_a)^{\beta+1}} J_{c_b-}^{\beta} Q(c_a)+\frac{Q\left(\Phi c_a+(1-\Phi) c_b\right)}{(1-2\Phi)(c_b-c_a)} \\
&\quad -\frac{\Gamma(\beta+1)}{(1-2\Phi)^2(c_b-c_a)^{\beta+1}} J_{c_a+}^{\beta} Q(c_b) \\
&= \frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)+Q\left(\Phi c_a+(1-\Phi) c_b\right)}{(1-2\Phi)(c_b-c_a)}-\frac{\Gamma(\beta+1)}{(1-2\Phi)^2(c_b-c_a)^{\beta+1}}\left[J_{c_a+}^{\beta} Q(c_b)+J_{c_b-}^{\beta} Q(c_a)\right].
\end{aligned}
$$

Thus by multiplying $\frac{(1-2\Phi)(c_b-c_a)}{2}$ on both sides we have conclusion (3.2).

$$
\begin{aligned}
&\frac{Q\left(\Phi c_b+(1-\Phi) c_a\right)+Q\left(\Phi c_a+(1-\Phi) c_b\right)}{2}-\frac{\Gamma(\beta+1)}{2(1-2\Phi)(c_b-c_a)^{\beta}}\left[J_{c_a+}^{\beta} Q(c_b)+J_{c_b-}^{\beta} Q(c_a)\right] \\
&= \frac{(1-2\Phi)(c_b-c_a)}{2} \int_0^1\left[(1-y)^{\beta}-y^{\beta}\right] Q'\left[y\left(\Phi c_a+(1-\Phi) c_b\right)+(1-y)\left(\Phi c_b+(1-\Phi) c_a\right)\right] dy.
\end{aligned}
$$

□

Corollary 3.2 *In Lemma 3.1, if we put $\Phi=0$, then have*

$$
\begin{aligned}
&\frac{Q(c_a)+Q(c_b)}{2}-\frac{\Gamma(\beta+1)}{2(c_b-c_a)^{\beta}}\left[J_{c_a+}^{\beta} Q(c_b)+J_{c_b-}^{\beta} Q(c_a)\right] \\
&= \frac{(c_b-c_a)}{2} \int_0^1\left[(1-y)^{\beta}-y^{\beta}\right] Q'\left(y c_a+(1-y) c_b\right) dy.
\end{aligned}
$$

Remark 3.2 In Lemma 3.1, if we put $\Phi = 0$ and $\beta = 1$, then equality (3.3), change into equality (1.2).

$$\begin{aligned}
&\frac{Q(c_a)+Q(c_b)}{2}-\frac{\Gamma(\beta+1)}{2(c_b-c_a)^{\beta}}\left[J^{\beta}_{c_a+}Q(c_b)+J^{\beta}_{c_b-}Q(c_a)\right]\\
=\;&\frac{(c_b-c_a)}{2}\int_0^1\left[(1-y)^{\beta}-y^{\beta}\right]Q'(yc_a+(1-y)c_b)\,dy.
\end{aligned}$$

Theorem 3.2 *Consider function which is differentiable $Q:[c_a,c_b]\to\Re$ on (c_a,c_b) with $c_a<c_b$. If $|Q'|$ is convex on $[c_a,c_b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{Q(\Phi c_b+(1-\Phi)c_a)+Q(\Phi c_a+(1-\Phi)c_b)}{2}-\frac{\tilde{\Gamma}(\beta+1)}{2(1-2\Phi)^{\beta}(c_b-c_a)^{\tilde{\beta}}}\left[J^{\tilde{\beta}}_{c_a+}Q(c_b)+J^{\tilde{\beta}}_{c_b-}Q(c_a)\right]\right|\\
\leq\;&\frac{(1-2\Phi)(c_b-c_a)}{2\left(\tilde{\beta}+1\right)}\left(1-\frac{1}{2^{\tilde{\beta}}}\right)\left[|Q'(\Phi c_b+(1-\Phi)c_a)|+|Q'(\Phi c_a+(1-\Phi)c_b)|\right]. \qquad (3.6)
\end{aligned}$$

Proof: Using lemma 3.1 and the convexity of $|Q'|$, we find

$$\begin{aligned}
&\left|\frac{Q(\Phi c_b+(1-\Phi)c_a)+Q(\Phi c_a+(1-\Phi)c_b)}{2}-\frac{\tilde{\Gamma}\left(\tilde{\beta}+1\right)}{2(1-2\Phi)(c_b-c_a)^{\tilde{\beta}}}\left[J^{\tilde{\beta}}_{c_a+}Q(c_b)+J^{\tilde{\beta}}_{c_b-}Q(c_a)\right]\right|\\
\leq\;&\frac{(1-2\Phi)(c_b-c_a)}{2}\int_0^1\left|(1-y)^{\tilde{\beta}}-y^{\tilde{\beta}}\right|Q'\left|y(\Phi c_b+(1-\Phi)c_a)+(1-y)(\Phi c_a+(1-\Phi)c_b)\right|dy\\
\leq\;&\frac{(1-2\Phi)(c_b-c_a)}{2}\int_0^1\left|(1-y)^{\tilde{\beta}}-y^{\tilde{\beta}}\right|\left[y\left|Q'(\Phi c_b+(1-\Phi)c_a)\right|+(1-y)\left|Q'(\Phi c_a+(1-\Phi)c_b)\right|\right]dy\\
=\;&\frac{(1-2\Phi)(c_b-c_a)}{2}\left\{\int_0^{\frac{1}{2}}\left[(1-y)^{\tilde{\beta}}-y^{\tilde{\beta}}\right]\left[y\left|Q'(\Phi c_b+(1-\Phi)c_a)\right|+(1-y)\left|Q'(\Phi c_a+(1-\Phi)c_b)\right|\right]dy\right.\\
&\left.+\int_{\frac{1}{2}}^1\left[(1-y)^{\tilde{\beta}}-y^{\tilde{\beta}}\right]\left[y\left|Q'(\Phi c_b+(1-\Phi)c_a)\right|+(1-y)\left|Q'(\Phi c_a+(1-\Phi)c_b)\right|\right]dy\right\}\\
K\;=\;&\frac{(1-2\Phi)(c_b-c_a)}{2}(K_1+K_2). \qquad (3.7)
\end{aligned}$$

Calculating K_1 and K_2, we have

$$\begin{aligned}
K_1\;=\;&|Q'(\Phi c_b+(1-\Phi)c_a)|\left[\int_0^{\frac{1}{2}}y(1-y)^{\tilde{\beta}}dy-\int_0^{\frac{1}{2}}y^{\tilde{\beta}+1}dy\right]\\
&+|Q'(\Phi c_a+(1-\Phi)c_b)|\left[\int_0^{\frac{1}{2}}(1-y)^{\tilde{\beta}+1}dy-\int_0^{\frac{1}{2}}(1-y)y^{\tilde{\beta}}dy\right]\\
K_1\;=\;&|Q'(\Phi c_b+(1-\Phi)c_a)|\left[\frac{1}{\left(\tilde{\beta}+1\right)\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right]\\
&+|Q'(\Phi c_a+(1-\Phi)c_b)|\left[\frac{1}{\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right]. \qquad (3.8)
\end{aligned}$$

and similarly

$$\begin{aligned}
K_2 &= \left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|\left[\int_0^{\frac{1}{2}} y^{\tilde{\beta}+1}dy-\int_0^{\frac{1}{2}} y(1-y)^{\tilde{\beta}}dy\right] \\
&\quad+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\left[\int_0^{\frac{1}{2}}(1-y)^{\tilde{\beta}}y^{\tilde{\beta}}dy-\int_0^{\frac{1}{2}}(1-y)^{\tilde{\beta}+1}dy\right] \\
K_2 &= \left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|\left[\frac{1}{\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right] \\
&\quad+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\left[\frac{1}{\left(\tilde{\beta}+1\right)\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right]. \qquad (3.9)
\end{aligned}$$

Putting the equalities (3.8) and (3.9) in (3.7), then we have

$$\begin{aligned}
K &= \frac{(1-2\Phi)(c_b-c_a)}{2}\Bigg\{\left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|\left[\frac{1}{\left(\tilde{\beta}+1\right)\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right] \\
&\quad+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\left[\frac{1}{\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right] \\
&\quad+\left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|\left[\frac{1}{\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right] \\
&\quad+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\left[\frac{1}{\left(\tilde{\beta}+1\right)\left(\tilde{\beta}+2\right)}-\frac{\left(\frac{1}{2}\right)^{\tilde{\beta}+1}}{\left(\tilde{\beta}+1\right)}\right]\Bigg\} \\
&= \frac{(1-2\Phi)(c_b-c_a)}{2\left(\tilde{\beta}+1\right)}\left(1-\frac{1}{2^{\tilde{\beta}}}\right)\left[\left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\right]
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left|\frac{Q\left(\Phi c_b+(1-\Phi)c_a\right)+Q\left(\Phi c_a+(1-\Phi)c_b\right)}{2}-\frac{\tilde{\Gamma}\left(\tilde{\beta}+1\right)}{2(1-2\Phi)^{\tilde{\beta}}(c_b-c_a)^{\tilde{\gamma}}}\left[J^{\tilde{\gamma}}_{c_a+}Q(c_b)+J^{\tilde{\gamma}}_{c_b-}Q(c_a)\right]\right| \\
\leq\ & \frac{(1-2\Phi)(c_b-c_a)}{2\left(\tilde{\beta}+1\right)}\left(1-\frac{1}{2^{\tilde{\gamma}}}\right)\left[\left|Q'\left(\Phi c_b+(1-\Phi)c_a\right)\right|+\left|Q'\left(\Phi c_a+(1-\Phi)c_b\right)\right|\right].
\end{aligned}$$

□

Corollary 3.3 *In Theorem 3.2, if we put $\Phi=0$, then we obtain*

$$\begin{aligned}
&\left|\frac{Q(c_a)+Q(c_b)}{2}-\frac{\tilde{\Gamma}\left(\tilde{\beta}+1\right)}{2(c_b-c_a)^{\tilde{\gamma}}}\left[J^{\tilde{\gamma}}_{c_a+}Q(c_b)+J^{\tilde{\gamma}}_{c_b-}Q(c_a)\right]\right| \\
\leq\ & \frac{(c_b-c_a)}{2(\tilde{\gamma}+1)}\left(1-\frac{1}{2^{\tilde{\gamma}}}\right)\left[\left|Q'(c_a)\right|+\left|Q'(c_b)\right|\right].
\end{aligned}$$

Remark 3.3 In Theorem 3.2, if we put $\Phi = 0$ and $\tilde{\beta} = 1$, then inequality (3.6), change into inequality (1.2).

$$\left| \frac{Q\left(c_a\right) + Q\left(c_b\right)}{2} - \frac{1}{c_b - c_a} \int_{c_a}^{c_b} Q\left(\tilde{\sigma}_1\right) d\tilde{\sigma}_1 \right| \leq \frac{\left(c_b - c_a\right)}{8} \left(\left|Q\,'\left(c_a\right)\right| + \left|Q\,'\left(c_a\right)\right| \right).$$

4. Graphical Illustration

Example 4.1 Suppose $Q(\tilde{\sigma}) = \tilde{\sigma}^4$ over a continuously varying interval$[c_a, c_b]$ in Corollary 1, where $c_a \in [0.1, 0.5]$ and $c_b = 1 - c_a$. The parameter $\beta = 0.6$ is fixed and the left hand side gives error of an approximation of the integral of the function and the right hand side tells about the error bound, the behavior of this has been shown graphically in figure 1.

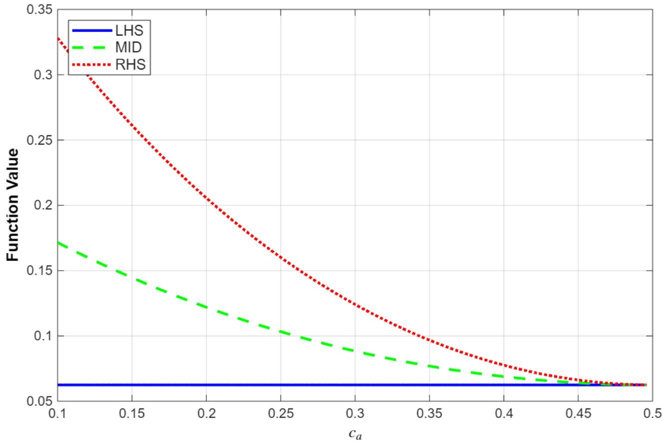

Figure 1: Graphical illustration of Example 1

Example 4.2 Suppose $Q(\tilde{\sigma}) = (\frac{\tilde{\sigma}}{2})^2$ in Corollary 1. We investigate the inequality components over a series of interval $[c_a, c_b]$, where $c_a \in [0.1, 0.5]$ and $c_b = c_a + 0.4$. The fractional order is chosen as $\beta = 0.6$, the behavior of this has been shown graphically in figure 2.

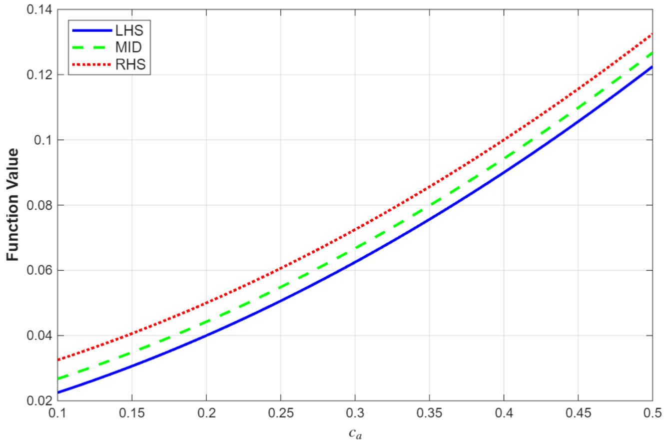

Figure 2: Graphical illustration of Example 2

Example 4.3 In this example, we analyze the behavior of fractional integral bounds in comparison with classical midpoint and endpoint-based approximation using the convex and rapidly increasing function $Q(\tilde{\sigma}) = e^{\tilde{\sigma}}$ in Corollary 1. We define a one-dimensional slice where the interval endpoint satisfy $c_b = c_a + 0.4$ and $c_a \in [0.1, 0.5]$, ensuring $c_a < c_a$. The fractional order is fixed at $\beta = 0.6$, the behavior of this has been shown graphically in figure 3.

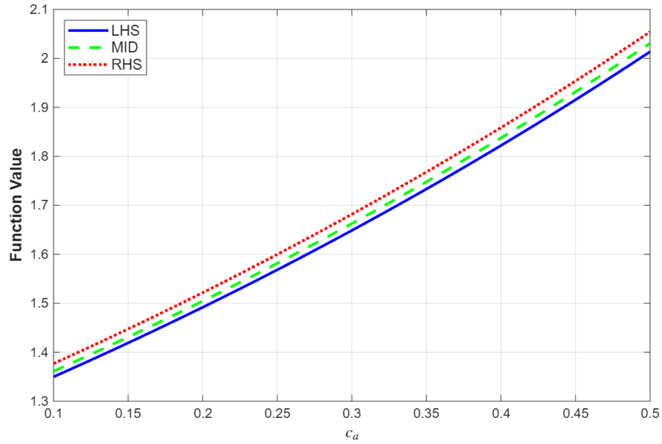

Figure 3: Graphical illustration of Example 3

Example 4.4 Suppose $Q(\tilde{\sigma}) = \tilde{\sigma}^2 + 1$ over a a family of interval $[c_a, c_b]$ in Corollary 1, where $c_a \in [0.1, 0.5]$ and $c_b = c_a + 0.4$, ensuring $c_a < c_a$. The fractional order is chosen as $\beta = 0.6$, the behavior of this has been shown graphically in figure 4.

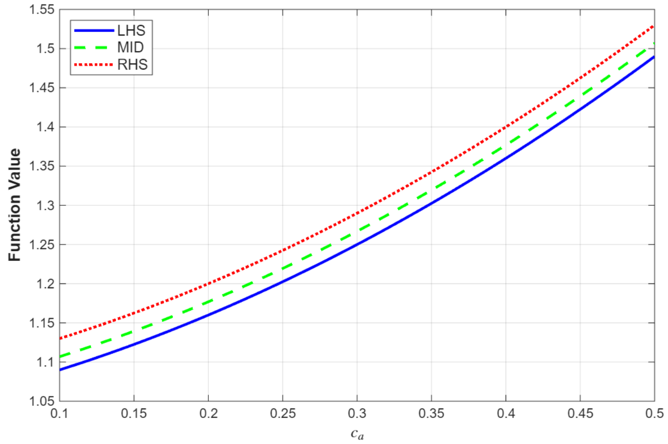

Figure 4: Graphical illustration of Example 4

5. Conclusion

In this work, we have established new Hermite–Hadamard type inequalities involving the Riemann–Liouville fractional integral operator under refined convexity assumptions on the derivatives of the underlying functions. The obtained results extend and sharpen several existing inequalities in the literature, thereby providing a unified and generalized framework that strengthens the connection between convexity theory and fractional calculus. The validity and effectiveness of the theoretical findings are further confirmed through graphical illustrations, which highlight the accuracy and consistency of the proposed bounds. Future research directions include the application of alternative analytical techniques, as well as the extension of the present approach to other fractional operators, particularly the Caputo-Fabrizio fractional operator.

References

1. Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis*, 3, 182.
2. Sarikaya, M. Z., Set, E., Yaldiz, H., & Başak, N. (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57(9–10), 2403–2407.
3. Tariq, M., Ahmad, H., Sahoo, S. K., Kashuri, A., Nofal, T. A., & Hsu, C. H. (2022). Inequalities of Simpson–Mercer type including Atangana–Baleanu fractional operators and their applications. *AIMS Mathematics*, 7(8), 15159–15181.
4. Rashid, S., Abdeljawad, T., Jarad, F., & Noor, M. A. (2019). Some estimates for generalized Riemann–Liouville fractional integrals of exponentially convex functions and their applications. *Mathematics*, 7(9), Article 807.
5. Dahmani, Z. (2010). On Minkowski and Hermite–Hadamard integral inequalities via fractional integration. *Annals of Functional Analysis*, 1(1), 51–58.
6. Dragomir, S. S., Bhatti, M. I., Iqbal, M., & Muddassar, M. (2015). Some new Hermite–Hadamard type inequalities. *Journal of Computational Analysis and Applications*, 18(4), 655–660.
7. Munir, A., Qayyum, A., Supadi, S. S. A., Budak, H., & Faiz, I. (2024). A study of improved error bounds for Simpson-type inequality via fractional integral operators. *Filomat*, 38, 3415–3427.
8. Alomari, M., Darus, M., & Dragomir, S. S. (2009). New inequalities of Simpson's type for s-convex functions with applications. *Research Report Collection*, 12(4).
9. Dragomir, S. S., & Ostrowski, F. (2020). Trapezoid type inequalities for generalized Riemann–Liouville fractional integrals of functions with bounded variation. *Acta Universitatis Sapientiae, Mathematica*, 12, 30–53.
10. Junjua, M. U. D., Qayyum, A., Munir, A., Budak, H., Saleem, M. M., & Supadi, S. S. (2024). A study of some new Hermite–Hadamard inequalities via specific convex functions with applications. *Mathematics*, 12(3), Article 478.
11. Sahoo, S. K., Hamed, Y. S., Mohammed, P. O., Kodamasingh, B., & Nonlaopon, K. (2023). New midpoint-type Hermite–Hadamard–Mercer inequalities pertaining to Caputo–Fabrizio fractional operators. *Alexandria Engineering Journal*, 65, 689–698.
12. Ahmed, A., Munir, A., Qayyum, A., Ghareeb, S., Talhat, R., & Alsulami, I. K. (2025). Hermite–Hadamard inequalities for exponential convexity classes with applications. *Filomat*, 39(29), 10235–10251.
13. Miller, K. S., & Ross, B. (1993). *An introduction to fractional calculus and fractional differential equations*. Wiley.
14. Hwang, H. R., Tseng, K. L., & Hsu, K. C. (2016). New inequalities for fractional integrals and their applications. *Turkish Journal of Mathematics*, 40(3), 471–486.
15. Dragomir, S. S., & Agarwal, R. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to the trapezoidal formula. *Applied Mathematics Letters*, 11(5), 91–95.
16. Kirmaci, U. S. (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to the midpoint formula. *Applied Mathematics and Computation*, 147(1), 137–146.
17. Muawwaz, M., Maaz, M., & Qayyum, A. (2024). Ostrowski's type inequalities using the modified two–step linear kernel. *European Journal of Mathematics and Applications*, 4, Article ID 6.
18. Sarikaya, M. Z., & Budak, H. (2016). Generalized Hermite–Hadamard type integral inequalities for fractional integrals. *Filomat*, 30(5), 1315–1326.

Aftab Ahmed,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: etcacademy007@gmail.com

and

Arslan Munir,
School of Mathematical Sciences,

University of Science and Technology of China,
Hefei 230026 People's Republic of China.
E-mail address: munirarslan999@gmail.com

and

Ather Qayyum,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan
and
Institute of Mathematical Sciences,
Faculty of Science, Universiti Malaya,
Malaysia.
E-mail address: atherqayyum@isp.edu.pk

and

Muhammad Muawwaz,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan
and
Department of Mathematics,
Riphah International University, Faisalabad Campus, Faisalabad,
Pakistan.
E-mail address: muawwaz123@gmail.com

and

Sara Ghareeb,
College of Basic Education- Kuwait.
E-mail address: sa.ghareeb@paaet.edu.kw