Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–28.
ISSN-0037-8712
doi:10.5269/bspm.79642

Variants of Compatible Mappings and Fixed Point Results in Perturbed Metric Spaces

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ABSTRACT: In this paper, we introduce variants of compatible mappings namely compatible mappings of types (K), (R) and (E) along with the concept of faintly compatible mappings in the setting of perturbed metric spaces. These classes extend and generalize the existing notions of compatible and weakly compatible mappings. For each of these classes, we establish common fixed point theorems for four self-mappings satisfying contraction-type conditions. The results obtained unify and extend several well-known fixed point results in the literature, providing a broader framework for the study of fixed points in perturbed metric spaces. Illustrative examples are included to demonstrate the applicability of the proved theorems.

Key Words: Perturbed metric space, common fixed point, compatible mappings and its variants, faintly compatible mappings.

Contents

1. Introduction

The measurement of the distance between two points is not always exact. During measurement, some errors may occur. These errors may be slightly positive, slightly negative, or sometimes zero. If error is zero, then it corresponds to the metric. To account for these, a positive error is subtracted and a negative error is added during determining the exact value of the distance function. These errors may play a significant role during measurement.
In order to overcome the difficulty, whenever error is added in metric, Mohamed Jleli and Bessem Samet [5] gave the notion of a perturbed metric space. Perturbed metric spaces represent a useful and practical improvement over the metric spaces. The significance of perturbed metric spaces lies across a wide range of mathematical and applied disciplines.
Even though for small positive errors, the structure of these spaces still retains the properties of metric spaces. In this way, perturbed metric spaces help to bridge the gap between the mathematical models and real-world situations, where exact distance are not measurable.

In 2025, Mohamed Jleli and Bessem Samet [5] introduced a more general form of distance function, known as perturbed metric space as follows :

Definition 1.1. Let $D, P : X \times X \to [0, \infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function

$$D - P : X \times X \to \mathbb{R},$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted October 19, 2025. Published December 19, 2025

defined by the relation

$$(D-P)(x,y)=D(x,y)-P(x,y),$$

for all $x,y,z\in X$, is a exact metric on X, i.e., for all $x,y,z\in X$, it satisfies the following conditions

(i) $(D-P)(x,y)\geq 0$;

(ii) $(D-P)(x,y)=0$ if and only if $x=y$;

(iii) $(D-P)(x,y)=(D-P)(y,x)$;

(iv) $(D-P)(x,y)\leq (D-P)(x,z)+(D-P)(z,y)$.

P is called a *perturbing function* and $D=d+P$ be an *perturbed metric.*
The set X endowed with D and *perturbed function* P denoted by (X,D,P) is known as *perturbed metric spaces.*

Notice that a perturbed metric on X is not necessarily a metric on X. But a metric is always perturbed metric when perturbed error is zero.

Example 1.1. Let $D:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be the mapping defined by

$$D(x,y)=|x-y|+x^2y^4,\ \textit{for all}\ x,y\in\mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P:\mathbb{R}\times\mathbb{R}\to[0,\infty)$$

given by

$$P(x,y)=x^2y^4,\quad x,y\in\mathbb{R}.$$

In this case, the exact metric is the mapping $d:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ defined by

$$d(x,y)=D(x,y)-P(x,y)\ ,\ \textit{where}$$

$$d(x,y)=|x-y|,\quad x,y\in\mathbb{R}.$$

Here we note that D is not necessarily a metric, because $D(1,1)=1\neq 0$ as $x=y$, but D is perturbed metric on X with respect to perturbed function P.

We now introduce topological structure in perturbed metric space.

The topological structure of the perturbed metric space (X,D,P) corresponds to the balls in metric spaces and is induced by the exact metric $d=D-P$. That is, the topology $\tau_{D,P}$ on X is defined as:

$$\tau_{D,P}:=\tau_d=\{U\subseteq X\mid \forall x\in U,\ \exists r>0 \text{ such that } B_d(x,r)\subseteq U\},$$

where the open ball with respect to d is given by:

$$B_d(x,r)=\{y\in X\mid d(x,y)=D(x,y)-P(x,y)<r\}.$$

Definition 1.2. Let (X,D,P) be a perturbed metric space with perturbed function P. A sequence $\{x_n\}$ in X is said to be

(i) *perturbed convergent sequence*, if $\{x_n\}$ is convergent in the metric space (X,d), where $d=D-P$ is the exact metric.

(ii) *perturbed Cauchy sequence*, if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

A mapping T defined on (X, D, P) is a *perturbed continuous mapping*, if T is continuous with respect to the exact metric d.

We recall some elementary properties of perturbed metric spaces [5] .

Proposition 1.1. [5] Let $D, P, Q : X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.

(i) If (X, D, P) and (X, D, Q) be two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space.

Here for the convenience of readers, we provide the proof of the proposition 1.1.

Proof.

(i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}[(D - P) + (D - Q)] = D - \frac{P + Q}{2}$$

is a metric on X, which proves (i).

(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D - P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii).

2. Variants of Compatible Mappings

In this section, we recall some basic definitions and results that will be used throughout the paper. We also introduce new classes of compatible mappings in the framework of perturbed metric spaces.

Definition 2.1 [2] Let S and T be two mappings of a perturbed metric space (X, D, P) into itself. Then S and T are called compatible if and only if

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Example 2.1 Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^4, \ \textit{for all}\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

Let $S, T : X \to X$ be defined by $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all $x \in X$, where $X = [0, \infty)$. Taking the sequence $\{x_n\}$ as $x_n = \frac{1}{n}, n > 0$, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X,$$

then S and T are said to be compatible

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0.$$

Definition 2.2 [2] A pair (S,T) of self-mappings of a perturbed metric space (X,D,P) is said to be compatible mappings of type (A) if and only if

$$\lim_{n\to\infty} D(SSx_n, TSx_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} D(STx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Proposition 2.1 [2] Let S and T be compatible mappings of type (A). If one of S or T is continuous, then S and T are compatible.

Proposition 2.2 [2] Let S and T be continuous mappings. If S and T are compatible, then they are compatible mappings of type (A).

The direct consequence of propositions 2.1 and 2.2 is in the form of following :

Proposition 2.3 [2] Let S and T be continuous mappings. Then S and T are compatible if and only if they are compatible mappings of type (A).

Proposition 2.4 [2] Let S and T be compatible mappings of type (A) of a perturbed metric space (X,D,P) into itself. If $Sz = Tz$ for some $z \in X$, then

$$STz = SSz = TTz = TSz.$$

Proposition 2.5 [2] Let S and T be compatible mappings of type (A) of a perturbed metric space (X,D,P) into itself. Suppose that

$$\lim_{n\to\infty} Tx_n = z \quad \text{and} \quad \lim_{n\to\infty} Sx_n = z \quad \text{for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} STx_n = Tz$ if T is continuous at z.

(b) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Definition 2.3 [2] A pair (S,T) of self-mappings of a perturbed metric space (X,D,P) is said to be compatible mappings of type (B) if and only if

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.4 [2] A pair (S,T) of self-mappings of a perturbed metric space (X,D,P) is said to be compatible mappings of type (C) if and only if

$$\lim_{n\to\infty} D(STx_n, TTx_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n) + \lim_{n\to\infty} D(Sz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, SSx_n) + \lim_{n\to\infty} D(Tz, TTx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Remark 2.1 *Compatible mappings of type (A) $\Longrightarrow$ compatible mappings of type (B) $\Longrightarrow$ compatible mappings of type (C), but the converse is not true in general.*

Example 2.2 Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \; for \; all \; x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Let $S, T : X \to X$ be defined by $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all $x \in X$, where $X = [0, \infty)$. Taking the sequence $\{x_n\}$ as $x_n = \frac{1}{n}, n > 0$. Then, S and T are compatible of type (A), compatible of type (B) and compatible of type (C) also. But the converse is not true in general.

Let $X = [1, 20]$, and $D : \mathbb{R} \times \mathbb{R} \to [1, 20]$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \; for \; all \; x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [1, 20]$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Defining $S, T : X \to X$ as below:

$$Sx = \begin{cases} 1, & \text{if } x = 1, \\ 3, & \text{if } 1 < x \leq 7, \\ x - 6, & \text{if } 7 < x \leq 20. \end{cases} \quad and \quad Tx = \begin{cases} 1, & \text{if } x = 1 \text{ or } x \in (7, 20], \\ 2, & \text{if } 1 < x \leq 7. \end{cases}$$

Taking sequence $\{x_n\}$ as $x_n = 7 + \frac{1}{n}, \; n > 0$. Then, S and T are compatible of type (C), but neither compatible nor compatible of type (A) nor compatible of type (B).

Definition 2.5 [2] A pair (S, T) of self-mappings of a metric space (X, D, P) is said to be compatible mappings of type (P) if and only if

$$\lim_{n\to\infty} D(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Proposition 2.6 [2] Every pair of compatible mappings of type (A) is compatible of type (B).

Proposition 2.7 [2] Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible mappings of type (B), then they are compatible of type (A).

Proposition 2.8 [2] Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible of type (B), then they are compatible.

Proposition 2.9 [2] Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible, then they are compatible of type (B).

Proposition 2.10 [2] Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. Then

(1) *S and T are compatible if and only if they are compatible of type (B);*

(2) *S and T are compatible of type (A) if and only if they are compatible of type (B).*

Proposition 2.11 [2] Let S and T be compatible mappings of a perturbed metric space (X, D, P) into itself. If $Sz = Tz$ for some $z \in X$, then $STz = SSz = TTz = TSz$.

Proposition 2.12 [2] Let S and T be compatible mappings of a perturbed metric space (X, D, P) into itself. Suppose that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} TSx_n = Sz$ *if S is continuous at z;*

(b) $\lim_{n\to\infty} STx_n = Tz$ *if T is continuous at z;*

(c) *$STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.*

Proposition 2.13 [2] Let S and T be compatible mappings of type (B) of a perturbed metric space (X, D, P) into itself. If $Sz = Tz$ for some $z \in X$, then

$$STz = SSz = TTz = TSz.$$

Proposition 2.14 [2] Let S and T be compatible mappings of type (B) of a perturbed metric space (X, D, P) into itself. Suppose that

$$\lim_{n\to\infty} Tx_n = z \quad \text{and} \quad \lim_{n\to\infty} Sx_n = z \quad \text{for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} TTx_n = Sz$ if S is continuous at z.

(b) $\lim_{n\to\infty} SSx_n = Tz$ if T is continuous at z.

(c) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Remark 2.2 In Proposition 2.13, assume that S and T be compatible mappings of type (C) or of type (P) instead of of type (B). The conclusion of Proposition 2.13 still holds.

Remark 2.3 In Proposition 2.14, assume that S and T be compatible mappings of type (C) or of type (P) instead of of type (B). The conclusion of Proposition 2.14 still holds.

In 2004, Rohan et al. [13] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 2.6 Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called compatible of type (R) if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(ffx_n, ggx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

In 2007, Singh and Singh [6] introduced the concept of compatible mappings of type (E) in a metric space as follows:

Definition 2.7 Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called compatible of type (E) if

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt \quad \text{and} \quad \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

In 2014, Jha et al. [3] introduced the concept of compatible mappings of type (K) in a metric space as follows:

Definition 2.8 Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called compatible of type (K) if

$$\lim_{n\to\infty} d(ffx_n, gt) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(ggx_n, ft) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

In 1999, R.P.Pant [8] introduced the notion of reciprocally continuous in metric spaces as follows:

Definition 2.9 Let A and S be mappings from a metric space (X, d) into itself. Then A and S are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \in X.$$

Now we introduce the analogues notions of compatible mappings and their variants in the setting of perturbed metric spaces.

Definition 2.10 Let f and g be two mappings of a perturbed metric space (X, D, P) into itself. Then f and g are called compatible of type (R) if

$$\lim_{n\to\infty} D(fgx_n, gfx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} D(ffx_n, ggx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

Definition 2.11 Let f and g be two mappings of a perturbed metric space (X, D, P) into itself. Then f and g are called compatible of type (E) if

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt \quad \text{and} \quad \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.
Splitting definition of compatible mapping of type (E) in two forms as:

Definition 2.11.1 A pair (f, g) of self-mappings of a perturbed metric space (X, D, P) is said to be **g**-compatible mappings of type (E) if and only if the following condition is satisfied:

$$\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = fz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.
Definition 2.11.2 A pair (f, g) of self-mappings of a perturbed metric space (X, D, P) is said to be **f**-compatible mappings of type (E) if and only if the following condition is satisfied:

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.

Definition 2.12 Let f and g be two mappings of a perturbed metric space (X, D, P) into itself. Then f and g are called compatible of type (K) if

$$\lim_{n\to\infty} D(ffx_n, gt) = 0 \quad \text{and} \quad \lim_{n\to\infty} D(ggx_n, ft) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Definition 2.13 Let A and S be mappings from a perturbed metric space (X, D, P) into itself. Then A and S are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \in X.$$

Remark 2.4 Continuous mappings are reciprocally continuous on (X, D, P) but the converse may not be true.

Example 2.3 Let $X = [2, 20]$ and D be the perturbed metric on X. Define mappings $A, S : X \to X$ by

$$Ax = 2 \quad \text{if } x = 2, \qquad Sx = 2 \quad \text{if } x = 2,$$

$$Ax = 3 \quad \text{if } x > 2, \qquad Sx = 6 \quad \text{if } x > 2.$$

It is noted that A and S are reciprocally continuous mappings but they are not continuous.
Now we give some properties related to compatible mappings of type (R) and type (E).

Proposition 2.15 Let f and g be compatible mappings of type (R) of a perturbed metric space (X, D, P) into itself. If $ft = gt$ for some $t \in X$, then

$$fgt = fft = ggt = gft.$$

Proof: Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t, n = 1, 2, \ldots$ for some $t \in X$ and $ft = gt$. We have $fx_n, gx_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (R), we have

$$D(fgt, gft) = \lim_{n\to\infty} D(fgx_n, gfx_n) = 0.$$

Hence we have $fgt = gft$. Therefore, since $ft = gt$, we have $fgt = fft = ggt = gft$. This completes the proof. □

Proposition 2.16 Let f and g be compatible mappings of type (R) of a perturbed metric space (X, D, P) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then

(a) $\lim_{n\to\infty} gfx_n = ft$ if f is continuous at t.

(b) $\lim_{n\to\infty} fgx_n = gt$ if g is continuous at t.

(c) $fgt = gft$ and $ft = gt$ if f and g are continuous at t.

Proof: (a) Suppose that f is continuous at t. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, we have $fx_n, fgx_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (R), we have

$$\lim_{n\to\infty} D(gfx_n, ft) = \lim_{n\to\infty} D(gfx_n, fgx_n) = 0.$$

Therefore, $\lim_{n\to\infty} gfx_n = ft$. (a) holds.

(b) The proof of $\lim_{n\to\infty} fgx_n = gt$ follows by similar arguments as in (a).

(c) Suppose that f and g are continuous at t and $\{x_n\}$ is a sequence in X defined $x_n = t$ $(n = 1, 2, \ldots)$ for some $t \in X$. Since $gx_n \to t$ as $n \to \infty$ and f is continuous at t, by (a), $gfx_n \to ft$ as $n \to \infty$. On the other hand, g is also continuous at t, and $\{x_n\}$ defined as above have $fx_n \to t$ as $n \to \infty$. So by (b) and by Proposition 2.15, we have $fgt = gft$. This completes the proof. □

Proposition 2.17 Let f and g be compatible mappings of type (E) of a perturbed metric space (X, D, P) into itself. Let one of f and g be continuous. Suppose that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

Then

(a) $ft = gt$ and $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$.

(b) If there exists $u \in X$ such that $fu = gu = t$, then $fgu = gfu$.

Proof: (a) Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$. Then by definition of compatible mappings of type (E), we have

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt.$$

If f is continuous, then we get

$$\lim_{n\to\infty} ffx_n = ft = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n,$$

which implies that $ft = gt$. Also,

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n.$$

Similarly, if g is continuous, then we get the same result.
(b) Next, suppose that $fu = gu = t$ for some $u \in X$. Then

$$fgu = f(gu) = ft \quad \text{and} \quad gfu = g(fu) = gt.$$

From (a), we have $ft = gt$. Hence $fgu = gfu$. This completes the proof. □

3. Fixed Point Results for Compatible Mappings and Variants of Compatible Mappings

In this section, we establish a sequence of common fixed point theorems for four self-mappings in the framework of perturbed metric spaces. The results are developed under different types of compatibility conditions, namely (A), (B), (C), (P), (K), (R) and (E) introduced in the previous section. Each theorem provides sufficient conditions for the existence and uniqueness of a common fixed point.
Theorem 3.1. Let A, B, S and T be mappings of a complete perturbed metric space (X, D, P) into itself satisfying the following conditions:

(3.1) $S(X) \subset B(X), \quad T(X) \subset A(X)$;

(3.2)
$$D(Sx, Ty) \le \lambda\big[\max\{D(Ax, By), D(Ax, Sx), D(By, Ty), D(Sx, By), D(Ax, Ty)\}\big],$$
for all $x, y \in X$, where $\lambda \in \big(0, \frac{1}{2}\big)$;

(3.3) one of the mappings A, B, S and T is continuous.

Assume that the pairs (A, S) and (B, T) are compatible. Then S, T, A and B have a unique common fixed point.
Proof: Since $S(X) \subset B(X)$, choose $x_0 \in X$ such that there exists $x_1 \in X$ with $Sx_0 = Bx_1 = y_0$. Similarly, select $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing this process, we can construct a sequence $\{y_n\}$ in X as follows:

$$Sx_{2n} = Bx_{2n+1} = y_{2n}, \quad Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}.$$

From the contractive condition, we obtain

$$\begin{aligned} D(y_{2n}, y_{2n+1}) &= D(Sx_{2n}, Tx_{2n+1}) \\ &\le \Bigg\{\lambda \max\Big[D(Ax_{2n}, Bx_{2n+1}),\ D(Ax_{2n}, Sx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad D(Sx_{2n}, Bx_{2n+1}),\ D(Ax_{2n}, Tx_{2n+1})\Big]\Bigg\}, \end{aligned}$$

$$\le \Bigg\{\lambda \max\Big[D(y_{2n-1}, y_{2n}), D(y_{2n-1}, y_{2n}), D(y_{2n}, y_{2n+1}), D(y_{2n}, y_{2n}), D(y_{2n-1}, y_{2n+1})\Big]\Bigg\},$$

$$\le \Bigg\{\lambda \max\Big[D(y_{2n-1}, y_{2n}), D(y_{2n}, y_{2n+1}), 0, [D(y_{2n-1}, y_{2n}) + D(y_{2n}, y_{2n+1})]\Big]\Bigg\}.$$

This implies that

$$D(y_{2n}, y_{2n+1}) \le \lambda[D(y_{2n-1}, y_{2n}) + D(y_{2n}, y_{2n+1})].$$

$$(1-\lambda)D(y_{2n}, y_{2n+1}) \le \lambda D(y_{2n-1}, y_{2n}).$$

$$D(y_{2n}, y_{2n+1}) \leq \frac{\lambda}{1-\lambda} D(y_{2n-1}, y_{2n}).$$

Let

$$\frac{\lambda}{1-\lambda} = h,$$

then

$$D(y_{2n}, y_{2n+1}) \leq hD(y_{2n-1}, y_{2n}).$$

We also obtain

$$D(y_{2n+1}, y_{2n+2}) \leq hD(y_{2n}, y_{2n+1}).$$

But

$$D(y_n, y_{n+1}) \leq hD(y_{n-1}, y_n) \leq \cdots \leq h^n D(y_0, y_1), \quad \forall n \geq 2.$$
$$D(y_n, y_{n+1}) \leq h^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric, we deduce that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq h^n D(y_0, y_1) \quad , n \in \mathbb{N}.$$

Since,

$$d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}) + P(y_n, y_{n+1}).$$

Therefore,

$$d(y_n, y_{n+1}) \leq h^n D(y_0, y_1) \quad , n \in \mathbb{N}.$$

Moreover, for every integer $m > 0$, we get

$$\begin{aligned} d(y_n, y_{n+m}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+m-1}, y_{n+m}) \\ &\leq h^n D(y_0, y_1) + h^{n+1} D(y_0, y_1) + \cdots + h^{n+m-1} D(y_0, y_1) \\ &= h^n D(y_0, y_1) (1 + h + h^2 + \cdots + h^{m-1}) \end{aligned}$$

$$d(y_n, y_{n+m}) \leq \frac{h^n}{1-h} D(y_0, y_1).$$

Therefore $\{y_n\}$ is a Cauchy sequence in metric space (X, d), so $\{y_n\}$ is also a perturbed Cauchy sequence in the perturbed metric space (X, D, P).

Taking limit as $n \to \infty$, we have $d(y_n, y_{n+m}) \to 0$. Therefore, $\{y_n\}$ is a perturbed Cauchy sequence in (X, D, P).
By the completeness of (X, D, P), there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of the sequence $\{y_n\}$ also converge to z.

Now suppose that A is continuous. Then AAx_{2n}, ASx_{2n} converge to Az as $n \to \infty$. Since (A, S) are compatible on X, it follows from Proposition 2.12 that SAx_{2n} converges to Az as $n \to \infty$.

We claim that $z = Az$. Consider

$$\begin{aligned} D(SAx_{2n}, Tx_{2n+1}) \leq \Big[\lambda \max \big\{&D(AAx_{2n}, Bx_{2n+1}),\, D(AAx_{2n}, SAx_{2n}),\, D(Bx_{2n+1}, Tx_{2n+1}), \\ &D(SAx_{2n}, Bx_{2n+1}),\, D(AAx_{2n}, Tx_{2n+1})\big\}\Big]. \end{aligned}$$

Letting $n \to \infty$, we have

$$D(Az, z) \leq \Big[\lambda \max\{D(Az, z), D(Az, Az), D(z, z), D(Az, z), D(Az, z)\}\Big] = \lambda(Az, z).$$

This implies that $D(Az, z) = 0$ implies $Az = z$. Next we claim that $Sz = z$. Consider

$$D(Sz, Tx_{2n+1}) \leq \Big[\lambda \max \big\{D(Az, Bx_{2n+1}),\, D(Az, Sz),\, D(Bx_{2n+1}, Tx_{2n+1}),\, D(Sz, Bx_{2n+1}),\, D(Az, Tx_{2n+1})\big\}\Big].$$

Letting $n \to \infty$, we have

$$D(Sz, z) \leq \Big[\lambda \max\{D(z, z), D(z, Sz), D(z, z), D(Sz, z), D(Sz, z)\}\Big] = \lambda(Sz, z).$$

This implies that $Sz = z$. Since $SX \subset BX$ and hence there exists a point $u \in X$ such that $z = Sz = Bu$.
We claim that $z = Tu$.

$$D(z, Tu) = D(Sz, Tu) \leq \Big[\lambda \max\big\{D(Az, Bu), D(Az, Sz), D(Bu, Tu), D(Sz, Bu), D(Az, Tu)\big\}\Big]$$

$$= \Big[\lambda \max\{D(z, z), D(z, z), D(z, Tu), D(z, z), D(z, Tu)\}\Big].$$

This implies that $z = Tu$. Since (B, T) is compatible in X and $Bu = Tu = z$, by Proposition 2.11, we have $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$. Also, we have

$$D(z, Bz) = D(Sz, Tz) \leq \Big[\lambda \max\big\{D(Az, Bz), D(Az, Sz), D(Bz, Tz), D(Sz, Bz), D(Az, Tz)\big\}\Big]$$

$$= \Big[\lambda \max\{D(z, Bz), D(z, z), D(Bz, Tz), D(z, Bz), D(z, Bz)\}\Big].$$

This implies that $z = Bz$. Hence, $z = Bz = Tz = Az = Sz$. Therefore, z is the common fixed point of S, T, A, and B.
Similarly, we can also complete the proof when B is continuous. Next, suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Az as $n \to \infty$. Since A and S are compatible on X, it follows from Proposition 2.12 that ASx_{2n} converges to Az as $n \to \infty$.
Consider

$$D(SSx_{2n}, Tx_{2n+1}) \leq \Big[\lambda \max\big\{D(ASx_{2n}, Bx_{2n+1}),\ D(ASx_n, SSx_{2n}),$$
$$D(Bx_{2n+1}, Tx_{2n+1}),\ D(SSx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, Tx_{2n+1})\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, z) \leq [\lambda \max\{D(Sz, z), D(Sz, Sz), D(z, z), D(Sz, z), D(Sz, z)\}] = \lambda D(Sz, z),$$

which implies that $Sz = z$. Since $SX \subset BX$, there exists a point $v \in X$ such that $z = Sz = Bv$.
Consider

$$D(SSx_{2n}, Tv) \leq [\lambda \max\{D(ASx_{2n}, Bv),\ D(ASx_{2n}, SSx_{2n}),\ D(Bv, Tv),\ D(SSx_{2n}, Bv),\ D(ASx_{2n}, Tv)\}].$$

Letting $n \to \infty$, we have

$$D(z, Tv) \leq \Big[\lambda \max\{D(z, z), D(z, z), D(z, Tv), D(z, z), D(z, Tv)\}\Big] = \lambda D(z, Tv).$$

This implies that $z = Tv$. Since B and T are compatible on X and $Bv = Tv = z$, by Proposition 2.11, we have $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$. Consider

$$D(Sx_{2n}, Tz) \leq \Big[\lambda \max\big\{D(Ax_{2n}, Bz), D(Ax_{2n}, Sx_{2n}), D(Bz, Tz), D(Sx_{2n}, Bz), D(Ax_{2n}, Tz)\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(z, Tz) \leq \Big[\lambda \max\{D(z, Tz), D(z, z), D(Tz, Tz), D(z, Tz), D(z, Tz)\}\Big] = \lambda D(z, Tz).$$

This implies that $Tz = z$. Since $TX \subset AX$, so there exists a point $w \in X$ such that $z = Tz = Aw$.
Consider

$$D(Sw, z) = D(Sw, Tz) \leq \Big[\lambda \max\big\{D(Aw, Bz), D(Aw, Sw), D(Bz, Tz), D(Sw, Bz), D(Aw, Tz)\big\}\Big]$$

$$= \Big[\lambda \max\{D(z,z), D(z,Sw), D(z,Tz), D(Sw,z), D(z,z)\}\Big].$$

This implies that $Sw = z$. Since S and A are compatible on X, $Sw = Aw = z$, by Proposition 2.11, we have $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. That is, $z = Az = Sz = Bz = Tz$. Therefore, z is common fixed point of S, T, A and B.

Similarly, we can complete the proof when T is continuous. Finally, suppose that z and w ($z \neq w$) are two common fixed points of S, T, A and B. Consider

$$D(z,w) = D(Sz,Tw) \leq \Big[\lambda \max\big\{D(Az,Bw), D(Az,Sz), D(Bw,Tw), D(Sz,Bw), D(Az,Tw)\big\}\Big]$$

$$= \Big[\lambda \max\{D(z,w), D(z,z), D(w,w), D(z,w), D(z,w)\}\Big] = \lambda D(z,w).$$

This implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proof.

Next we give the following theorem for compatible mappings of type (A).

Theorem 3.2. Let S, T, A and B be mappings of a complete perturbed metric space (X, D, P) into itself satisfying (3.1)–(3.3). Assume that the pairs (A, S) and (B, T) are compatible of type (A). Then S, T, A and B have a unique common fixed point.

Proof: Suppose that A is continuous. Since (A, S) are compatible of type (A), by Proposition 2.1, the pair (A, S) is compatible, so the result easily follows from Theorem 3.1. Similarly, if B is continuous and (B, T) is compatible of type (A), then (B, T) is compatible, so the result easily follows from Theorem 3.1. Similarly, we can complete the proof when S or T is continuous. This completes the proof.

Also we give the following theorem for compatible mappings of type (B).

Theorem 3.3. Let S, T, A and B be mappings of a complete perturbed metric space (X, D, P) into itself satisfying (3.1)–(3.3).
Assume that the pairs (A, S) and (B, T) are compatible of type (B). Then S, T, A and B have a unique common fixed point.

Proof: From the proof of Theorem 3.1, $\{y_n\}$ is a perturbed Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converge to z.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since the pair (A, S) is compatible of type (B), it follows from Proposition 2.14 that AAx_{2n} converges to Sz as $n \to \infty$.

Consider

$$D(SAx_{2n}, Tx_{2n+1}) \leq \lambda \max\Big\{D(AAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, SAx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),$$
$$D(SAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, Tx_{2n+1})\Big\}.$$

Letting $n \to \infty$, we get

$$D(Sz,z) \leq \Big[\lambda \max\{D(Sz,z), D(Sz,Sz), D(z,z), D(Sz,z), D(Sz,z)\}\Big].$$

This implies that $Sz = z$. Since $SX \subset BX$, there exists a point $u \in X$ such that $z = Sz = Bu$. Consider

$$D(SAx_{2n}, Tu) \leq \Big[\lambda \max\big\{D(AAx_{2n}, Bu), D(AAx_{2n}, SAx_{2n}), D(Bu, Tu), D(SAx_{2n}, Bu), D(AAx_{2n}, Tu)\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, Tu) \leq \lambda D(Sz, Tu).$$

This implies that $Tu = Sz$ ($z = Tu$). Since the pair (B, T) is compatible of type (B) and $Bu = z = Tu$, by Proposition 2.13 we have $TBu = BTu$ and so $Bz = BTu = TBu = Tz$. Consider

$$D(Sx_{2n}, Tz) \leq \Big[\lambda \max\big\{D(Ax_{2n}, Bz), D(Ax_{2n}, Sx_{2n}), D(Bz, Tz), D(Sx_{2n}, Bz), D(Ax_{2n}, Tz)\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(z, Tz) \le \lambda D(z, Tz),$$

which implies that $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$. Consider

$$D(Sv, Tz) \le \Big[\lambda \max\big\{D(Av, Bz), D(Av, Sv), D(Bz, Tz), D(Sv, Bz), D(Av, Tz)\big\}\Big],$$

which implies that

$$D(Sv, z) \le \lambda D(Sv, z).$$

This implies $Sv = z$. Since the pair (A, S) is compatible of type (B) and $Sv = z = Av$, it follows from Proposition 2.13 that $Sz = SAv = ASv = Az$. Therefore, $Az = Bz = Sz = Tz = z$ and hence z is the common fixed point of S, T, A and B. Now suppose that A is continuous. Then AAx_{2n} and ASx_{2n} converge to Az as $n \to \infty$. Since (A, S) is compatible of type (B), it follows from Proposition 2.14 that SSx_{2n} converges to Az as $n \to \infty$. Consider

$$D(SSx_{2n}, Tx_{2n+1}) \le \lambda \max\Big\{D(ASx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, SSx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\ D(SSx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, Tx_{2n+1})\Big\}.$$

Letting $n \to \infty$, we get

$$D(Az, z) \le \lambda D(Az, z).$$

This implies $Az = z$. Consider

$$D(Sz, Tx_{2n+1}) \le \Big[\lambda \max\big\{D(Az, Bx_{2n+1}), D(Az, Sz), D(Bx_{2n+1}, Tx_{2n+1}), D(Sz, Bx_{2n+1}), D(Az, Tx_{2n+1})\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, z) \le \lambda D(Sz, z).$$

This implies $Sz = z$. Since $SX \subset BX$, there exists a point $w \in X$ such that $z = Sz = Bw$. Consider

$$D(z, Tw) = D(Sz, Tw) \le \Big[\lambda \max\big\{D(Az, Bw), D(Az, Sz), D(Bw, Tw), D(Sz, Bw), D(Az, Tw)\big\}\Big]$$

$$= \Big[\lambda \max\{D(z, z), D(z, z), D(z, Tw), D(z, z), D(z, Tw)\}\Big].$$

This implies that $z = Tw$. Since (B, T) is compatible of type (B) and $Bw = z = Tw$, from Proposition 2.13, $TBw = BTw$ and so $Bz = BTw = TBw = Tz$. Consider

$$D(Sz, Tz) \le \Big[\lambda \max\big\{D(z, Tz), D(z, z), D(Tz, Tz), D(z, Tz), D(z, Tz)\big\}\Big] = \lambda D(z, Tz).$$

This implies that $z = Tz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly, we can complete the proof when B or T is continuous.

Finally, if z and w $(z \ne w)$ are two common fixed points, then we have

$$D(z, w) = D(Sz, Tw) \le \Big[\lambda \max\big\{D(Az, Bw), D(Az, Sz), D(Bw, Tw), D(Sz, Bw), D(Az, Tw)\big\}\Big] = \lambda D(z, w).$$

This implies $z = w$. This implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proofs.

Now we give the following theorem for compatible mappings of type (C).

Theorem 3.4. Let S, T, A and B be mappings of a perturbed metric space (X, D, P) into itself satisfying (3.1)–(3.3).
Assume that the pair (A, S) and (B, T) are compatible of type (C). Then S, T, A and B have a unique

common fixed point.

Proof: From the proof of Theorem 3.1, $\{y_n\}$ is a perturbed Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since the pair (A, S) is compatible of type (C), it follows from Remark 2.3 that AAx_{2n} converges to Sz as $n \to \infty$.

We claim that $Sz = z$. Consider

$$D(SAx_{2n}, Tx_{2n+1}) \leq \Big[\lambda \max\Big\{D(AAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, SAx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\ D(SAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, Tx_{2n+1})\Big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, z) \leq \Big[\lambda \max\{D(Sz, z), D(Sz, Sz), D(z, z), D(Sz, z), D(Sz, z)\}\Big] = \lambda D(Sz, z).$$

This implies that $Sz = z$. Since $SX \subset BX$, there exists a point $u \in X$ such that $z = Sz = Bu$. Consider

$$D(SAx_{2n}, Tu) \leq \Big[\lambda \max\big\{D(AAx_{2n}, Bu), D(AAx_{2n}, SAx_{2n}), D(Bu, Tu), D(SAx_{2n}, Bu), D(AAx_{2n}, Tu)\big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, Tu) \leq \Big[\lambda \max\{D(Sz, Sz), D(Sz, Sz), D(Sz, Tu), D(Sz, Tu), D(Sz, Tu)\}\Big] = \lambda D(Sz, Tu).$$

This implies that $Sz = Tu$ ($z = Tu$). Since the pair (B, T) is compatible of type (C) and $Bu = z = Tu$, by Remark 2.2, we get $TBu = BTu$ and so $Bz = BTu = TBu = Tz$. Consider

$$D(Sx_{2n}, Tz) \leq \Big[\lambda \max\big\{D(Ax_{2n}, Bz), D(Ax_{2n}, Sx_{2n}), D(Bz, Tz), D(Sx_{2n}, Bz), D(Ax_{2n}, Tz)\big\}\Big].$$

Letting $n \to \infty$, we have

$$D(z, Tz) \leq \Big[\lambda \max\{D(z, Tz), D(z, z), 0, D(z, Tz), D(z, Tz)\}\Big] = \lambda D(z, Tz).$$

This implies that $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$. Consider

$$D(Sv, z) = D(Sv, Tz) \leq \Big[\lambda \max\big\{D(Av, Bz), D(Av, Sv), D(Bz, Tz), D(Sv, Bz), D(Av, Tz)\big\}\Big]$$

$$\leq \Big[\lambda \max\{D(z, z), D(Sv, z), D(z, z), D(Sv, z), D(z, z)\}\Big] = \lambda D(Sv, z).$$

This implies that $z = Sv$. Since the pair (A, S) is compatible of type (C) and $Sv = z = Av$, by Remark 2.2, $ASv = SAv$. We have $Sz = SAv = ASv = Az$. Therefore, $Bz = Az = Tz = Sz = z$ and hence z is the common fixed point of S, T, A and B.

Suppose that A is continuous. Then AAx_{2n} and ASx_{2n} converge to Az as $n \to \infty$. Since the pair (A, S) is compatible of type (C), it follows from Remark 2.3 that SSx_{2n} converges to Az as $n \to \infty$. Also we have

$$D(SSx_{2n}, Tx_{2n+1}) \leq \Big[\max\Big\{\lambda D(ASx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, SSx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\ D(SSx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, Tx_{2n+1})\Big\}\Big].$$

Letting $n \to \infty$, we get

$$D(Az, z) \leq \Big[\lambda \max\{D(Az, z), D(Az, Av), D(z, z), D(Az, z), D(Az, z)\}\Big] = \lambda D(Az, z).$$

This implies that $Az = z$. Consider

$$D(Sz, Tx_{2n+1}) \leq \Big[\lambda \max\left\{D(Az, Bx_{2n+1}), D(Az, Sz), D(Bx_{2n+1}, Tx_{2n+1}), D(Sz, Bx_{2n+1}), D(Az, Tx_{2n+1})\right\}\Big].$$

Letting $n \to \infty$, we get

$$D(Sz, z) \leq \lambda D(Sz, z).$$

This implies $Sz = z$. Since $SX \subset BX$, there exists a point $w \in X$ such that $z = Sz = Bw$. Again, we have

$$D(z, Tw) = D(Sz, Tw) \leq \Big[\lambda \max\left\{D(Az, Bw), D(Az, Sz), D(Bw, Tw), D(Sz, Bw), D(Az, Tw)\right\}\Big] = \lambda D(z, Tw).$$

This implies that $Tw = z$. Since (B, T) is compatible of type (C) and $Bw = z = Tw$, by Remark 2.2 we have $TBw = BTw$ and so $Bz = BTw = TBw = Tz$.

Also consider

$$D(z, Tz) = D(Sz, Tz) \leq \lambda D(z, Tz).$$

Thus implies that $Tz = z$. Hence $Tz = Bz = Sz = Az = z$. Therefore, z is the common fixed point of S, T, A and B.

Similarly, we can complete the proof when B or T is continuous. Uniqueness follows easily. This completes the proofs.

Next, we give the following theorem for compatible mappings of type (P).

Theorem 3.5. Let S, T, A and B be mappings of a complete perturbed metric space (X, D, P) into itself satisfying (3.1)–(3.3).
Assume that the pairs (A, S) and (B, T) are compatible of type (P). Then S, T, A and B have a unique common fixed point.

Proof: From the proof of Theorem 3.1, $\{y_n\}$ is a perturbed Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converge to z as $n \to \infty$.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since (A, S) is compatible of type (P), it follows from Remark 2.3 that AAx_{2n} converges to Sz as $n \to \infty$. We claim $Sz = z$. Consider

$$\begin{aligned} D(SAx_{2n}, Tx_{2n+1}) \leq \Big[\lambda \max\Big\{&D(AAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, SAx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\ &D(SAx_{2n}, Bx_{2n+1}),\ D(AAx_{2n}, Tx_{2n+1})\Big\}\Big]. \end{aligned}$$

Letting $n \to \infty$, we have

$$D(Sz, z) \leq \Big[\lambda \max\{D(Sz, z), D(Sz, Sz), D(z, z), D(Sz, z), D(Sz, z)\}\Big] = \lambda D(Sz, z).$$

This implies that $Sz = z$. Since $SX \subset BX$, so there exists a point $u \in X$ such that $z = Sz = Bu$.

Now we claim that $Tu = z$. Consider

$$D(Sx_{2n}, Tu) \leq \Big[\lambda \max\left\{D(Ax_{2n}, Bu), D(Ax_{2n}, Sx_{2n}), D(Bu, Tu), D(Sx_{2n}, Bu), D(Ax_{2n}, Tu)\right\}\Big].$$

Letting $n \to \infty$, we get

$$D(z, Tu) \leq \Big[\lambda \max\{D(z, z), D(z, z), D(z, Tu), D(z, z), D(z, Tu)\}\Big] = \lambda D(z, Tu).$$

This implies that $z = Tu$. Therefore, $Bu = Tu = z$. Since (B, T) is compatible of type (P), by Remark 2.2, we have $TTu = BBu$, which implies that $D(Bz, Tz) = 1$. Hence $Tz = Bz$.

Now we claim that $Tz = z$. Consider

$$D(Sx_{2n}, Tz) \leq \Big[\lambda \max\left\{D(Ax_{2n}, Bz), D(Ax_{2n}, Sx_{2n}), D(Bz, Tz), D(Sx_{2n}, Bz), D(Ax_{2n}, Tz)\right\}\Big].$$

Letting $n \to \infty$, we have

$$D(z, Tz) \leq \lambda D(z, Tz),$$

which implies that $Tz = z$. Therefore, $Bz = Tz = z$. Since $TX \subset AX$, so there exists a point $v \in X$ such that $z = Tz = Av$.

Now we claim that $Sv = z$. Consider

$$D(Sv, z) = D(Sv, Tz) \leq \Big[\lambda \max\big\{D(Av, Bz), D(Av, Sv), D(Bz, Tz), D(Sv, Bz), D(Av, Tz)\big\}\Big] = \lambda D(Sv, z).$$

This implies that $z = Sv$. Therefore $z = Sv = Av$. Since (A, S) is compatible of type (P), by Remark 2.2, we have $SSv = AAv$, which implies that $D(Sz, Az) = 0$. Hence $Sz = Az$. Since $Az = Bz = Sz = Tz = z$, z is a common fixed point of S, T, A and B.

Similarly, we can complete the proof when A or B or T is continuous. The uniqueness follows easily. This completes the proofs.

Next, we give the following theorem for compatible mappings of type (R).

Theorem 3.6. Let A, B, S and T be mappings of a complete perturbed metric space (X, D, P) into itself satisfying the conditions (3.1) , (3.2) and (3.3).

Assume that the pairs (A, S) and (B, T) are compatible mappings of type (R). Then A, B, S and T have a unique common fixed point.

Proof: From the proof of Theorem 3.1, the sequence $\{y_n\}$ is a perturbed Cauchy sequence in X, and hence it converges to some point $z \in X$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ also converge to z.

Now suppose that A is continuous. Since A and S are compatible of type (R), by Proposition 2.16, AAx_{2n} and SAx_{2n} converge to Az as $n \to \infty$.

We claim that $z = Az$. Putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in inequality (3.2), we have

$$\begin{aligned} D(SAx_{2n}, Tx_{2n+1}) \leq \lambda \max\{&D(AAx_{2n}, Bx_{2n+1}), D(AAx_{2n}, SAx_{2n}), D(Bx_{2n+1}, Tx_{2n+1}), \\ &D(SAx_{2n}, Bx_{2n+1}), D(AAx_{2n}, Tx_{2n+1})\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$D(Az, z) \leq \lambda \max\{D(Az, z), D(Az, Az), D(z, z), D(Az, z), D(Az, z)\} = \lambda D(Az, z),$$

which implies that $Az = z$.

Next, we claim that $Sz = z$. Putting $x = z$ and $y = x_{2n+1}$ in (3.2), we have

$$D(Sz, Tx_{2n+1}) \leq [\lambda \max\{D(Az, Bx_{2n+1}), D(Az, Sz), D(Bx_{2n+1}, Tx_{2n+1}), D(Sz, Bx_{2n+1}), D(Az, Tx_{2n+1})\}].$$

Letting $n \to \infty$, we obtain

$$D(Sz, z) \leq [\lambda \max\{D(z, z), D(z, Sz), D(z, z), D(Sz, z), D(z, z)\}] = \lambda\ D(Sz, z),$$

which implies that $Sz = z$.

Since $S(X) \subset B(X)$, there exists a point $u \in X$ such that $z = Sz = Bu$.
We claim that $z = Tu$. Putting $x = z$ and $y = u$ in (3.2), we get

$$\begin{aligned} D(z, Tu) = D(Sz, Tu) &\leq [\lambda \max\{D(Az, Bu), D(Az, Sz), D(Bu, Tu), D(Sz, Bu), D(Az, Tu)\}], \\ &= [\lambda \max\{D(z, z), D(z, z), D(z, Tu), D(z, z), D(z, Tu)\}], \\ &= \lambda D(z, Tu). \end{aligned}$$

Substituting limits, we have

$$D(z, Tu) \leq \lambda D(z, Tu),$$

which implies that $z = Tu$. Since B and T are compatible of type (R) and $Bu = Tu = z$, by Proposition 2.15, $BTu = TBu$, and hence $Bz = BTu = TBu = Tz$. Also we have

$$\begin{aligned}D(z, Bz) &= D(Sz, Tz)\\&\leq \left\{\lambda \max\left[D(Az, Bz), D(Az, Sz), D(Bz, Tz), D(Sz, Bz), D(Az, Tz)\right]\right\}\\&= \left\{\lambda \max\left[D(z, Bz), D(z, z), D(Bz, Bz), D(z, Bz), D(z, Bz)\right]\right\}\\&= \lambda D(z, Bz),\end{aligned}$$

which implies that $z = Bz$. Hence, $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of A, S, B, and T.

Similarly, we can complete the proof when B is continuous.

Next, suppose that S is continuous. Since A and S are compatible of type (R), by Proposition 2.16, SSx_{2n} and Sx_{2n} converge to Sz as $n \to \infty$.

We claim that $z = Sz$. Putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (3.2), we have

$$\begin{aligned}D(SSx_{2n}, Tx_{2n+1}) \leq \Bigg\{\lambda \max\Big[&D(ASx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, SSx_{2n}),\ D(Bx_{2n+1}, Tx_{2n+1}),\\&D(SSx_{2n}, Bx_{2n+1}),\ D(ASx_{2n}, Tx_{2n+1})\Big]\Bigg\}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}D(Sz, z) &\leq \left\{\lambda \max\left[D(Sz, z), D(Sz, Sz), D(z, z), D(Sz, z), D(Sz, z)\right]\right\}\\&= \lambda D(Sz, z),\end{aligned}$$

which implies that $Sz = z$. Since $SX \subset BX$, hence there exists a point $v \in X$ such that $z = Sz = Bv$.

We claim that $z = Tv$. Putting $x = Sx_{2n}$ and $y = v$ in (3.2), we have

$$D(SSx_{2n}, Tv) \leq \left\{\lambda \max\left[D(ASx_{2n}, Bv), D(ASx_{2n}, SSx_{2n}), D(Bv, Tv), D(SSx_{2n}, Bv), D(ASx_{2n}, Tv)\right]\right\}.$$

Letting $n \to \infty$, we have

$$\begin{aligned}D(z, Tv) &\leq \left\{\lambda \max\left[D(z, z), D(z, z), D(z, Tv), D(z, z), D(z, Tv)\right]\right\}\\&= \lambda D(z, Tv),\end{aligned}$$

which implies that $z = Tv$. Since B and T are compatible of type (R) and $Bv = Tv = z$, by Proposition 2.15, $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$.

We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$ in (3.2), we have

$$D(Sx_{2n}, Tz) \leq \left\{\lambda \max\left[D(Ax_{2n}, Bz), D(Ax_{2n}, Sx_{2n}), D(Bz, Tz), D(Sx_{2n}, Bz), D(Ax_{2n}, Tz)\right]\right\}.$$

Letting $n \to \infty$, we have

$$\begin{aligned}D(z, Tz) &\leq \left\{\lambda \max\left[D(z, Tz), D(z, z), D(Tz, Tz), D(z, Tz), D(z, Tz)\right]\right\}\\&= \lambda D(z, Tz),\end{aligned}$$

which implies that $z = Tz$. Since $TX \subset AX$, there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$. Putting $x = w$ and $y = z$ in (3.2), we have

$$\begin{aligned}D(Sw, z) &= D(Sw, Tz)\\&\leq \left\{\lambda \max\left[D(Aw, Bz), D(Aw, Sw), D(Bz, Tz), D(Sw, Bz), D(Aw, Tz)\right]\right\}\\&= \left\{\lambda \max\left[D(z, z), D(z, Sw), D(Tz, Tz), D(Sw, z), D(z, z)\right]\right\}\\&= \lambda D(z, Sw),\end{aligned}$$

which implies that $Sw = z$.

Since A and S are compatible of type (R) and $Sw = Aw = z$, by Proposition 2.15, $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. That is, $z = Az = Sz = Bz = Tz$. Therefore, z is a common fixed point of A, S, B, and T.

Similarly, we can complete the proof when T is continuous. Uniqueness follows easily, which completes the proof.

Next, we prove the following theorem for compatible mappings of type (K).

Theorem 3.7. Let A, B, S and T be mappings of a complete perturbed metric space (X, D, P) into itself satisfying the conditions (3.1) and (3.2). Suppose that the pairs (A, S) and (B, T) are reciprocally continuous. Assume that the pairs (A, S) and (B, T) are compatible of type (K). Then A, B, S, and T have a unique common fixed point.

Proof: From the proof of Theorem 3.1, the sequence $\{y_n\}$ is a perturbed Cauchy sequence in X, and hence it converges to some point $z \in X$. Consequently, the subsequences $\{Sx_n\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ also converge to z.

Since the pairs (A, S) and (B, T) are compatible of type (K), we have $AAx_{2n} \to Sz$, $SSx_{2n} \to Az$, and $BBx_{2n} \to Tz$, $TTx_{2n+1} \to Bz$ as $n \to \infty$.

We claim that $Bz = Az$. Putting $x = Sx_{2n}$ and $y = Tx_{2n+1}$ in (3.2), we have

$$D(SSx_{2n}, TTx_{2n+1}) \leq \Big\{\lambda \max\Big[D(ASx_{2n}, BTx_{2n+1}),\ D(ASx_{2n}, SSx_{2n}),\ D(BTx_{2n+1}, TTx_{2n+1}),\ D(SSx_{2n}, BTx_{2n+1}),\ D(ASx_{2n}, TTx_{2n+1})\Big]\Big\}.$$

Letting $n \to \infty$ and using reciprocal continuity of the pairs A, S and B, T, we have

$$\begin{aligned} D(Az, Bz) &\leq \{\lambda \max[D(Az, Bz), D(Az, Az), D(Bz, Bz), D(Az, Bz), D(Az, Bz)]\} \\ &= \lambda D(Az, Bz), \end{aligned}$$

which implies that $D(Az, Bz) = 0$ and hence $Az = Bz$.

Next, we claim that $Sz = Bz$. Putting $x = z$ and $y = Tx_{2n+1}$ in (3.2), we have

$$D(Sz, TTx_{2n+1}) \leq \Big\{\lambda \max\Big[D(Az, BTx_{2n+1}),\ D(Az, Sz),\ D(BTx_{2n+1}, TTx_{2n+1}),\ D(Sz, BTx_{2n+1}),\ D(Az, TTx_{2n+1})\Big]\Big\}.$$

Letting $n \to \infty$, and using reciprocal continuity of the pairs A, S and B, T, we have

$$\begin{aligned} D(Sz, Bz) &\leq \{\lambda \max[D(Bz, Bz), D(Bz, Sz), D(Bz, Bz), D(Sz, Bz), D(Bz, Bz)]\} \\ &= \lambda\ D(Sz, Bz), \end{aligned}$$

which implies that $Sz = Bz$.

We claim that $Sz = Tz$. Putting $x = z$ and $y = z$ in (3.2), we have

$$\begin{aligned} D(Sz, Tz) &\leq \{\lambda \max[D(Az, Bz), D(Az, Sz), D(Bz, Tz), D(Sz, Bz), D(Az, Tz)]\} \\ &= \{\lambda \max[0, 0, D(Sz, Tz), 0, D(Sz, Tz)]\} \\ &= \lambda D(Sz, Tz), \end{aligned}$$

which implies that $Sz = Tz$.

We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$ in (3.2), we have

$$D(Sx_{2n}, Tz) \leq \left\{\lambda \max \left[D(Ax_{2n}, Bz),\ D(Ax_{2n}, Sx_{2n}),\ D(Bz, Tz),\ D(Sx_{2n}, Bz),\ D(Ax_{2n}, Tz)\right]\right\}.$$

Letting $n \to \infty$, we have

$$\begin{aligned} D(z, Tz) &\leq \left\{\lambda \max \left[D(z, Bz), D(z, z), D(z, Bz), D(z, Bz), D(z, Tz)\right]\right\} \\ &= \{\lambda \max[D(z, Tz), 0, 0, D(z, Tz), D(z, Tz)]\} \\ &= \lambda D(z, Tz), \end{aligned}$$

which implies that $z = Tz$.

Hence, $z = Az = Bz = Sz = Tz$. Therefore, z is a common fixed point of A, B, S, and T. Uniqueness follows easily, which completes the proof.

Finally, we prove the following theorem for compatible mappings of type (E).

Theorem 3.8. Let A, B, S and T be mappings of a complete perturbed metric space (X, D, P) into itself satisfying the conditions (3.1) and (3.2). Suppose that one of A and S is continuous, and one of B and T is continuous. Assume that the pairs (A, S) and (B, T) are compatible of type (E). Then A, B, S and T have a unique common fixed point.

Proof: From the proof of Theorem 3.1, $\{y_n\}$ is a perturbed Cauchy sequence in X, and hence it converges to some point $z \in X$. Consequently, the subsequences $\{Sx_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ also converge to z.

Now suppose that A and S are compatible of type (E) and one of the mappings A and S is continuous. Then by Proposition 2.17, we have $Az = Sz$. Since $SX \subset BX$, there exists a point $w \in X$ such that $Sz = Bw$. Putting $x = z$ and $y = w$ in (3.2), we have

$$\begin{aligned} D(Sz, Tw) &\leq \left\{\lambda \max \left[D(Az, Bw),\ D(Az, Sz),\ D(Bw, Tw),\ D(Sz, Bw),\ D(Az, Tw)\right]\right\} \\ &= \left\{\lambda \max \left[D(Az, Sz),\ D(Sz, Sz),\ D(Sz, Tw),\ D(Sz, Bw),\ D(Sz, Tw)\right]\right\} \\ &= \lambda D(Sz, Tw), \end{aligned}$$

which implies that $D(Sz, Tw) = 0$ and hence $Sz = Tw$. Thus, we have $Az = Sz = Tw = Bw$.

Putting $x = z$ and $y = x_{2n+1}$ in (3.2), we have

$$\begin{aligned} D(Sz, Tx_{2n+1}) \leq \Big\{\lambda \max \Big[&D(Az, Bx_{2n+1}),\ D(Az, Sz),\ D(Bx_{2n+1}, Tx_{2n+1}), \\ &D(Sz, Bx_{2n+1}),\ D(Az, Tx_{2n+1})\Big]\Big\}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} D(Sz, z) &\leq \left\{\lambda \max \left[D(Sz, z), 0, D(z, z), D(Sz, z), D(Sz, z)\right]\right\} \\ &= \lambda D(Sz, z), \end{aligned}$$

which implies that $D(Sz, z) = 0$, that is, $Sz = z$.
Therefore, z is a common fixed point of A and S.

Again, suppose that B and T are compatible of type (E) and one of the mappings B and T is continuous. Then we get $Bw = Tw = z$. By Proposition 2.3, we have $BBw = BTw = TBw = TTw$, that is, $Bz = Tz$.

Putting $x = x_{2n}$ and $y = z$ in (3.2), we have

$$D(Sx_{2n}, Tz) \leq \left\{\lambda \max\Big[D(Ax_{2n}, Bz),\ D(Ax_{2n}, Sx_{2n}),\ D(Bz, Tz),\ D(Sx_{2n}, Bz),\ D(Ax_{2n}, Tz)\Big]\right\}.$$

Letting $n \to \infty$, we have

$$\begin{aligned} D(z, Tz) &\leq \big\{\lambda \max\big[D(z, Tz), D(z, z), D(Tz, Tz), D(z, Tz), D(z, Tz)\big]\big\} \\ &= \lambda D(z, Tz), \end{aligned}$$

which implies that $D(z, Tz) = 0$, that is, $Tz = z$. Thus, we have $Tz = Bz = z$. Therefore, z is a common fixed point of B and T. Hence, z is a common fixed point of A, B, S, and T.

Uniqueness can be easily shown. This completes the proof.

4. Faintly Compatible Mappings

In this section, we introduce the concept of faintly compatible mappings in the framework of perturbed metric spaces.

Before proceeding further, we recall some relevant concepts.

Definition 4.1. [12] Two self-mappings A and B of a metric space (X, d) are said to be semi-compatible if

(i) $Ax = Bx \implies ABx = BAx$;

(ii) $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X \implies \lim_{n\to\infty} d(ABx_n, Bx_n) = 0$.

Definition 4.2. [1] Two self-mappings A and B of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, that is, if $ABx = BAx$ whenever $Ax = Bx$, $x \in X$.

Definition 4.3. [9] Two self-mappings A and B of a metric space (X, d) are said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad \text{for some } z \in X,$$

but

$$\lim_{n\to\infty} d(ABx_n, BAx_n)$$

is either non-zero or non-existent.

Definition 4.4. [4] A pair (S, T) of self-mappings of a metric space (X, d) is said to be occasionally weakly compatible (OWC) if

$$STx = TSx \quad \text{for some } x \in C(S, T).$$

In the sense of Jungck and Rhodes, a pair (S, T) of self-mappings of a metric space (X, d) is said to be occasionally weakly compatible (OWC) if there exists at least one coincidence point at which S and T commute, i.e., if $ST = TS$ for some $x \in X$, then $STx = TSx$.

Definition 4.5. [**10**] A pair (A, S) of self-mappings of a metric space (X, d) is said to be conditionally compatible mappings if and only if whenever the set of sequences $\{y_n\}$ satisfying

$$\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} S(y_n)$$

is nonempty, there exists another sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = t \quad \text{and} \quad \lim_{n\to\infty} d(A(Sz_n), S(Az_n)) = 0.$$

Definition 4.6. Two self-mappings A and S of a metric space (X, d) will be called to be faintly compatible if A and S are conditionally compatible and A and S commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.
Now we introduce the analogues notions of faintly compatible mappings and other variants in the setting of perturbed metric spaces.

Definition 4.7. Two self-mappings A and B of a perturbed metric space (X, D, P) are said to be semi-compatible if

(i) $Ax = Bx \implies ABx = BAx$;

(ii) $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X \implies \lim_{n\to\infty} D(ABx_n, Bx_n) = 0$.

Definition 4.8. Two self-mappings A and B of a perturbed metric space (X, D, P) are said to be weakly compatible if they commute at their coincidence points, that is, if $ABx = BAx$ whenever $Ax = Bx$, $x \in X$.
Definition 4.9. [9] Two self-mappings A and B of a perturbed metric space (X, D, P) are said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad \text{for some } z \in X,$$

but

$$\lim_{n\to\infty} D(ABx_n, BAx_n)$$

is either non-zero or non-existent.
Definition 4.10. [4] A pair (S, T) of self-mappings of a perturbed metric space (X, D, P) is said to be occasionally weakly compatible (OWC) if

$$STx = TSx \quad \text{for some } x \in C(S, T).$$

Definition 4.11. :A pair (A, S) of self-mappings of a perturbed metric space (X, D, P) is said to be conditionally compatible mappings if and only if whenever the set of sequences $\{y_n\}$ satisfying

$$\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} S(y_n)$$

is nonempty, there exists another sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = t \quad \text{and} \quad \lim_{n\to\infty} D(A(Sz_n), S(Az_n)) = 0.$$

It may be observed that compatibility is independent of the notion of conditional compatibility, and in the setting of a unique common fixed point (or unique point of coincidence), conditional compatibility does not reduce to the class of compatibility. The following example illustrate these facts.
Example 4.1 Let $D : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + |x - y|^{1/2}, \ for\ all\ x, y \in \mathbb{X}.$$

Then D is a perturbed metric on $\mathbb{X}$ with respect to the perturbed function

$$P : \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

given by

$$P(x, y) = |x - y|^{1/2}, \quad x, y \in \mathbb{X}.$$

Let $X = [1, 8]$ with the perturbed metric

$$D(x, y) = |x - y| + |x - y|^{1/2}.$$

Define $A, S : X \to X$ by

$$A(x) = \begin{cases} 2, & x \le 2, \\ 5, & x > 2, \end{cases} \qquad S(x) = \begin{cases} 6 - 2x, & x \le 2, \\ 8, & x > 2. \end{cases}$$

Take the constant sequence $z_n \equiv 2$. Then

$$A(z_n) = 2, \qquad S(z_n) = 2,$$

so

$$A(S(z_n)) = A(2) = 2, \qquad S(A(z_n)) = S(2) = 2,$$

and therefore

$$\lim_{n\to\infty} D\big(A(S(z_n)), S(A(z_n))\big) = D(2, 2) = 0.$$

If now consider $y_n = 2 - \frac{1}{n}$ (so $y_n \to 2$). For large n we have

$$A(y_n) = 2, \qquad S(y_n) = 6 - 2y_n = 2 + \tfrac{2}{n} > 2,$$

hence $A(S(y_n)) = 5$ while $S(A(y_n)) = S(2) = 2$. Therefore

$$D\big(A(S(y_n)), S(A(y_n))\big) = D(5, 2) = |5 - 2| + |5 - 2|^{1/2} = 3 + \sqrt{3} \neq 0.$$

Then it can be verified that A and S are not compatible. But conditionally compatible.

Definition 4.12. Two self-mappings A and S of a perturbed metric space (X, D, P) will be called to be faintly compatible if A and S are conditionally compatible and A and S commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

We now proceed to prove some theorems related to faintly compatible mappings.

Theorem 4.1. [11] Let A and S be non-compatible faintly compatible self-mappings of a metric space (X, d) satisfying:

(4.1) $A(X) \subseteq S(X)$,

(4.2) $d(A(x), A(y)) \le k\, d(S(x), S(y)), \quad 0 \le k < 1.$

If either A or S is continuous, then A and S have a unique common fixed point.

Theorem 4.2. Let A and S be non-compatible faintly compatible self-mappings of a perturbed metric space (X, D, P) satisfying:

(4.3) $A(X) \subseteq S(X)$,

(4.4) $D(A(x), A(y)) \le k\, D(S(x), S(y)), \quad 0 \le k < 1.$

If either A or S is continuous, then A and S have a unique common fixed point.
Proof. Non-compatibility of A and S implies that there exists some sequence $\{x_n\}$ in X such that

$$A(x_n) \to t \quad \text{and} \quad S(x_n) \to t$$

for some $t \in X$, but

$$\lim_{n\to\infty} D(A(S(x_n)), S(A(x_n)))$$

is either nonzero or non-existent.

Since A and S are faintly compatible and $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = t$, there exists a sequence $\{z_n\}$ in X satisfying

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = u$$

such that

$$\lim_{n\to\infty} D(A(S(z_n)), S(A(z_n))) = 0.$$

Further, since A is continuous, then

$$\lim_{n\to\infty} A(A(z_n)) = A(u) \quad \text{and} \quad \lim_{n\to\infty} A(S(z_n)) = A(u).$$

The last three limits together imply

$$\lim_{n\to\infty} S(A(z_n)) = A(u).$$

Since $A(X) \subseteq S(X)$, this implies that $A(u) = S(v)$ for some $v \in X$, and

$$A(A(z_n)) \to S(v), \quad S(A(z_n)) \to S(v).$$

Also, using (4.4), we get

$$D(A(v), A(A(z_n))) \le k\, D(S(v), S(A(z_n))).$$

On letting $n \to \infty$, we get $A(v) = S(v)$. Thus v is a coincidence point of A and S.

Further, faint compatibility implies

$$A(S(v)) = S(A(v)),$$

and hence

$$A(S(v)) = A(A(v)) = S(S(v)).$$

If $A(v) \neq A(A(v))$, then using (4.4) we get

$$D(A(v), A(A(v))) \le k\, D(S(v), S(A(v))) = k\, D(A(v), A(A(v))),$$

a contradiction. Hence $A(v)$ is a common fixed point of A and S.

The same conclusion is obtained if S is assumed to be continuous instead of A. The uniqueness of the common fixed point theorem is an easy consequence of the condition (4.4). Hence the result is proved. □

Example 4.2. Let $D : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + |x - y|^{1/2}, \ \textit{for all} \ x, y \in \mathbb{X}.$$

Then D is a perturbed metric on $\mathbb{X}$ with respect to the perturbed function

$$P : \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

given by

$$P(x, y) = |x - y|^{1/2}, \quad x, y \in \mathbb{X}.$$

Let $X = [0, 20]$ with the perturbed metric

$$D(x, y) = |x - y| + |x - y|^{1/2}, \qquad x, y \in X.$$

Define mappings $A, S : X \to X$ by

$$A(x) = \begin{cases} \dfrac{x}{10}, & \text{if } x < 10, \\ \dfrac{20 - x}{10}, & \text{if } x \ge 10, \end{cases} \qquad S(x) = \begin{cases} 0, & \text{if } x = 0, \\ 20 - x, & \text{if } x > 0. \end{cases}$$

Then A and S satisfy all the conditions of Theorem 4.2 and have a unique common fixed point at $x = 0$. It can be verified in this example that A and S satisfy condition (4.4) with $k = \frac{1}{10}$. Furthermore, A and S are faintly compatible. Also, A and S are noncompatible. To see that, let us consider an increasing sequence $\{x_n\}$ in $X = [2, 20]$ such that $x_n \to 20$. Then

$$A(x_n) \to 0, \qquad S(x_n) \to 0, \qquad A(S(x_n)) \to 0, \qquad S(A(x_n)) \to 20 \text{ as } n \to \infty.$$

Therefore, A and S are noncompatible.

It is well known that the strict contractive condition

$$D(A(x), A(y)) < D(S(x), S(y))$$

does not ensure the existence of common fixed points unless the space taken is compact or some sequence of iterates is assumed to be a Cauchy sequence. The next theorem illustrates the applicability of faintly compatible mappings satisfying the strict contractive condition.

Theorem 4.3. Let A and S be non-compatible faintly compatible self-mappings of a perturbed metric space (X, D, P) satisfying the condition (4.3) of Theorem 4.2 and

(4.5) $D(A(x), A(y)) < D(S(x), S(y))$ whenever $Sx \neq Sy$.

If either A or S is continuous, then A and S have a unique common fixed point.

Proof. Non-compatibility of A and S implies that there exists some sequence $\{x_n\}$ in X such that

$$A(x_n) \to t \quad \text{and} \quad S(x_n) \to t$$

for some $t \in X$, but

$$\lim_{n\to\infty} D(A(S(x_n)), S(A(x_n))) \neq 0$$

or is nonexistent.

Since A and S are faintly compatible, there exists a sequence $\{z_n\}$ in X satisfying

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = v$$

such that

$$\lim_{n\to\infty} D(A(S(z_n)), S(A(z_n))) = 0.$$

Further, since A is continuous, we have

$$\lim_{n\to\infty} A(A(z_n)) = A(v) \quad \text{and} \quad \lim_{n\to\infty} A(S(z_n)) = A(v).$$

The last three limits together imply

$$\lim_{n\to\infty} S(A(z_n)) = A(v).$$

Since $A(X) \subseteq S(X)$, this implies that $A(v) = S(w)$ for some $w \in X$, and

$$A(A(z_n)) \to S(w), \quad S(A(z_n)) \to S(w).$$

Also, using (4.5), we get

$$D(A(w), A(A(z_n))) < D(S(w), S(A(z_n))).$$

On letting $n \to \infty$, we get $A(w) = S(w)$.

Again, in view of faint compatibility of A and S, we get

$$A(S(w)) = S(A(w)),$$

and hence

$$A(S(w)) = S(A(w)) = S(S(w)).$$

We claim that $A(w) = A(A(w))$. If not, then by (4.5) we get

$$D(A(w), A(A(w))) < D(S(w), S(A(w))) = D(A(w), A(A(w))),$$

which is a contradiction, implying that

$$A(w) = A(A(w)) = S(A(w)).$$

Hence $A(w)$ is a common fixed point of A and S.

The same conclusion can be drawn when S is assumed to be continuous instead of A. Thus, the uniqueness of the common fixed point of A and S is obtained. $\square$

Theorem 4.4. Let A and S be non-compatible faintly compatible self-mappings of a perturbed metric space (X, D, P) satisfying the condition (4.3) of Theorem 4.2 and

(4.6) $D(A(x), A(y)) \leq k\, D(S(x), S(y)), \quad k \geq 0;$

(4.7) $D(A(x), A(A(x))) \neq \max\{D(A(x), S(A(x))),\ D(S(A(x)), A(A(x)))\}$ whenever the right-hand side is nonzero.

Suppose either A or S is continuous. Then A and S have a common fixed point.

Proof. Non-compatibility of A and S implies that there exists some sequence $\{x_n\}$ in X such that

$$A(x_n) \to t \quad \text{and} \quad S(x_n) \to t$$

for some $t \in X$, but

$$\lim_{n\to\infty} D(A(S(x_n)), S(A(x_n))) \neq 0$$

or is nonexistent.

Since A and S are faintly compatible and $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = t$, there exists a sequence $\{z_n\}$ in X satisfying

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} S(z_n) = v \text{ (say)}$$

such that

$$\lim_{n\to\infty} D(A(S(z_n)), S(A(z_n))) = 0.$$

Further, since A is continuous, we have

$$\lim_{n\to\infty} A(A(z_n)) = A(v) \quad \text{and} \quad \lim_{n\to\infty} A(S(z_n)) = A(v).$$

The last three limits together imply

$$\lim_{n\to\infty} S(A(z_n)) = A(v).$$

Since $A(X) \subseteq S(X)$, this implies that $A(v) = S(w)$ for some $w \in X$, and

$$A(A(z_n)) \to S(w), \quad S(A(z_n)) \to S(w).$$

Also, using (4.6), we get

$$D(A(w), A(A(z_n))) \leq k\, D(S(w), S(A(z_n))).$$

On letting $n \to \infty$, we get $A(w) = S(w)$. This implies that w is a coincidence point of A and S.

In view of faint compatibility of A and S, we get

$$A(S(w)) = S(A(w)) = A(A(w)) = S(S(w)).$$

We claim that $A(w) = A(A(w))$. If not, by virtue of (4.7) we get

$$D(A(w), A(A(w))) \neq \max\{D(A(w), S(A(w))),\ D(S(A(w)), A(A(w)))\} = D(A(w), A(A(w))),$$

which is a contradiction, implying that

$$A(w) = A(A(w)) = A(S(w)).$$

Hence, $A(w)$ is a common fixed point of A and S.

The same conclusion can be drawn when S is assumed to be continuous instead of A. Thus, the uniqueness of the common fixed point of A and S follows. □

As an application of faint compatible mappings, we now prove a common fixed point theorem under a more general condition that may hold for mappings satisfying contractive, as well as non-expansive and Lipschitz-type conditions.

Theorem 4.5. Let A and S be non-compatible faintly compatible self-mappings of a perturbed metric space (X, D, P) satisfying

(4.8) $D(A(x), A(A(x))) \neq \max\{D(A(x), S(A(x))),\ D(S(A(x)), A(A(x)))\}$ whenever the right-hand side is nonzero.

Suppose A and S are continuous. Then A and S have a common fixed point.

Proof. Non-compatibility of A and S implies that there exists some sequence $\{x_n\}$ in X such that

$$A(x_n) \to t \quad \text{and} \quad S(x_n) \to t$$

for some $t \in X$, but

$$\lim_{n\to\infty} D(A(S(x_n)), S(A(x_n))) \neq 0$$

or is nonexistent.

The continuity of A and S implies that

$$\lim_{n\to\infty} A(S(x_n)) = A(t) \quad \text{and} \quad \lim_{n\to\infty} S(A(x_n)) = S(t).$$

In view of faint compatibility and continuity of A and S, we can easily obtain a common fixed point as has been proved in the corresponding part of Theorem 4.4. □

Remark 4.1. Theorem 4.5 remains true if one replaces the condition (4.8) by any one of the following conditions:

(4.9) $D(S(x), S(S(x))) \neq \max\{D(S(x), A(S(x))), D(A(S(x)), S(S(x)))\}$,

(4.10) $D(A(x), A(A(x))) \neq D(A(x), S(x)) + D(S(x), A(A(x)))$,

(4.11) $D(S(x), S(S(x))) \neq D(S(x), A(x)) + D(A(x), S(S(x)))$ whenever the right-hand side is non-zero.

Remark 4.2. Faint compatibility is a necessary condition for the existence of common fixed points of given mappings A and S satisfying contractive or more general Lipschitz-type mapping pairs. Let A and S be a Lipschitz-type pair of self-mappings of a perturbed metric space (X, D, P) and let A and S have a common fixed point x. Then $A(x) = S(x) = A(S(x)) = S(A(x)) = x$.

If we choose the constant sequence $x_n = x$, then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} A(S(x_n)) = \lim_{n\to\infty} S(A(x_n)) = x,$$

and

$$\lim_{n\to\infty} D(A(S(x_n)), S(A(x_n))) = D(x, x) = 0,$$

that is, A and S are faintly compatible.

5. Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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