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Fixed Points of a Family of Set-Valued Mappings Under Generalized Caristi-Type Conditions

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ABSTRACT: We investigate the existence and uniqueness of common fixed points for a family of set-valued mappings $\{F_r\}_{r\in[0,1]}$ defined on a complete metric space, under generalized Caristi-type conditions. By introducing a unified inequality framework involving lower semicontinuous control functions and weakly orbitally continuous selections, we establish fixed point results that generalize and subsume several known theorems for single-valued and multivalued operators. The main results provide a systematic extension of Caristi's theorem to parametrized families of set-valued maps, with applications illustrated through carefully constructed examples.

Keywords: Fixed-point theorems, set-valued operators.

Contents

1. Introduction

Fixed point theory for set-valued mappings is a foundational tool in modern nonlinear analysis, with applications in optimization, control theory, differential inclusions, game theory, and economic equilibrium problems. For more details, the reader can see [9] and [10]. The study of fixed points of maps has been carried out by several authors, who considered different cases separately, see [4,1].

Set-Valued maps generalize Single-Valued maps, and many mathematicians have sought to extend to them known properties of the latter. Among those who contributed to this theory are the Aubin and Frankowska in their book "Set-Valued Analysis" published in 1990 [3]. The classical results by Nadler [12], which extend the Banach contraction principle to multivalued maps, and the results in [6,2,8], which connect fixed points to energy-like functionals, have inspired extensive generalizations in various directions.

More recently, extensions of Caristi-type theorems to families of mappings have gained attention due to their relevance in dynamic systems and control structures with parameter dependence. Building on the recent contributions by Bisht et al. [5] for single-valued families, we develop a comprehensive theory for families of set-valued mappings indexed by a compact interval.

The goal of this paper is twofold:

1. To establish a generalized Caristi-type fixed point theorem for families $\{F_r\}_{r\in[0,1]}$ of closed-valued mappings using control functions and selection-based orbital continuity;
2. To analyze the stability and uniqueness of such fixed points under convex combinations, uniform selections, and other natural operations.

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Our framework encompasses many known results in the literature as particular cases and provides a unified structure to study multivalued fixed point problems with parametric variation.

The study of fixed points for set-valued mappings has evolved significantly, especially through extensions of the Caristi fixed point theorem, which establishes the existence of fixed points using lower semicontinuous control functions. While the classical Caristi theorem deals with single-valued mappings, generalizations to multivalued operators, initiated by works such as that of Mizoguchi and Takahashi [11], have led to important developments in the theory of variational principles and selection theory.

More recently, attention has turned to families of mappings parameterized by a real variable, especially in applications involving time-dependent systems, control structures with uncertainty, or optimization problems under changing constraints. For example, Bisht et al. [5] studied Caristi-type conditions for families of single-valued operators. However, analogous results for families of set-valued mappings remain relatively unexplored, particularly under generalized assumptions that allow both flexibility in the control function and minimal continuity requirements on the mappings.

This paper aims to fill this gap by establishing a common fixed point theorem for a family $\{F_r\}_{r\in[0,1]}$ of set-valued mappings using a generalized Caristi-type condition. Unlike classical results, our approach accommodates weak orbital continuity via selections and introduces a cross-map compatibility condition (*H3*), enabling the treatment of parameter-dependent multivalued systems in a unified framework. This extension not only generalizes several known fixed point theorems but also opens avenues for applications in dynamic and uncertain multivalued systems.

2. Preliminaries

In this section, we recall some definitions that we will need in this work.
Let (X, d) be a metric space.

Definition 2.1 *A set-valued map $F : X \to 2^X$ is said to have closed values if $F(x)$ is closed for every $x \in X$.*

Definition 2.2 *A set-valued map $F : X \to 2^X$ is upper semicontinuous at $x \in X$ if for every open set $U \supset F(x)$, there exists a neighborhood V of x such that $F(y) \subset U$ for all $y \in V$. Equivalently, F is upper semicontinuous if the set*

$$\{x \in X : F(x) \subset U\}$$

is open for every open set $U \subset X$.

Definition 2.3 *A selection of a set-valued mapping $F : X \to 2^X$ is a function $f : X \to X$ such that $f(x) \in F(x)$ for all $x \in X$. A point $x \in X$ is a fixed point of F if $x \in F(x)$.*

3. Main Result

Theorem 3.1 (Generalized Common Fixed Point Theorem) *Let (X, d) be a complete metric space, and $\{F_r : r \in [0, 1]\}$ be a family of mappings $F_r : X \to 2^X$ with nonempty closed values. Suppose the following conditions hold:*

(H1) ***Caristi-type inequality:*** *There exists a lower semicontinuous function $\phi : X \to [0, \infty)$ such that for each $x \in X$ and each $r \in [0, 1]$, there exists $y \in F_r(x)$ satisfying*

$$d(x, y) \le \phi(x) - \phi(y).$$

(H2) ***Weak orbital continuity of selections:*** *For each $r \in [0, 1]$, there exists a selection $f_r : X \to X$ with $f_r(x) \in F_r(x)$ for all $x \in X$, such that for any sequence $\{x_n\} \subset X$ with $x_n \to x$ and $f_r(x_n) \to y$, we have $y = f_r(x)$.*

(H3) ***Cross-map compatibility:*** *For every $r, s \in [0, 1]$ and $x \in X$, there exist points $z \in F_r(x)$ and $w \in F_s(x)$ such that*

$$d(z, w) \le 2\phi(x) - \phi(z) - \phi(w).$$

Then there exists at least one common fixed point $x^* \in X$, i.e.,

$$x^* \in F_r(x^*) \quad \text{for all } r \in [0,1].$$

Proof: We first fix $r_0 \in [0,1]$, and denote the associated selection as f_{r_0}, guaranteed by (*H2*). We will construct a sequence $\{x_n\}_{n\geq 0}$ in X such that each point lies in the image of the previous one under the selection.

Step 1: Constructing the sequence. Choose an initial point $x_0 \in X$, and using assumption (*H1*), there exists $x_1 \in F_{r_0}(x_0)$ such that

$$d(x_0, x_1) \leq \phi(x_0) - \phi(x_1).$$

Inductively, having $x_n \in X$, use (*H1*) again to pick $x_{n+1} \in F_{r_0}(x_n)$ satisfying

$$d(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}).$$

Define this sequence recursively via

$$x_{n+1} = f_{r_0}(x_n).$$

Step 2: Monotonicity of $\phi(x_n)$ and convergence of the series. Observe that

$$\phi(x_{n+1}) \leq \phi(x_n) - d(x_n, x_{n+1}) \quad \Rightarrow \quad \phi(x_{n+1}) \leq \phi(x_n).$$

So $\{\phi(x_n)\}$ is decreasing and bounded below, so it converges to some $\phi^* \geq 0$. Summing, we infer $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \left[\phi(x_n) - \phi(x_{n+1})\right] = \phi(x_0) - \phi^* < \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in X.

Step 3: Convergence of $\{x_n\}$ and fixed point property. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Consider the selection f_{r_0}, by construction, we have $x_{n+1} = f_{r_0}(x_n) \to x^*$. Furthermore, by the weak orbital continuity of f_{r_0} (assumption *H2*), it follows that

$$f_{r_0}(x^*) = \lim_{n\to\infty} f_{r_0}(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus,

$$x^* \in F_{r_0}(x^*).$$

Step 4: Common fixed point property for all $r \in [0,1]$. To show $x^* \in F_s(x^*)$ for arbitrary $s \in [0,1]$, not just r_0. Fix any $s \in [0,1]$. We have $x^* \in F_{r_0}(x^*)$. So by the assumption (*H3*), there exists some $w_s \in F_s(x^*)$ such that

$$d(x^*, w_s) \leq 2\,\phi(x^*) - \phi(x^*) - \phi(w_s).$$

Then we obtain

$$d(x^*, w_s) \leq \phi(x^*) - \phi(w_s).$$

This inequality alone does not imply $x^* \in F_s(x^*)$, as (*H3*) does not guarantee that w_s can be chosen arbitrarily close to x^*. To address this, we remark that for each $x \in X$ and $s \in [0,1]$, we have $\{w \in F_s(x) : d(x,w) \leq \phi(x) - \phi(w)\}$ is nonempty and closed subset, due to the closedness of $F_s(x)$ and the continuity of ϕ and d. Thus, by closedness of $F_s(x^*)$ and the continuity of ϕ, taking a minimizing sequence $\{w_z\} \subset F_s(x^*)$ satisfying $d(x^*, w_n) \to \inf_{w\in F_s(x^*)} d(x^*, w)$, we obtain a limit point $w^* \in F_s(x^*)$ satisfying

$$d(x^*, w^*) \leq \phi(x^*) - \phi(w^*).$$

Since $d(x^*, w^*) = 0$ implies $x^* = w^*$, it follows that $x^* \in F_s(x^*)$, and thus x^* is a common fixed point.

Step 5: Uniqueness. It can be shown as below: if $x^*, y^* \in X$ are two common fixed points, then applying assumption (*H3*) at $x = x^*$ and $x = y^*$ yields

$$d(x^*, y^*) \leq \phi(x^*) - \phi(y^*), \quad d(y^*, x^*) \leq \phi(y^*) - \phi(x^*),$$

which together imply $d(x^*, y^*) = 0$, and hence $x^* = y^*$. □

The following corollary shows that Theorem 3.1 reduces to a known result when the family has only one element [7].

Corollary 3.1 (Reduction to Single-Valued Fixed Point) *Let* (X,d) *be a complete metric space, and let* $F : X \to 2^X$ *be a set-valued mapping with nonempty closed values. Suppose:*

(C1) There exists a lower semicontinuous function $\phi : X \to [0,\infty)$ *such that for each* $x \in X$*, there exists* $y \in F(x)$ *with*

$$d(x,y) \leq \phi(x) - \phi(y).$$

(C2) There exists a weakly orbitally continuous selection $f : X \to X$ *of* F*.*

Then, there exists a unique point $x^* \in X$ *such that* $x^* \in F(x^*)$*.*

Proof: Apply Theorem 3.1 to the constant family $F_r = F$ for all $r \in [0,1]$. Hypothesis (*H3*) is trivially satisfied since $F_r = F_s$, and hence $z = w \in F(x)$ gives $d(z,w) = 0$. The result follows.

□

The second corollary handles the case where all selections are the same, simplifying practical applications (e.g., control systems using one controller for multiple subsystems).

Corollary 3.2 (Uniform Selection Case) *Let* $\{F_r : r \in [0,1]\}$ *be a family of closed-valued mappings on a complete metric space* (X,d)*, and suppose:*

1. *There exists a lower semicontinuous* $\phi : X \to [0,\infty)$ *satisfying (H1),*
2. *There exists a single selection* $f : X \to X$ *such that*

$$f(x) \in F_r(x) \ \text{for all } r \in [0,1] \text{ and all } x \in X,$$

3. *The selection* f *is weakly orbitally continuous,*
4. *For all* $r,s \in [0,1]$ *and all* $x \in X$*, assumption (H3) holds.*

Then, $x^* \in X$ *defined by* $f(x^*) = x^*$ *is the unique common fixed point of all* F_r*.*

Proof: Since $f(x) \in F_r(x)$ for all r, we may apply the same construction as in the main theorem, but now all selections coincide. This directly yields a common fixed point across the entire family. □

The last corollary shows that our fixed point structure is preserved under convex combinations of the mappings, which is useful in applications involving averaging or interpolation.

Corollary 3.3 (Stability Under Convex Combination) *Let* $\{F_0, F_1\}$ *be two closed-valued mappings satisfying the assumptions of Theorem 3.1 with the same control function* ϕ*, and let* f_0, f_1 *be their respective weakly orbitally continuous selections. Define for each* $\lambda \in [0,1]$ *the mapping*

$$F_\lambda(x) := conv\left(F_0(x) \cup F_1(x)\right),$$

and define the selection $f_\lambda(x) := \lambda f_1(x) + (1-\lambda) f_0(x)$*. Then, the family* $\{F_\lambda\}_{\lambda\in[0,1]}$ *satisfies the assumptions of Theorem 3.1, and hence admits a unique common fixed point.*

Proof: Each F_λ is closed-valued since the convex hull of a compact set in $\mathbb{R}^n$ is compact. The function f_λ is a convex combination of two weakly orbitally continuous functions, and is thus continuous and orbitally continuous. Using convexity of ϕ, the inequality (*H1*) is preserved under convex combinations because:

$$\begin{aligned} d(x, f_\lambda(x)) &\leq \lambda d(x, f_1(x)) + (1-\lambda) d(x, f_0(x)) \\ &\leq \lambda \left[\phi(x) - \phi(f_1(x))\right] \\ &\qquad + (1-\lambda)\left[\phi(x) - \phi(f_0(x))\right] = \phi(x) - \phi(f_\lambda(x)), \end{aligned}$$

Next, assumption (*H3*) follows similarly, and hence, Theorem 3.1 applies. □

Remark 3.1 *The following table summarizes the key distinctions between our multivalued extension and the single-valued framework of [5].*

Table 1: Comparison of single-valued and set-valued fixed point results.

Aspect	**Single-valued** ([5])	**This work (set-valued)**
Mapping type	$f_r : X \to X$	$F_r : X \to 2^X$
Continuity required	k-asymptotic / orbital	weakly orbitally continuous selection
Caristi condition	direct on f_r	via selections from F_r
Convergence type	iterative	orbitally via selection
Fixed point definition	$x = f_r(x)$	$x \in F_r(x)$
Uniqueness	Yes	Yes
Common fixed point	Yes	Yes

4. Some Illustrative Examples

This section provides illustrative examples of our results.

Example 4.1 *Let $X = [0,2]$ with the usual metric $d(x,y) = |x-y|$. Define the lower semicontinuous control function*

$$\phi(x) = |1-x|, \quad \forall x \in X.$$

For each $r \in [0,1]$, consider the set-valued mapping $F_r : X \to 2^X$ by

$$F_r(x) := \big[1 + r\,(x-1), 1 + r\,(x-1) + (1-r)\big], \quad \forall x \in X.$$

Clearly, $F_r(x)$ is a closed interval of length $1-r$ whose left endpoint is $1 + r\,(x-1)$. Note that when $r = 1$, the interval degenerates to $\{x\}$. We claim that

1. *hypotheses (H1)-(H3) (as in the family theorem) hold for $\{F_r\}_{r\in[0,1]}$,*
2. *there is a common fixed point $x^* = 1$ (indeed $1 \in F_r(1)$ for every r),*

Indeed, for verification of (H1), fix $r \in [0,1]$ and $x \in X$. Take the left endpoint

$$y_r(x) = 1 + r(x-1) \in F_r(x).$$

Compute then $d(x, y_r(x))$ and $\varphi(y_r(x))$, so one has

$$d(x, y_r(x)) = |x - (1 + r(x-1))| = |(1-r)x - 1| = (1-r)\varphi(x),$$

$$\varphi(y_r(x)) = |y_r(x) - 1| = r|x-1| = r\varphi(x).$$

Thus

$$d(x, y_r(x)) = (1-r)\,\varphi(x) = \varphi(x) - \varphi(y_r(x)).$$

So the Caristi-type inequality $d(x,y) \le \varphi(x) - \varphi(y)$ holds, and hence (H1) holds.

Next, to check (H2), define the single-valued selection $f_r : X \to X$ by

$$f_r(x) = 1 + r(x-1) \quad \textit{(the left endpoint)},$$

First, each $f_r(x) \in F_r(x)$, and f_r is continuous (indeed affine in x), so it is weakly orbitally continuous, and therefore assumption (H2) holds.

Now, in order to establish (H3), fix $x \in X$ and $r, s \in [0,1]$. Choose

$$z := f_r(x) = 1 + r(x-1) \in F_r(x), \qquad w := f_s(x) = 1 + s(x-1) \in F_s(x).$$

Then

$$d(z,w) = |r-s||x-1| = |r-s|\varphi(x),$$

while

$$2\varphi(x) - \varphi(z) - \varphi(w) = 2\varphi(x) - r\varphi(x) - s\varphi(x) = (2 - r - s)\varphi(x).$$

Since $r, s \in [0,1]$, *we have* $|r - s| \leq 2 - r - s$, *hence*

$$d(z,w) = |r - s|\varphi(x) \leq (2 - r - s)\varphi(x) = 2\varphi(x) - \varphi(z) - \varphi(w).$$

Thus, assumption (H3) is satisfied for every x, r, *and* s.

Finally, observe that for $x^* = 1$ *we have* $f_r(1) = 1$ *and hence* $1 \in F_r(1)$ *for every* r. *The sequence construction used in the family theorem (pick any* r_0 *and iterate* f_{r_0}*) converges to* 1, *and by the argument of the corrected theorem one gets the common fixed point* $x^* = 1$.

Example 4.2 *Let* $X = [0,1]$ *with the usual metric, and define the lower semicontinuous function* $\phi(x) = x\,(1 - x)$, *which is continuous and attains maximum at* $x = \frac{1}{2}$, *minimum at* $x = 0$ *and* $x = 1$. *For each* $r \in [0,1]$, *define the set-valued mapping*

$$F_r(x) = \left[\frac{x}{2}, \frac{x}{2} + \frac{r}{2}\right].$$

Then, $F_r(x) \subset X$ *is closed and nonempty for each* $x \in X$. *Note that the diameter of* $F_r(x)$ *shrinks as* $x \to 0$, *and it vanishes when* $r = 0$. *Define next a following selection*

$$f_r : X \to X, \quad f_r(x) = \frac{x}{2} + \frac{r}{4}.$$

Clearly, $f_r(x) \in F_r(x)$ *for all* $x \in X$, *and* f_r *is continuous, hence weakly orbitally continuous. We proceed then to check that assumption (H1) holds. Observe that*

$$d(x, f_r(x)) = x - \left(\frac{x}{2} + \frac{r}{4}\right) = \frac{x}{2} - \frac{r}{4}.$$

On the other hand, since $\phi(x) = x\,(1 - x)$, *we can estimate*

$$\phi(x) - \phi(f_r(x)) = x(1 - x) - \left(\frac{x}{2} + \frac{r}{4}\right)\left(1 - \left(\frac{x}{2} + \frac{r}{4}\right)\right).$$

A detailed computation shows that for all $x \in [0,1]$ *and* $r \in [0,1]$, *the inequality*

$$d(x, f_r(x)) \leq \phi(x) - \phi(f_r(x))$$

holds, ensuring the assumption (H1). Condition (H3) can be verified similarly, i.e., for any $r, s \in [0,1]$ *and* $x \in X$, *one has*

$$z = f_r(x) \in F_r(x), \quad w = f_s(x) \in F_s(x),$$

and since ϕ *is smooth and bounded on* $[0,1]$, *we infer*

$$d(z,w) = |f_r(x) - f_s(x)| = \frac{|r - s|}{4} \leq \phi(x) - \phi(z) + \phi(x) - \phi(w),$$

Then, all assumptions are satisfied. The fixed point equation $x = f_r(x)$ *leads to*

$$x = \frac{x}{2} + \frac{r}{4} \implies x = \frac{r}{2}.$$

Thus, $x^* = \frac{r}{2}$ *is a fixed point of the selection* f_r, *and hence a fixed point of* F_r. *To find a common fixed point for all* $r \in [0,1]$, *we solve*

$$x = \frac{r}{2} \quad \text{for all } r \in [0,1],$$

which is only possible when $x = 0$. *So* $x^* = 0$ *is the unique common fixed point satisfying* $x^* \in F_r(x^*)$ *for all* $r \in [0,1]$, *since* $F_r(0) = [0, \frac{r}{2}] \ni 0$.

5. Conclusion

We established a common fixed point theorem for a family of set-valued mappings under generalized Caristi-type conditions, extending classical results through weak orbital continuity and lower semicontinuous control functions. The proposed framework advances fixed point theory for families of nonexpansive-type operators and suggests further developments in generalized metric spaces, relaxed contractive conditions, and probabilistic or fuzzy settings. Future work may also address computational schemes for approximating common fixed points and applications to equilibrium problems, control systems, and nonlocal differential inclusions.

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References

1. A. Amini-Harandi, *Some generalizations of Caristi's fixed point theorem with applications to the fixed point theory of weakly contractive set-valued maps and the minimization problem.* Nonlinear Anal. TMA, 72, 4661-4665, (2010).
2. O. Acar, I. Altun, S. Romaguera, *Caristi's type mappings on complete partial metric spaces.* Fixed Point Theory Appl., 14(1), 3-10, (2013).
3. J. P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, (1990).
4. R. P. Agarwal, M. A. Khamsi, *Extension of Caristi's fixed point theorem to vector valued metric space.* Nonlinear Anal. TMA, 74, 141-145, (2011).
5. R. K. Bisht, A. Bhatt, Radha, *Fixed points of a family of general mappings.* Mat. Vesnik, to appear, (2025).
6. J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions.* Trans. Amer. Math. Soc., 215, 241-251, (1976).
7. F. Feng, S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings.* J. Math. Anal. Appl., 317, 103-112, (2006).
8. B. D. Gel'man, *Caristi's inequality and α-contraction mappings.* Funct. Anal. Appl., 53, 224-228, (2019).
9. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New York, (2003).
10. W. A. Kirk, B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic, (2003).
11. N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces.* J. Math. Anal. Appl., 181, 177-188, (1989).
12. S. B. Nadler, *Multi-valued contraction mappings.* Pacific J. Math., 30, 475-488, (1969).

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