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Employment Neoteric Double Integral Transformation to Solve Applied Equations

Huda Ali Falih and Athraa Neamah Albukhuttar

ABSTRACT: In this work, a novel double transform in one dimension is presented. The double transform is derived from the ZZ and Shehu transforms, the existence, uniqueness, and some special properties are established. Moreover, it is supported by solving some applied partial differential equations such as Laplace equation, Poisson and telegraph.

Keywords: Single integral transform, Double ZZ-Shehu transform, initial conditions.

Contents

1. Introduction

Initial value problems are one of the most important mathematical tools of expressing models for complex occurrences in medicine, engineering and physics. Therefore, the solutions of these problems are a field in which researchers compete due to their effective in this aspect that can be obtained using various integral transforms methods. Many integral transforms are presented in literatures as Laplace, ZZ, Ezaki, Shehu transforms...etc [6,3,12,10], that play a vital role in solving many types of ordinary equations, partial, delay so on with its applications. The solutions to initial and boundary value problems can be obtained using various integral transforms [1,2,7,9], which are important applied tools in other sciences.

Recently, double integral transforms have attracted significant attention from researchers due to their extensive applications in various Scientifics and the smoothness of its solution mechanism [5,8,4,11]. In this study, we offer a combination of two single transformations called double ZZ-Shehu transformation. Moreover, the most important fundamental properties and transformation formulas for the functions are discussed. Several related theorems are proved and used to solve applied partial differential equations that support the method.

2. Preliminaries

Some definitions and theorems that useful in this work are introduced:

Definition 2.1 *Let be $\mathcal{A}(n)$ a function defined for all $n \geq 0$, then ZZ transform of $\mathcal{A}(n)$ is defined by [12]:*

$$Z(\mathcal{H},\mathcal{B}) = Z\{\mathcal{A}(n)\} = \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty \mathcal{A}(n)e^{-\frac{\mathcal{H}}{\mathcal{B}}n}dn$$

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Definition 2.2 *The Shehu transformation [10] is defined by:*

$$S(\mathcal{I},\mu) = S\mathcal{A}(m) = \int_0^\infty \mathcal{A}(m)\, e^{-\frac{\mathcal{I}}{\mu}m} dm$$

Definition 2.3 *The double ZZ-Shehu* (ZS) *of function* $\mathcal{A}(n,m)$ *is denoted by* $\mathfrak{t}_2$

$$\mathfrak{t}_2\mathcal{A}(n\ ,m) = ZS\left[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)\right] = \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}\mathcal{A}(n,m)dndm$$

the inverse double ZZ- Shehu transform of $\mathfrak{t}_2$ *is defined as:*

$$\begin{aligned}\mathfrak{t}_2^{-1}\{\mathfrak{t}_2\mathcal{A}(n\ ,m)\} &= ZS^{-1}\ \left(ZS\left[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)\right]\right)\\ &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\mathcal{H}}{\mathcal{B}}\ ZS\left[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)\right]e^{-\frac{\mathcal{H}}{\mathcal{B}}n}d\mathcal{B}\cdot\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{\mu}\\ &\qquad S\left[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)\right]e^{-\frac{\mathcal{I}}{\mu}m}\ d\mathcal{I}\end{aligned}$$

where ZS^{-1} *is the inverse of transform operator.*

2.1. Essential the Double ZZ-Shehu Transformation

The following two theorems are showed the existence and uniqueness of $\mathfrak{t}_2$

Theorem 2.1 *For a continuous with exponential order mapping* $a,b\in R$.

$$\sup_{n,m>0}\frac{|\mathcal{A}(n,m)|}{e^{(an+bm)}} < \infty, a,b\in R.$$

Then $\mathfrak{t}_2$ *of* $\mathcal{A}(n,m)$ *exists.*

Proof: From Definition 2.3

$$\begin{aligned}|\mathfrak{t}_2\mathcal{A}(n,m)| &= |\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}\mathcal{A}(n,m)\,|dndm|\\ &\le \frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}|\mathcal{A}(n,m)\,|dndm\\ &\le k\frac{\theta}{\omega}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}e^{(an+bm)}dndm\\ &= k\frac{\theta}{\omega}\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}-a)n}dn\int_0^\infty e^{-(\frac{\mathcal{I}}{\mu}-b)m}dm\\ &= \frac{k\ \mathcal{H}\mu}{(\mathcal{H}-a\mathcal{B})(\mathcal{I}-b\mu)}\end{aligned}$$

□

Theorem 2.2 *Let* $\mathfrak{t}_2(k_1(n,m))$ *and* $\mathfrak{t}_2(k_2(n,m))$ *be double ZZ-Shehu transformation for* $k_1(n,m)$ *and* $k_2(n,m)$ *respectively. If* $\mathfrak{t}_2(k_1(n,m)) = \mathfrak{t}_2(k_2(n,m))$ *then*

$$k_1(n,m) = k_2(n,m)$$

Proof: Assume σ and τ to be sufficiently large then since

$$f(n,m) = \mathfrak{t}_2^{-1}\left[\mathfrak{t}_2\left(f\left(n,m\right)\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\mathcal{H}}{\mathcal{B}}\ (e^{-\frac{\mathcal{H}}{\mathcal{B}}n})\right)d\mathcal{B}\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{\mu}(e^{-}\frac{\mathcal{I}}{\mu}m)\mathfrak{t}_2\left(f\left(n,m\right)\right]d\mathcal{I}$$

□

we deduce

$$\begin{aligned} k_1(n,m) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\mathcal{H}}{\mathcal{B}}(e^{-\frac{\mathcal{H}}{\mathcal{B}}n})d\mathcal{B}\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{\mu}(e^{-\frac{\mathcal{I}}{\mu}m})\mathfrak{t}_2(k_1(n,m))d\mathcal{I} \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\mathcal{H}}{\mathcal{B}}(e^{-\frac{\mathcal{H}}{\mathcal{B}}n})d\mathcal{B}\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{\mu}(e^{-\frac{\mathcal{I}}{\mu}m})\mathfrak{t}_2(k_2(n,m))d\mathcal{I} \\ &= k_2(n,m) \end{aligned}$$

3. Double ZZ Transform for Some Functions

1. If $\mathcal{A}(n,m)=1$, then

$$\begin{aligned} \mathfrak{t}_2\{(\mathcal{A}(n,m)\} = ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}\,dndm \\ &= \int_0^\infty e^{-\frac{\mathcal{I}}{\mu}m}dm = \frac{\mu}{\mathcal{I}} \end{aligned}$$

2. If $\mathcal{A}(n,m)=e^{(an+bm)}$, then

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} = ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{(an+bm)}\,e^{-(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m)}\,dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-(\frac{\mathcal{H}}{\mathcal{B}}-a)n}e^{-(\frac{\mathcal{I}}{\mu}-b)m}\,dndm = \frac{\mathcal{H}\mu}{(\mathcal{H}-a\mathcal{B})(\mathcal{I}-b\mu)} \end{aligned}$$

3. If $\mathcal{A}(n,m)=n^\epsilon m^\rho \epsilon,\rho=0,1,2,\ldots$, then

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} = ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty n^\epsilon m^\rho e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty e^{-\frac{\mathcal{H}}{\mathcal{B}}n}\,n^\epsilon dn\int_0^\infty e^{-\frac{\mathcal{I}}{\mu}m}m^\rho dm = \epsilon!\rho!\left(\frac{\mathcal{H}}{\mathcal{B}}\right)^\epsilon\left(\frac{\mathcal{I}}{\mu}\right)^{\rho+1} \end{aligned}$$

4. If $\mathcal{A}(n,m)$= $n^\epsilon m^\rho \epsilon,\rho\geq -1$ then

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} = ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty n^\epsilon m^\rho e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty n^\epsilon\,m^\rho e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \end{aligned}$$

By letting $d=\frac{\mathcal{H}}{\mathcal{B}}n, andh=\frac{\mathcal{I}}{\mu}m \rightarrow = \Gamma(\in +1)\left(\frac{\mathcal{H}}{\mathcal{B}}\right)^\in \Gamma(\rho+1)\left(\frac{\mathcal{I}}{\mu}\right)^{\rho+1}$

5. If $\mathcal{A}(n,m)=\cos(an+bm)$, then

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} &= ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] \\ &= e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\int_0^\infty\int_0^\infty \cos(an+bm)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\left(\frac{e^{i(an+bm)}+e^{-i((an+bm))}}{2}\right)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \\ &= \frac{1}{2}(\frac{\mathcal{H}}{\mathcal{B}}\int\int_0^\infty e^{i(an+bm)}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm+\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-i((an+bm))}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm) \\ &= \frac{\mathcal{H}\mu(\mathcal{H}\mathcal{I}+a\mu\mathcal{I}b)}{(\mathcal{H}^2+a^2\mathcal{B}^2)\;(\mathcal{I}^2+b^2\mu^2)} \end{aligned}$$

6. If $\mathcal{A}(n,m) = \sin(an+bm)$

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} &= ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty sin(an+bm)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \left(\frac{e^{i(an+bm)}-e^{-i((an+b\mathfrak{y}))}}{2i}\right)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm \\ &= \frac{1}{2i}\Big(\frac{\mathcal{H}}{\mathcal{B}}\int\int_0^\infty e^{i(an+bm)}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm-\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-i((an+bm))e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}}dndm \\ &= \frac{\mathcal{H}\mu(\mathcal{H}\mu b+\mathcal{B}\mathcal{I}a)}{(\mathcal{H}^2+a^2\mathcal{B}^2)\ (\mathcal{I}^2+\mu^2b^2)} \end{aligned}$$

7. If $\mathcal{A}(n,m) = \cos h\,(an+bm)$, then

$$\begin{aligned} \mathfrak{t}_2\{\mathcal{A}(n,m)\} &= ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \cosh(an+bm)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \left(\frac{e^{(an+bm)}+e^{-((an+bm))}}{2}\right)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm \\ &= \frac{1}{2}\Big(\frac{\mathcal{H}}{\mathcal{B}}\int\int_0^\infty e^{(an+bm)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}}dndm+\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-((an+bm))e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}}dndm \\ &= \frac{\mathcal{H}\mu(\mathcal{I}+\alpha\mu\mathcal{B}b)}{(\mathcal{H}^2-\alpha^2\mathcal{B}^2)(\mathcal{I}^2-b^2\mu^2)} \end{aligned}$$

8. $\mathcal{A}(n,m) = sinh\,(an+bm)$, then

$$\begin{aligned} \mathfrak{t}_2\{\ \mathcal{A}(n,m)\} &= ZS[(\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)] \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \sinh(an+bm)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm \\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \left(\frac{e^{(an+bm)}-e^{-((an+bm))}}{2}\right)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm \\ &= \frac{1}{2}\Big(\frac{\mathcal{H}}{\mathcal{B}}\int\int_0^\infty e^{(an+bm)}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm-\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-((an+bm))}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm\Big) \\ &= \frac{\mathcal{H}\mu(\mu\mathcal{H}b+a\mathcal{B}\mathcal{I})}{(\mathcal{H}^2-\alpha^2\mathcal{B}^2)(\mathcal{I}^2-b^2\mu^2)} \end{aligned}$$

Table 1: Double ZZ-Shehu transformation for some special functions

No.	$\mathcal{A}(n,m)$	$\mathfrak{t}_2\{\mathcal{A}(n,m)\}$
1	1	$\frac{\mu}{\mathcal{I}}$
2	$e^{(an+bm)}$	$\frac{\mathcal{H}\mu}{(\mathcal{H}-a\mathcal{B})(\mathcal{I}-b\mu)}$
3	$n^\epsilon m^\rho$	$\epsilon!\rho!\left(\frac{\mathcal{H}}{\mathcal{B}}\right)^\epsilon\left(\frac{\mathcal{I}}{\mu}\right)^{\rho+1}$
4	$n^\epsilon m^\rho \epsilon,\rho\geq -1$	$\Gamma(\in+1)\left(\frac{\mathcal{H}}{\mathcal{B}}\right)^\in\Gamma(\rho+1)\left(\frac{\mathcal{I}}{\mu}\right)^{\rho+1}$
5	$\cos(an+bm)$	$\frac{\mathcal{H}\mu(\mathcal{H}\mathcal{I}+a\mu\mathcal{I}b)}{(\mathcal{H}^2+a^2\mathcal{B}^2)(\mathcal{I}^2+b^2\mu^2)}$
6	$\sin(an+bm)$	$\frac{\mathcal{H}\mu(\mathcal{H}\mu b+\mathcal{B}\mathcal{I}a)}{(\mathcal{H}^2+a^2\mathcal{B}^2)(\mathcal{I}^2+\mu^2b^2)}$
7	$\cos h\,(an+bm)$	$\frac{\mathcal{H}\mu(\mathcal{I}+\alpha\mu\mathcal{B}b)}{(\mathcal{H}^2-\alpha^2\mathcal{B}^2)(\mathcal{I}^2-b^2\mu^2)}$
8	$\sinh(an+bm)$	$\frac{\mathcal{H}\mu(\mu\mathcal{H}b+a\mathcal{B}\mathcal{I})}{(\mathcal{H}^2-\alpha^2\mathcal{B}^2)(\mathcal{I}^2-b^2\mu^2)}$

4. Properties of $\mathfrak{t}_2$ Transformation

4.1. Linear properties of the double ZZ - Shehu transformation

$$\mathfrak{t}_2\left\{\alpha_1\mathcal{A}_1\left(n,m\right)+\alpha_2\mathcal{A}_2(n,m)+\ldots+\alpha_k\mathcal{A}_k(n,m)\right\}=\alpha_1\mathfrak{t}_2\{\mathcal{A}_1\left(n,m\right)\}+\ldots+\alpha_k\mathfrak{t}_2\{\mathcal{A}_k(n,m)\}$$

From definition

$$=\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\{\alpha_1\mathcal{A}_1(n,m)\}+\ldots+\{\alpha_k\mathcal{A}_k(n,m)\}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}$$
$$=\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\alpha_1\mathcal{A}_1(n,m)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm+\ldots+\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\alpha_k\mathcal{A}_k(n,m)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dn\,dm$$
$$=\alpha_1\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\mathcal{A}_1(n,m)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm+\alpha_k\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\mathcal{A}_k(n,m)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dn\,dm$$
$$=\alpha_1\mathfrak{t}_2\left(\mathcal{A}_1(n,m)\right)+\ldots+\alpha_k\mathfrak{t}_2\mathcal{A}_k((n,m))$$

4.2. Changing of scale property

If $\mathfrak{t}_2\left(\mathcal{A}(n,m)\right)=ZS\left[\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)\right]$ then

$$\mathfrak{t}_2\left(\mathcal{A}\left(an,bm\right)\right)=\frac{1}{ab}\frac{\mathcal{H}}{\mathcal{B}}ZS\left(\left(\frac{\mathcal{H}}{a},\frac{\mathcal{I}}{b}\right),\left(\mathcal{B},\mu\right)\right)$$

From definition

$$\mathfrak{t}_2\left(\mathcal{A}\left(an,bm\right)\right)=\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\mathcal{A}\left(an,bm\right)e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dn\,dm$$

Let $an=y$, $bm=z$ we get

$$=\frac{1}{ab}\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\left(\mathcal{A}\left(y,z\right)\right)e^{-\left(\frac{\mathcal{H}y}{\mathcal{B}a}+\frac{\mathcal{I}z}{\mu b}\right)}dy\,dz$$
$$=\frac{1}{ab}\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\left(\mathcal{A}\left(y,z\right)\right)e^{-\left(\frac{\frac{\mathcal{H}}{a}}{\mathcal{B}}y+\frac{\frac{\mathcal{I}}{b}}{\mu}\,z\right)}\,dy\,dz$$
$$=\frac{1}{ab}\frac{\mathcal{H}}{\mathcal{B}}H\left(\left(\frac{\mathcal{H}}{a},\frac{\mathcal{B}}{b}\right),\left(\mathcal{B}.\mu\right)\right)$$

Where H is the Heaviside function.

4.3. Shifting Property

If $\mathfrak{t}_2\left(\mathcal{A}(n,m)\right)=ZS\left[\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)\right]$ then

$$\mathfrak{t}_2\left[\left(\mathcal{A}(n,m)e^{(an+bm)}\right)\right]=\frac{\mathcal{H}}{\mathcal{B}}ZS[(\mathcal{H}-a\mathcal{B})(\mathcal{I}-b\mu)(\mathcal{B},\mu)]$$

From definition

$$\begin{aligned}\mathfrak{t}_2(\mathcal{A}(n,m))e^{(an+bm)}&=\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\mathcal{A}\left(n,m\right)e^{(an+bm)}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm\\&=\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty\mathcal{A}(n,m)e^{-\left(\left(\frac{\mathcal{H}-a\mathcal{B}}{\mathcal{B}}\right)n+\left(\frac{\mathcal{I}-b\mu}{\mu}\right)m\right)}\,dndm\\&=\frac{\theta}{\omega}ZS\left(\left(\mathcal{H}-\mathcal{B}a,\mathcal{I}-b\mu\right),\left(\mathcal{B},\mu\right)\right)\end{aligned}$$

Theorem 4.1 *Let* $\mathfrak{t}_2(\mathcal{A}\left(n,m\right))=ZS\left[\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)\right]$, *then*

$$\mathfrak{t}_2[\mathcal{A}\left(n,m\right)]=(-1)^m\omega^m\frac{d^m}{d\mathcal{H}^m}\mathfrak{t}_2[\mathcal{A}\left(n,m\right)]$$

Proof: If $m = 1$, then

$$\begin{aligned}\frac{d}{d\mathcal{H}}\mathfrak{t}_2(\mathcal{A}(n,m)) &= \frac{d}{d\mathcal{H}}\left(\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \mathcal{A}(n,m)\, e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm\right)\\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \frac{\partial}{\partial\mathcal{H}}e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\mathcal{A}(n,m)\ dndm\\ &= (-1)\,\frac{\mathcal{H}}{\mathcal{B}^2}\int_0^\infty\int_0^\infty (n\mathcal{A}(n,m)\, e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\ dndm\\ &= \frac{-1}{\mathcal{B}}\mathfrak{t}_2\left\{n\mathcal{A}(n,m)\right\} = \mathfrak{t}_2\left\{n\mathcal{A}(n,m)\right\} = -\mathcal{B}\frac{d}{d\mathcal{H}}\ \mathfrak{t}_2\left\{\mathcal{A}(n,m)\right\}\end{aligned}$$

If $m = 2$, then

$$\begin{aligned}&= \frac{d^2}{d\mathcal{H}^2}\left(\frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \mathcal{A}(n,m)\, e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}dndm\right)\\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial\mathcal{H}^2}\mathcal{A}(n,m)\, e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\ dndm\\ &= \frac{\mathcal{H}}{\mathcal{B}^2}\int_0^\infty\int_0^\infty \left(n^2\mathcal{A}(n,m)\right)\ e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\ dndm\\ &= \frac{1}{\mathcal{B}^2}\ \mathfrak{t}_2\left\{n^2\ \mathcal{A}(n,m)\right\} = \mathfrak{t}_2\left\{n^2\mathcal{A}(n,m)\right\} = \mathcal{B}^2\frac{d^2}{d\mathcal{H}^2}\ \mathfrak{t}_2\left\{\mathcal{A}(n,m)\right\},\end{aligned}$$

then,

$$\mathfrak{t}_2\left\{m^b\mathcal{A}(n,m)\right\} =\ (-1)^b\mathcal{B}^b\frac{d^n}{d\mathcal{H}^n}\mathfrak{t}_2\mathcal{A}(n,m)$$

□

Theorem 4.2 *If $f(n,m)$ and $h(n,m)$ have $\mathfrak{t}_2$ transformation, then*

$$\mathfrak{t}_2(f**h)\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)) = \frac{\mathcal{H}}{\mathcal{B}}\mathfrak{t}_2\left\{f\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)\right\}\mathfrak{t}_2h\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)$$

Proof: By definition double ZZ-Shehu transform, we have

$$b = \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\left(\int_0^{\mathfrak{y}}\int_0^{\mathfrak{x}} f(a,b)\, h(n-a)(m-b)dadb)\right) dndm$$

$$\mathfrak{t}_2\{(f**h)((\mathcal{H},\mathcal{I}),(\mathcal{B},\mu)\} = \frac{\mathcal{H}}{\mathcal{B}}\ \mathfrak{t}_2\{\ f\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)\}\ \mathfrak{t}\ _2h\left(\mathcal{H},\mathcal{I}\right),\left(\mathcal{B},\mu\right)$$

□

Theorem 4.3 *Double ZZ-Shuhe transformation for partial derivatives is*

1. $\mathfrak{t}_2\ \left\{\frac{\partial\mathcal{A}(n,m)}{\partial n}\right\} = \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - (HB)S(\mathcal{A}(0,m))$
2. $\mathfrak{t}_2\left\{\frac{\partial^2\mathcal{A}(n,m)}{\partial^2 n}\right\} = \left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - \left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)S\left(\mathcal{A}(0,m)\right) - \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\left(\frac{\partial}{\partial n}Z\mathcal{A}(0,m)\right)$
3. $\mathfrak{t}_2\left\{\frac{\partial\mathcal{A}(n,m)}{\partial m}\right\} = \left(\frac{\mathcal{I}}{\mu}\right)t2(An,m) - Z(\mathcal{A}(n\ ,0))$
4. $\mathfrak{t}_2\ \left\{\frac{\partial^2\mathcal{A}(n,m)}{\partial m^2}\right\} = \left(\frac{\mathcal{I}^2}{\mu^2}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - \left(\frac{\mathcal{I}}{\mu}\right)Z\left(\mathcal{A}(n,0)\right) - Z\left(\frac{\partial}{\partial m}\mathcal{A}(n,0)\right)$

Proof: The proof is obtained directly by definition of $\mathfrak{t}_2$

$$\begin{aligned}\mathfrak{t}_2\left\{\frac{\partial\mathcal{A}(n,m)}{\partial n}\right\} &= \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - (\frac{\mathcal{H}}{\mathcal{B}})S(\mathcal{A}(0,m)\\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\int_0^\infty \frac{d}{dn}\mathcal{A}(n,m)\,e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dndm\\ &= \frac{\mathcal{H}}{\mathcal{B}}\int_0^\infty\left[\int_0^\infty \frac{d}{dn}\mathcal{A}(n,m)\,e^{-\left(\frac{\mathcal{H}}{\mathcal{B}}n+\frac{\mathcal{I}}{\mu}m\right)}\,dn\right]dm\\ &= \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - (\frac{\theta}{\omega})S(\mathcal{A}(0,m))\end{aligned}$$

As with the proof of the first point, proofs for the remaining points can be deduced based on the definition. □

5. Application

In this section, some supported examples are solved.

Example 5.1 To solve Laplace equation
$\mathcal{A}_{nn}+\mathcal{A}_{mm}=0$, with
$\mathcal{A}(n\ .0)=0 \qquad \mathcal{A}_m=(n\ .0)=\cos n$
$\mathcal{A}(0,m)=\sinh m \qquad \mathcal{A}_{\ n}=((0,m)=0$

Solution: Applying the double ZZ-Shehu transformation

$$\begin{aligned}&\left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - \left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)S(\mathcal{A}(0,m)) - \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\left(\frac{\partial}{\partial n}Z\mathcal{A}(0,m)\right)\\ &\quad+\left(\frac{\mathcal{I}^2}{\mu^2}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) - \left(\frac{\mathcal{I}}{\mu}\right)Z(\mathcal{A}(n,0)) - Z\left(\frac{\partial}{\partial m}\mathcal{A}(n\ ,0)\right)\\ &\quad\left(\frac{\mathcal{H}^2\mu^2+\mathcal{I}^2\mathcal{B}^2}{\mathcal{B}^2\mu^2}\right)\mathfrak{t}_2(\mathcal{A}(n,m)) = \frac{\mathcal{H}^4\mu^2+\mathcal{H}^2\mu^2\mathcal{B}^2+\mathcal{H}^2\mathcal{B}^2\mathcal{I}^2-\mathcal{H}^2\mathcal{B}^2\mu^2}{\mathcal{B}^2(\mathcal{I}^2-\mu^2)(\mathcal{H}^2+\mathcal{B}^2)}\\ &\quad(\frac{\mathcal{H}^2\mu^2+\mathcal{I}^2\mathcal{B}^2}{\mathcal{B}^2\mu^2})\mathfrak{t}_2(\mathcal{A}(n,m)) = \frac{(\mathcal{H}^4\mu^2+\mathcal{H}^2\mathcal{B}^2\mathcal{I}^2)}{\mathcal{B}^2(\mathcal{I}^2-\mu^2)(\mathcal{H}^2+\mathcal{B}^2)}\end{aligned}$$

Simplification

$$\begin{aligned}\mathfrak{t}_2(\mathcal{A}(n,m))) &= \frac{\mathcal{H}^2(\mathcal{H}^2\mu^2+\omega^2\mathcal{I}^2)}{\mathcal{B}^2(\mathcal{I}^2-\mu^2)(\mathcal{H}^2+\mathcal{B}^2)}\ \frac{\mathcal{B}^2\mu^2}{\mathcal{B}^2\mu^2+\mathcal{I}^2\mathcal{B}^2}\\ \mathfrak{t}_2(\mathcal{A}(n,m)) &= \frac{\mathcal{H}^2\mu^2}{(\mathcal{I}^2-\mu^2)(\mathcal{H}^2+\omega^2)}\\ \mathcal{A}(n,m) &= \mathfrak{t}_2^{-1}[\ \frac{\mathcal{H}^2\mu^2}{(\mathcal{H}^2+\omega^2)(\mathcal{I}^2-\mu^2)}] = \ \cos n\ \ \sinh m\end{aligned}$$

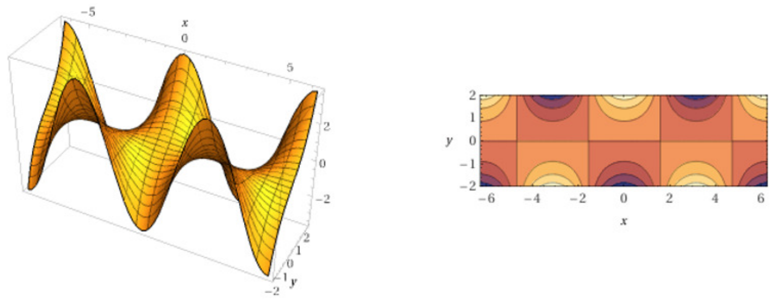

Figure 1: The solution $\mathcal{A}(n,m)$ of Example 5.1

Example 5.2 Consider the Poisson equation
$\mathcal{A}_{nn} - \mathcal{A}_{mm} - \mathcal{A}_m - \mathcal{A} = 0$, with
$\mathcal{A}\,(n\cdot 0) = e^n \qquad \mathcal{A}_m\,(n.0) = -e^n$
$\mathcal{A}\,(0,\mathfrak{y}) = e - m \qquad \mathcal{A}_n\,(0,m) = e^{-m}$

Solution: By taking $\mathfrak{t}_2$ to both sides

$$\left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)\mathfrak{t}_2\,(\mathcal{A}\,(n,m)) - \left(\frac{\mathcal{H}}{\mathcal{B}}\right)\left(\frac{\partial}{\partial n}\;Z\mathcal{A}\,(0,m)\right) + \left(\frac{\mathcal{I}^2}{\mu^2}\right)\mathfrak{t}_2\,(\mathcal{A}\,(n,m)) - \left(\frac{\mathcal{I}}{\mu}\right)Z\,(\mathcal{A}\,(n,0)) - Z\,\left(\frac{\partial}{\partial m}\;\mathcal{A}\,(n\;,0)\right)$$

Simple calculation

$$\left(\frac{\mathcal{H}^2\mu^2 + \mathcal{I}^2\mathcal{B}^2}{\mathcal{B}^2\mu^2}\right)\mathfrak{t}_2\,(\mathcal{A}\,(n,m)) = \frac{\mathcal{I}^2\mathcal{H}\mathcal{B}^2 - \mathcal{H}^3\mu^2}{\mathcal{I}^2\mathcal{B}\,(\mathcal{H}^2 + \mathcal{B}^2)}$$

Then

$$\mathfrak{t}_2\,(\mathcal{A}\,(n,m)) = \frac{-\mathcal{H}\mu^2\mathcal{B}}{\mathcal{I}^2\,(\mathcal{H}^2 + \mathcal{B}^2)}$$

After taking invers of double ZZ-Shehu transform

$$\mathcal{A}\,(n,m) = \mathfrak{t}_2^{-1}\left[\frac{-\mathcal{H}\mu^2\mathcal{B}}{\mathcal{I}^2\,(\mathcal{H}^2 + \mathcal{B}^2)}\right] = -m\;\sin n$$

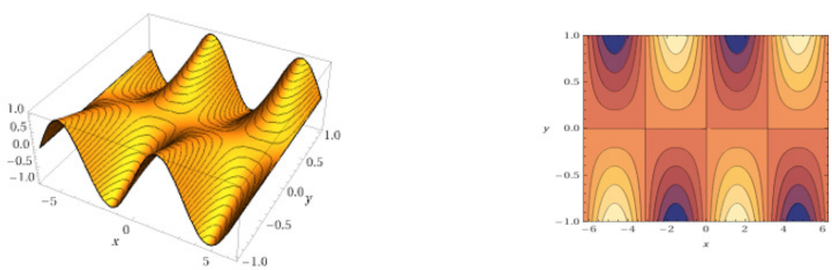

Figure 2: The solution $\mathcal{A}(n,m)$ of Example 5.2

Example 5.3 Consider the Telegraph equation
$\mathcal{A}_{nn} - \mathcal{A}_{mm} - \mathcal{A}_m - \mathcal{A} = 0$, with
$\mathcal{A}\,(n\;.0) = e^n \qquad \mathcal{A}_m\,(n.0) = -e^n$
$\mathcal{A}\,(0,\mathfrak{y}) = e - m \qquad \mathcal{A}_n\,(0,m) = e^{-m}$

Solution: Applying the formable tranform to all conditions, we have

$$\begin{aligned}\left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)\mathfrak{t}_2\left(\mathcal{A}\left(n,m\right)\right)-\left(\frac{\mathcal{H}^2}{\mathcal{B}^2}\right)S\left(\mathcal{A}\left(0,m\right)\right)-\left(\frac{\mathcal{H}}{\mathcal{B}}\right)\left(\frac{\partial}{\partial n}Z\mathcal{A}\left(0,m\right)\right)&\\ -\left(\frac{\mathcal{I}^2}{\mu^2}\right)\mathfrak{t}_2(\mathcal{A}\left(n,m\right))-\left(\frac{\mathcal{I}}{\mu}\right)Z\left(\mathcal{A}\left(n\ ,0\right)\right)-Z\ \left(\frac{\partial}{\partial m}\mathcal{A}\left(n\ ,0\right)\right)&\\ -\left(\frac{\mathcal{I}}{\mu}\right)\mathfrak{t}_2(\mathcal{A}\left(n,m\right))-Z(\mathcal{A}\left(n\ ,0\right))-\mathfrak{t}_2\left(\mathcal{A}\left(n,m\right)\right)&\\ =0.&\end{aligned}$$

Simple calculation to get:

$$\left(\frac{\mathcal{H}^2\mu^2-\mathcal{I}^2\mathcal{B}^2-\mathcal{I}^2\mathcal{B}\mu-\mathcal{B}^2\mu^2}{\mathcal{B}^2\mu^2}\right)\mathfrak{t}_2\left(\mathcal{A}\left(n,m\right)\right)=\frac{\mathcal{H}(\mathcal{H}^2\mu^2-\mu^2\mathcal{B}^2-\mathcal{B}^2\mathcal{I}^2-\mathcal{B}^2\mathcal{I}\mu)}{\mu\mathcal{B}^2(\mathcal{I}+\mu)(\mathcal{H}-\mu)}$$

$$\mathfrak{t}_2\left(\mathcal{A}\left(n,m\right)\right)=\frac{\mathcal{H}(\mathcal{H}^2\mu^2-\mu^2\mathcal{B}^2-\mathcal{B}^2\mathcal{I}^2-\mathcal{B}^2\mathcal{I}\mu)}{\mu\mathcal{B}^2(\mathcal{I}+\mu)(\mathcal{H}-\mu)}\frac{\mathcal{B}^2\mu^2}{\mathcal{H}^2\mu^2-\delta^2\mathcal{B}^2-\delta\mathcal{B}^2\mu-\mathcal{B}^2\mu^2}$$

$$\mathfrak{t}_2\left(\mathcal{A}\left(n,m\right)\right)=\frac{\mathcal{H}\mu}{(\mathcal{H}-\mathcal{B})(\mathcal{I}-\mathcal{B})}$$

Taking the inverse double ZZ-Shehu transform

$$\mathcal{A}\left(n,m\right)=\mathfrak{t}_2^{-1}\left[\frac{\mathcal{H}\mu}{(\mathcal{H}-\mathcal{B})(\mathcal{I}-\mathcal{B})}\right]=e^{n-m}$$

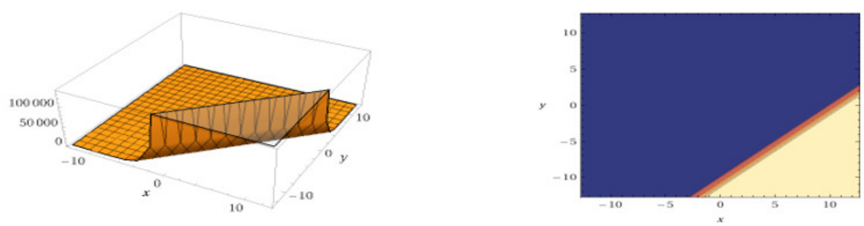

Figure 3: The solution $\mathcal{A}(n,m)$ of Example 5.3

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Huda Ali Falih,
Department of Mathematics,
Faculty of Education for Women, University of Kufa,
Iraq.
E-mail address: hudaa.aleidani@student.uokufa.edu.iq

and

Athraa Neamah Albukhuttar,
Department of Mathematics,
College of Education, University of Al-Qadisiyah,
Iraq.
E-mail address: athraan.kadhim@uokufa.edu.iq