Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.79698

Study of the Structure of Near-Rings with n-Derivations

A. Zerbane, A. Boua and A. Raji

ABSTRACT: This paper aims to formulate a set of theorems that guarantee the commutativity of the near-ring $\mathcal{N}$ under specific conditions. We show that if $\mathcal{N}$ admits an n-derivation D with certain properties, then $\mathcal{N}$ must be a commutative ring. Our results are illustrated with concrete examples, highlighting their crucial role in the logical framework of the proofs.

Keywords: 3-prime near-rings, n-derivations, commutativity.

Contents

1. Introduction

Throughout this paper, $\mathcal{N}$ will be a left near-ring with multiplicative center $\mathcal{Z}(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero symmetric if $0x = 0$, for all $x \in \mathcal{N}$ (recall that the left distributive law yields $x0 = 0$), and $\mathcal{N}$ is called 2-torsion free if for each $r \in \mathcal{N}$, $2r = 0$ implies $r = 0$. Further, $\mathcal{N}$ is called 3-prime if $r\mathcal{N}t = \{0\}$ for all $r, t \in \mathcal{N}$, implies $r = 0$ or $t = 0$. Let α be any mapping from $\mathcal{N}$ into itself; for any pair of elements $r, t \in \mathcal{N}$, we define $[r, t]_\alpha = \alpha(r)t - t\alpha(r)$ and $(r \circ t)_\alpha = \alpha(r)t + t\alpha(r)$. In particular, $[r, t]_1 = [x, y]$ and $(r \circ t)_1 = r \circ t$ in the usual sense. An additive mapping H: $\mathcal{N} \to \mathcal{N}$ is said to be a left (resp. right) multiplier if $H(rt) = H(r)t$ (resp. $H(rt) = rH(t)$) holds for all $r, t \in \mathcal{N}$. H is said to be a multiplier if it is both a left and a right multiplier. Let n be a fixed positive integer. An n-additive (i.e., additive in each argument) mapping $D : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n\text{-times}} \longrightarrow \mathcal{N}$ is said to be an n-derivation of $\mathcal{N}$ if the following equations hold for all $r_1, r_1^{'}, r_2, r_2^{'}, ..., r_n, r_n^{'} \in \mathcal{N}$

$$\begin{aligned}
D(r_1r_1^{'}, r_2, ..., r_i, ..., r_n) &= D(r_1, r_2, ..., r_i, ..., r_n)r_1^{'} + r_1D(r_1^{'}, r_2, ..., r_i, ..., r_n)\\
D(r_1, r_2r_2^{'}, ..., r_i, ..., r_n) &= D(r_1, r_2, ..., r_i, ..., r_n)r_2^{'} + r_2D(r_1, r_2^{'}, ..., r_i, ..., r_n)\\
&\vdots\\
D(r_1, r_2, ..., r_i, ..., r_nr_n^{'}) &= D(r_1, r_2, ..., r_i, ..., r_n)r_n^{'} + r_nD(r_1, r_2^{'}, ..., r_i, ..., r_n^{'}).
\end{aligned}$$

For an example of n-derivation, let $\mathcal{S}$ be a commutative left near-ring.

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid 0, x, y, z \in \mathcal{S} \right\}.$$

Define $D : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n\text{-times}} \longrightarrow \mathcal{N}$ such that

$$D\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & z_1z_2\ldots z_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2020 *Mathematics Subject Classification:* 16N60, 16W25, 16Y30.
Submitted October 22, 2025. Published June 05, 2026.

It can be easily verified that D is an n-derivation of $\mathcal{N}$. An additive map $d\colon \mathcal{N} \longrightarrow \mathcal{N}$ is termed a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. Building on existing research, concepts like symmetric bi-derivation, permuting tri-derivation, and permuting n-derivation have been explored in ring theory by researchers such as G. Maksa in [4], M. A. Ozturk in [5], and K. H. Park in [6]. Building on previous studies, M. A. Ozturk and K. H. Park have investigated symmetric bi-derivations and permuting tri-derivations within near-rings, with their findings presented in [7] and [8]. Based on the concept of n-derivations in [3], our research demonstrates that prime near-rings with n-derivations, under specific conditions, exhibit commutative ring properties. There has been significant research focus on the commutativity of near-rings that satisfy specific properties and identities, particularly those involving derivations, generalized derivations, and permuting n-derivations, as documented in various studies (see [3,9,10,11,12] for further details). Now our purpose is to examine the commutativity properties of prime near-rings that admit n-derivations under certain conditions.

2. Preliminary Lemmas

We now state the following two results (see [1, Lemma 1.2] for Lemma 2.1 and [2, Corollary 3.5] for Lemma 2.2).

Lemma 2.1 *Let $\mathcal{N}$ be a 3-prime near-ring. If $z \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$ and x is an element of $\mathcal{N}$ such that $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.*

Lemma 2.2 *Let $\mathcal{N}$ be a 3-prime near-ring, and α and β nonzero left multipliers on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[x, y]_{\alpha,\beta} \in \mathcal{Z}(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;

(ii) $-[x, y]_{\alpha,\beta} \in \mathcal{Z}(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;

(iii) $(x \circ y)_{\alpha,\beta} \in \mathcal{Z}(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;

(iv) $-(x \circ y)_{\alpha,\beta} \in \mathcal{Z}(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;

(v) *$\mathcal{N}$ is a commutative ring.*

We now prove some preliminary results.

Lemma 2.3 *Let $\mathcal{N}$ be a near-ring, D is a n-additive mapping. Then D is a n-derivation of $\mathcal{N}$ if and only if*

$$D(r_1, r_2, ..., r_i t, ..., r_n) = r_i D(r_1, r_2, ..., t, ..., r_n) + D(r_1, r_2, ..., r_i, ..., r_n)t$$

for all $t, r_1, r_2, ..., r_n \in \mathcal{N}$.

Proof: For all $t, r_1, r_2, ..., r_n \in \mathcal{N}$, we have

$$\begin{aligned} D(r_1, r_2, ..., r_i(t+t), ..., r_n) &= D(r_1, r_2, ..., r_i, ..., r_n)(t+t) + r_i D(r_1, r_2, ..., t+t, ..., r_n) \\ &= D(r_1, r_2, ..., r_i, ..., r_n)t + D(r_1, r_2, ..., r_i, ..., r_n)t \\ &\quad + r_i D(r_1, r_2, ..., t, ..., r_n) + r_i D(r_1, r_2, ..., t, ..., r_n), \end{aligned}$$

and

$$\begin{aligned} D(r_1, r_2, ..., r_i t + r_i t, ..., r_n) &= D(r_1, r_2, ..., r_i t, ..., r_n) + D(r_1, r_2, ..., r_i t, ..., r_n) \\ &= D(r_1, r_2, ..., r_i, ..., r_n)t + r_i D(r_1, r_2, ..., t, ..., r_n) \\ &\quad + D(r_1, r_2, ..., r_i, ..., r_n)t + r_i D(r_1, r_2, ..., t, ..., r_n). \end{aligned}$$

Combining above two equalities we obtain that

$$D(r_1, r_2, ..., r_i, ..., r_n)t + r_i D(r_1, r_2, t, ..., r_n) = r_i D(r_1, r_2, ..., t, ..., r_n) + D(r_1, r_2, ..., r_i, ..., r_n)t$$

which means that,

$$D(r_1, r_2, ..., r_i t, ..., r_n) = r_i D(r_1, r_2, ..., t, ..., r_n) + D(r_1, r_2, ..., r_i, ..., r_n)t.$$

Converse can be demonstrated similarly. □

While the right distributive law generally fails in left near-rings, we can still derive specific partial distributive properties.

Lemma 2.4 *Let $\mathcal{N}$ be a near-ring. Let D be an n-derivation of $\mathcal{N}$. Then*

$$\Big(D(r_1, r_2, ..., r_i, ..., r_n)r_i^{'} + r_i D(r_1, r_2, ..., r_i^{'}, ..., r_n)\Big)r_i^{''} = D(r_1, r_2, ..., r_i, ..., r_n)r_i^{'}r_i^{''} + r_i D(r_1, r_2, ..., r_i^{'}, ..., r_n)r_i^{''}$$

for all $r_i^{'}, r_i^{''}, r_1, r_2, ..., r_n \in \mathcal{N}$.

Proof: Using the definition of D, we have

$$\begin{aligned} D(r_1, r_2, ..., (r_i r_i^{'})r_i^{''}, ..., r_n) &= D(r_1, r_2, ..., r_i r_i^{'}, ..., r_n)r_i^{''} + (r_i r_i^{'})D(r_1, r_2, ..., r_i^{''}, ..., r_n) \\ &= \Big(D(r_1, r_2, ..., r_i, ..., r_n)r_i^{'} + r_i D(r_1, r_2, ..., r_i^{'}, ..., r_n)\Big)r_i^{''} \\ &\quad + r_i r_i^{'} D(r_1, r_2, ..., r_i^{''}, ..., r_n) \text{ for all } r_i^{'}, r_i^{''}, r_1, r_2, ..., r_n \in \mathcal{N}. \end{aligned}$$

And

$$\begin{aligned} D(r_1, r_2, ..., r_i(r_i^{'}r_i^{''}), ..., r_n) &= D(r_1, r_2, ..., r_i, ..., r_n)(r_i^{'}r_i^{''}) + r_i D(r_1, r_2, ..., r_i^{'}r_i^{''}, ..., r_n) \\ &= D(r_1, r_2, ..., r_i, ..., r_n)r_i^{'}r_i^{''} + r_i D(r_1, r_2, ..., r_i^{'}, ..., r_n)r_i^{''} \\ &\quad + r_i r_i^{'} D(r_1, r_2, ..., r_i^{''}, ..., r_n) \text{ for all } r_i^{'}, r_i^{''}, r_1, r_2, ..., r_n \in \mathcal{N}. \end{aligned}$$

From two expressions of $D(r_1, r_2, ..., r_i r_i^{'} r_i^{''}, ..., r_n)$, we get the required result. □

Lemma 2.5 *Let $\mathcal{N}$ be a near-ring, and let D be an n-derivation of $\mathcal{N}$. Then*

$$D(r_1, ..., r_{i-1}, \mathcal{Z}(\mathcal{N}), r_{i+1}, ..., r_n) \subset \mathcal{Z}(\mathcal{N}) \quad \textit{for all} \quad r_1, ..., r_{i-1}, r_{i+1}, ..., r_n \in \mathcal{N}.$$

Proof: For all $t \in \mathcal{Z}(\mathcal{N}), r_1, r_2, ..., r_n \in \mathcal{N}$, we have

$$D(r_1, ..., r_{i-1}, tr_i, r_{i+1}, ..., r_n) = D(r_1, ..., r_{i-1}, r_i t, r_{i+1}, ..., r_n),$$

and hence for all $t \in \mathcal{Z}(\mathcal{N}), r_1, r_2, ..., r_n \in \mathcal{N}$, we have

$$D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)r_i + tD(r_1, r_2, ..., r_i, ..., r_n) = D(r_1, r_2, ..., r_i, ..., r_n)t + r_i D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n).$$

From Lemma 2.3, we get $D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)r_i = r_i D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)$ $r_1, r_2, ..., r_n \in \mathcal{N}$. Hence $D(r_1, ..., r_{i-1}, \mathcal{Z}(\mathcal{N}), r_{i+1}, ..., r_n) \subset \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{Z}(\mathcal{N})$, $r_1, ..., r_{i-1}, r_{i+1}, ..., r_n \in \mathcal{N}$. □

Lemma 2.6 *Let $\mathcal{N}$ be a 3-prime near-ring, and α and β are nonzero multipliers. Then the following properties are satisfied:*

1. $[x, y]_\alpha = [\alpha(x), y] = [x, \alpha(y)] = \alpha([x, y])$ *for all $x, y \in \mathcal{N}$,*
2. $[[x, y]_\alpha, z]_\beta = [[x, y]_\beta, z]_\alpha$ *for all $x, y, z \in \mathcal{N}$.*

Proof:

1. For all $x, y \in \mathcal{N}$, we have

$$[x, y]_\alpha = \alpha(x)y - y\alpha(x) = x\alpha(y) - \alpha(y)x = \alpha(xy - yx).$$

2. For all $x, y, z \in \mathcal{N}$, we have

$$\begin{aligned}[[x,y]_\alpha, z]_\beta &= [\beta([x,\alpha(y)]), z] \\ &= [[\beta(x),\alpha(y)], z] \\ &= [\alpha([\beta(x), y]), z] \\ &= [[x,y]_\beta, z]_\alpha \text{ for all } x,y,z \in \mathcal{N}.\end{aligned}$$

□

Lemma 2.7 *A near-ring $\mathcal{N}$ admits an n-derivation D. Then $\mathcal{N}$ is zero-symmetric.*

Proof: Assume that $\mathcal{N}$ has an n-derivation D. We have for all $t_1, r_1, r_2, r_3, ..., r_n \in \mathcal{N}$

$$\begin{aligned}D((r_1.0).t_1, r_2, r_3, ..., r_n) &= D(r_1.0, r_2, r_3, ..., r_n)t_1 + (r_1.0)D(t_1, r_2, r_3, ..., r_n) \\ &= 0.t_1 + 0.D(t_1, r_2, r_3, ..., r_n).\end{aligned}$$

On the other side, we have

$$\begin{aligned}D(r_1(0.t_1), r_2, r_3, ..., r_n) &= D(r_1, r_2, r_3, ..., r_n)(0.t_1) + r_1 D(0.t_1, r_2, r_3, ..., r_n) \\ &= 0.t_1 + 0.t_1 + 0.D(t_1, r_2, r_3, ..., r_n).\end{aligned}$$

Now, compare the two expressions of $D(r_1.0.t_1, r_2, r_3, ..., r_n)$ we conclude that N is zero-symmetric. □

3. Main Results

In this section, we give some new results and examples concerning the existence of n-derivations in near-rings. We will also apply Lemma 2.7 several times without mentioning it. We begin with the following interesting result.

Theorem 3.1 *Let $\mathcal{N}$ be a 3-prime near-ring, D is a nonzero n-derivation on $\mathcal{N}$ and α is a nonzero left multipliers on $\mathcal{N}$. Then the following assertions are equivalent:*

(i) $D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) = 0$, *for all* $y_i, r_1, r_2, ..., r_n \in \mathcal{N}$,

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof: For $(ii) \Rightarrow (i)$, it is obvious.
$(i) \Rightarrow (ii)$. By our hypothesis, we have

$$D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) = 0 \quad \text{for all} \quad y_i, r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.1}$$

Substituting $\alpha(r_i)y_i$ for y_i in (3.1), and using the fact that $[r_i, \alpha(r_i)y_i]_\alpha = \alpha(r_i)[r_i, y_i]_\alpha$, we obtain

$$D(r_1, ..., r_{i-1}, \alpha(r_i)[r_i, y_i]_\alpha, r_{i+1}, ..., r_n) = 0 \quad \text{for all} \quad y_i, r_1, r_2, ..., r_n \in \mathcal{N}.$$

This implies that $D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)[r_i, y_i]_\alpha + \alpha(r_i)D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) = 0$ for all $y_i, r_1, r_2, ..., r_n \in \mathcal{N}$. Using (3.1) in the last relation, we get $D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)[r_i, y_i]_\alpha = 0$. Therefore, $(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)\alpha(r_i)y_i = D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)y_i\alpha(r_i)$. Putting $y_i = y_i t$ in the last relation and using it, we obtain $D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)y_i[\alpha(r_i), t] = 0$ for all $t, y_i, r_1, r_2, ..., r_n \in \mathcal{N}$, which can be rewritten as $D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)\mathcal{N}[\alpha(r_i), t] = 0$ for all $t, r_1, r_2, ..., r_n \in \mathcal{N}$, Since $\mathcal{N}$ is 3-prime, we find that

$$D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n) = 0 \in \mathcal{Z}(\mathcal{N}) \quad \text{or} \quad \alpha(r_i) \in \mathcal{Z}(\mathcal{N}) \quad \text{for all } r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.2}$$

But $\alpha(r_i) \in \mathcal{Z}(\mathcal{N})$ also implies that $D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N})$ for all $r_1, r_2, ..., r_n \in \mathcal{N}$, by Lemma 2.5, hence (3.2) reduces to

$$D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N}) \text{ for all } r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.3}$$

For $r_i = r_i z$, where $z \in \mathcal{N}$ in (3.3), we get

$$D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)z + \alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N}). \tag{3.4}$$

Thus, for all $z, r_1, r_2, ..., r_n \in \mathcal{N}$,

$$\begin{aligned} &z\Big(D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)z + \alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n)\Big) \\ &= \Big(D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)z + \alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n)\Big)z. \end{aligned}$$

Now using Lemma 2.3 and (3.3) in the above equation, we infer that

$$\begin{aligned} &D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)z^2 + z\alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n) \\ &= D(r_1, ..., r_{i-1}, \alpha(r_i), r_{i+1}, ..., r_n)z^2 + \alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n)z. \end{aligned}$$

Hence, $z\alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n) = \alpha(r_i)D(r_1, ..., r_{i-1}, z, r_{i+1}, ..., r_n)z$ for all $z, r_1, r_2, ..., r_n \in \mathcal{N}$. Replacing z by $\alpha(z)$ and using (3.3), we obtain $D(r_1, ..., r_{i-1}, \alpha(z), r_{i+1}, ..., r_n)[\alpha(z), \alpha(r_i)] = 0$, which can we rewritten as $D(r_1, ..., r_{i-1}, \alpha(z), r_{i+1}, ..., r_n)\mathcal{N}[\alpha(z), \alpha(r_i)] = \{0\}$ for all $z, r_1, r_2, ..., r_n \in \mathcal{N}$, Since $\mathcal{N}$ is 3-prime, we implies that

$$D(r_1, ..., r_{i-1}, \alpha(z), r_{i+1}, ..., r_n) = 0 \quad \text{or} \quad [\alpha(z), \alpha(r_i)] = 0 \quad \text{for all } z, r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.5}$$

Suppose there exists $z_0 \in \mathcal{N} \setminus \{0\}$ such that

$$D(r_1, ..., r_{i-1}, \alpha(z_0), r_{i+1}, ..., r_n) = 0 \text{ for all } r_1, r_2, ..., r_{i-1}, r_{i+1}, ..., r_n \in \mathcal{N}.$$

Replacing r_i by z_0 and z by $\alpha(t)$ in (3.4), we infer that $\alpha(z_0)D(r_1, ..., r_{i-1}, \alpha(t), r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N})$ for all $t, r_1, r_2, ..., r_n \in \mathcal{N}$. By Lemma 2.1, we obtain $\alpha(z_0) \in \mathcal{Z}(\mathcal{N})$ or $D(r_1, ..., r_{i-1}, \alpha(t), r_{i+1}, ..., r_n) = 0$ for all $t, r_1, r_2, ..., r_n \in \mathcal{N}$. In this case, (3.5) becomes

$$D(r_1, ..., r_{i-1}, \alpha(t), r_{i+1}, ..., r_n) = 0 \quad \text{or} \quad [\alpha(z), \alpha(r_i)] = 0 \text{ for all } t, z, r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.6}$$

If $D(r_1, ..., r_{i-1}, \alpha(t), r_{i+1}, ..., r_n) = 0$ for all $t, r_1, r_2, ..., r_{i-1}, r_{i+1}, ..., r_n \in \mathcal{N}$. Replacing t by tyr_i in the (3.6) and using it, we obtain $\alpha(t)\mathcal{N}D(r_1, ..., r_{i-1}, r_i, r_{i+1}, ..., r_n) = \{0\}$ for all $t, r_1, r_2, ..., r_n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $D = \alpha = 0$ is a contradiction. Then (3.6) becomes $[\alpha(z), \alpha(r_i)] = 0$ for all $z, r_i \in \mathcal{N}$, then $\alpha(z)\alpha(r_i) = \alpha(r_i)\alpha(z)$ for all $z, r_i \in \mathcal{N}$. By substituting $r_i t$ in place of r_i, we obtain $\alpha(z)\alpha(r_i)t = \alpha(r_i)t\alpha(z)$ for all $z, t, r_i \in \mathcal{N}$. Using the two above expressions, we arrive at $\alpha(z)[\alpha(r_i), t] = 0$ for all $z, t, r_i, t \in \mathcal{N}$. Putting zs instead of z, we get $\alpha(z)s[\alpha(r_i), t] = 0$ for all $s, t, z, r_i \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $\alpha \neq 0$, we obtain $[\alpha(r_i), t] = [r_i, t]_\alpha = 0 \in \mathcal{Z}(\mathcal{N})$ for all $r_i, t \in \mathcal{N}$, which shows by Lemma 2.2 that $\mathcal{N}$ is a commutative ring. □

Theorem 3.2 *Let $\mathcal{N}$ be a 2-torsion-free 3-prime near-ring, D be a nonzero n-derivation, and α be a nonzero multiplier on $\mathcal{N}$. Then the following assertions are equivalent:*

(i) $D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N})$ *for all* $y_i, r_1, r_2, ..., r_n \in \mathcal{N}$,

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof: For $(ii) \Rightarrow (i)$, it is obvious.
$(i) \Rightarrow (ii)$. By hypothesis given, we have

$$D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N}) \ \text{ for all } y_i, r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.7}$$

Now we have two cases,

- If $\mathcal{Z}(\mathcal{N}) = \{0\}$. So, our hypothesis becomes

$$D(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) = 0 \text{ for all } y_i, r_1, r_2, ..., r_n \in \mathcal{N}.$$

By the previous theorem, we conclude that $\mathcal{N} = \mathcal{Z}(\mathcal{N}) = \{0\}$, which is a contradiction.

- If $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. Putting tr_i instead of r_i, where $t \in \mathcal{Z}(\mathcal{N})$, in relation (3.7), we infer that $D(r_1, ..., r_{i-1}, t[r_i, y_i]_\alpha, r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{Z}(\mathcal{N}), y_i, r_1, r_2, ..., r_n \in \mathcal{N}$. Expanding the last equation, we get $D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)[r_i, y_i]_\alpha + tD(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n) \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{Z}(\mathcal{N}), y_i, r_1, r_2, ..., r_n \in \mathcal{N}$, and therefore

$$\begin{aligned} &x\Big(D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)[r_i, y_i]_\alpha + tD(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n)\Big) \\ &= \Big(D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)[r_i, y_i]_\alpha + tD(r_1, ..., r_{i-1}, [r_i, y_i]_\alpha, r_{i+1}, ..., r_n)\Big)x \end{aligned}$$

Using Lemmas 2.3, 2.4, and 2.5 together (3.7), we get $D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)[[r_i, y_i]_\alpha, x] = 0$. Left multiplying by s, where $s \in \mathcal{N}$, and invoking Lemma 2.5, the above expression gives $D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)s[[r_i, y_i]_\alpha, x] = \{0\}$ for all $t \in \mathcal{Z}(\mathcal{N}), x, s, y_i, r_1, r_2, ..., r_n \in \mathcal{N}$, and therefore $D(r_1, ..., r_{i-1}, t, r_{i+1}, ..., r_n)\mathcal{N}[[r_i, y_i]_\alpha, x] = \{0\}$ for all $t \in \mathcal{Z}(\mathcal{N}), x, y_i, r_1, r_2, ..., r_n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that

$$D(r_1, ..., r_{i-1}, \mathcal{Z}(\mathcal{N}), r_{i+1}, ..., r_n) = \{0\} \quad \text{or} \quad [r_i, y_i]_\alpha \in \mathcal{Z}(\mathcal{N}) \text{ for all } y_i, r_1, r_2, ..., r_n \in \mathcal{N}. \tag{3.8}$$

Assume that $D(t_1, ..., t_{i-1}, \mathcal{Z}(\mathcal{N}), t_{i+1}, ..., t_n) = \{0\}$ for all $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n \in \mathcal{N}$. By (3.7), we can see that

$$D(t_1, ..., t_{i-1}, D(t_1, ..., t_{i-1}, [t_i, s_i]_\alpha, t_{i+1}, ..., t_n), t_{i+1}, ..., t_n) = 0 \text{ for all } s_i, t_1, ..., t_n \in \mathcal{N}. \tag{3.9}$$

Replacing s_i by $t_i s_i$ in (3.9) and using the fact that $[x, xt]_\alpha = x[x, t]_\alpha$, we get

$$\begin{aligned} 0 &= D(t_1, ..., t_{i-1}, D(t_1, ..., t_{i-1}, t_i[t_i, s_i]_\alpha, t_{i+1}, ..., t_n), t_{i+1}, ..., t_n) \\ &= D\big(t_1, ..., t_{i-1}, D(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n), t_{i+1}, ..., t_n\big)[t_i, s_i]_\alpha \\ &\quad + 2D(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n)D\big(t_1, ..., t_{i-1}, [t_i, s_i]_\alpha, t_{i+1}, ..., t_n\big) \\ &\quad + t_i D\big(t_1, ..., t_{i-1}, D(t_1, ..., t_{i-1}, [t_i, s_i]_\alpha, t_{i+1}, ..., t_n), t_{i+1}, ..., t_n\big) \text{ for all } s_i, t_1, t_2, ..., t_n \in \mathcal{N}. \end{aligned}$$

By (3.9), preceding relation forces

$$\begin{aligned} 0 &= D\big(t_1, ..., t_{i-1}, D(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n), t_{i+1}, ..., t_n\big)[t_i, s_i]_\alpha \\ &\quad + 2D(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n)D\big(t_1, ..., t_{i-1}, [t_i, s_i]_\alpha, t_{i+1}, ..., t_n\big) \text{ for all } s_i, t_1, t_2, ..., t_n \in \mathcal{N}. \end{aligned}$$

Substituting $[t_i, x_i]_\alpha$ for t_i in the last expression and using (3.9) together with 2-torsion freeness, we get $D\big(t_1, ..., t_{i-1}, [t_i, x_i]_\alpha, t_{i+1}, ..., t_n\big)D\big(t_1, ..., t_{i-1}, [[t_i, x_i]_\alpha, s_i]_\alpha, t_{i+1}, ..., t_n\big) = 0$ for all $x_i, s_i, t_1, ..., t_n \in \mathcal{N}$. Thus, for all $x_i, s_i, t_1, t_2, ..., t_n \in \mathcal{N}$

$$D\big(t_1, ..., t_{i-1}, [t_i, x_i]_\alpha, t_{i+1}, ..., t_n\big)\mathcal{N}D\big(t_1, ..., t_{i-1}, [[t_i, x_i]_\alpha, s_i]_\alpha, t_{i+1}, ..., t_n\big) = \{0\}.$$

Since $\mathcal{N}$ is 3-prime, we get

$$\begin{aligned} &D(t_1, ..., t_{i-1}, [t_i, x_i]_\alpha, t_{i+1}, ..., t_n) = 0 \quad \textit{or} \quad D\big(t_1, ..., t_{i-1}, [[t_i, x_i]_\alpha, s_i]_\alpha, t_{i+1}, ..., t_n\big) = 0 \\ &\text{for all } x_i, s_i, t_1, t_2, ..., t_n \in \mathcal{N}. \end{aligned} \tag{3.10}$$

Suppose there exist $x_{0_i}, t_{0_1}, t_{0_2}, ..., t_{0_n} \in \mathcal{N}$ such that

$$D\big(t_{0_1}, ..., t_{0_{i-1}}, [[t_{0_i}, x_{0_i}]_\alpha, s_i]_\alpha, t_{0_{i+1}}, ..., t_{0_n}\big) = 0 \text{ for all } s_i \in \mathcal{N}.$$

Using the fact that $[x,y]_\alpha = \alpha(x)y - x\alpha(y) = x\alpha(y) - \alpha(y)x$, we infer that

$$D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big) = D\big(t_{0_1},...,t_{0_{i-1}},\alpha(s_i)[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)$$

for all $s_i \in \mathcal{N}$. Now invoking (3.7) and Lemma 2.3 in the latter expression, we get

$$\begin{aligned}&D\big(t_{0_1},...,t_{0_{i-1}},\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big)[t_{0_i},x_{0_i}]_\alpha\\ &= [t_{0_i},x_{0_i}]_\alpha D\big(t_{0_1},...,t_{0_{i-1}},\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big) \text{ for all } s_i \in \mathcal{N}.\end{aligned} \tag{3.11}$$

Replacing s_i by $[t_{0_i},x_{0_i}]_\alpha s_i$ in the left-hand side of the last equation together using Lemma 2.4 and (3.11), we obtain

$$\begin{aligned}&D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big)[t_{0_i},x_{0_i}]_\alpha\\ &= D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)\alpha(s_i)[t_{0_i},x_{0_i}]_\alpha\\ &\ +[t_{0_i},x_{0_i}]_\alpha^2 D\big(t_{0_1},...,t_{0_{i-1}},\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big) \text{ for all } s_i \in \mathcal{N}.\end{aligned}$$

Putting $s_i = [t_{0_i},x_{0_i}]_\alpha s_i$ in the right-hand side of equation (3.11) and using it with (3.7), we get

$$\begin{aligned}&[t_{0_i},x_{0_i}]_\alpha D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big)\\ &= D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)[t_{0_i},x_{0_i}]_\alpha\alpha(s_i)\\ &\ +[t_{0_i},x_{0_i}]_\alpha^2 D\big(t_{0_1},...,t_{0_{i-1}},\alpha(s_i),t_{0_{i+1}},...,t_{0_n}\big) \text{ for all } s_i \in \mathcal{N}.\end{aligned}$$

Comparing the last two equations, we get

$$\begin{aligned}&D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)[t_{0_i},x_{0_i}]_\alpha\alpha(s_i)\\ &= D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)\alpha(s_i)[t_{0_i},x_{0_i}]_\alpha \text{ for all } s_i \in \mathcal{N}.\end{aligned}$$

Substituting $s_i z$ for s_i in the above equation, we get

$$\begin{aligned}&D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)[t_{0_i},x_{0_i}]_\alpha\alpha(s_i)z\\ &= D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)\alpha(s_i)z[t_{0_i},x_{0_i}]_\alpha \text{ for all } z,s_i \in \mathcal{N}.\end{aligned}$$

From the last two equations, we obtain

$$D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)\alpha(s_i)\big[[t_{0_i},x_{0_i}]_\alpha,z\big] = 0 \text{ for all } z,s_i \in \mathcal{N}. \tag{3.12}$$

Substituting us_iv for s_i in (3.12) and since α is a multiplier, we conclude that

$$D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)u\alpha(s_i)v\big[[t_{0_i},x_{0_i}]_\alpha,z\big] = 0 \text{ for all } u,v,z,s_i \in \mathcal{N}.$$

This gives us

$$D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big)\mathcal{N}\alpha(s_i)\mathcal{N}\big[[t_{0_i},x_{0_i}]_\alpha,z\big] = \{0\} \text{ for all } z,s_i \in \mathcal{N}.$$

Since $\alpha \neq 0$ and $\mathcal{N}$ is a 3-prime, the latter relation gives

$$D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big) = 0 \text{ or } [t_{0_i},x_{0_i}]_\alpha \in \mathcal{Z}(\mathcal{N}) \text{ for all } s_i \in \mathcal{N}. \tag{3.13}$$

By the hypothesis and Lemma 2.5, (3.13) becomes $D\big(t_{0_1},...,t_{0_{i-1}},[t_{0_i},x_{0_i}]_\alpha,t_{0_{i+1}},...,t_{0_n}\big) = 0$ and therefore (3.10) reduces to $D(t_1,...,t_{i-1},[t_i,x_i]_\alpha,t_{i+1},...,t_n) = 0$ for all $x_i,t_1,t_2,...,t_n \in \mathcal{N}$. So that (3.8), becomes $D(t_1,...,t_{i-1},[t_i,x_i]_\alpha,t_{i+1},...,t_n) = 0$ or $[r_i,y_i]_\alpha \in \mathcal{Z}(\mathcal{N})$ for all $x_i,r_i,y_i,t_1,...,t_n \in \mathcal{N}$. Which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.2 and Theorem 3.1.

□

The conclusion of Theorem 3.2 remains valid if we replace $[r_i,y_i]_\alpha$ by $[[r_i,y_i]_\alpha,z_i]_\beta$. In fact, we obtain the following result.

Theorem 3.3 *Let $\mathcal{N}$ be a 2-torsion-free 3-prime near-ring, D be a nonzero n-derivation, and α and β be nonzero multipliers on $\mathcal{N}$. Then the following assertions are equivalent:*

(i) $D(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)\in\mathcal{Z}(\mathcal{N})$ *for all* $y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}$,

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof: For $(ii)\Rightarrow(i)$, it is obvious.
$(i)\Rightarrow(ii)$.

Suppose that $\mathcal{Z}(\mathcal{N})=\{0\}$, then our hypothesis becomes

$$D(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)=0 \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}. \tag{3.14}$$

Substituting $[r_i,y_i]_\alpha z_i$ for z_i in (3.14), we get

$$\begin{aligned}&[r_i,y_i]_\alpha D(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)\\&+D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n))[[r_i,y_i]_\alpha,z_i]_\beta=0 \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}.\end{aligned} \tag{3.15}$$

From (3.14), (3.15) gives

$$D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n))[[r_i,y_i]_\alpha,z_i]_\beta=0 \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}.$$

Expanding the last equation, we obtain

$$\begin{aligned}&D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)[r_i,y_i]_\alpha\beta(z_i)\\&=D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)\beta(z_i)[r_i,y_i]_\alpha \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}.\end{aligned} \tag{3.16}$$

Now changing z_i by z_it, where $t\in\mathcal{N}$, in the last expression and using it, we find

$$D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)\beta(z_i)[[r_i,y_i]_\alpha,t]=0 \text{ for all } y_i,z_i,t,r_1,...,r_n\in\mathcal{N}.$$

Replacing uz_iu by z_i, where $u,v\in\mathcal{N}$, in (3.15) and using the fact that $H(xyz)=xH(y)z$, we get

$$D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)\mathcal{N}\beta(z_i)\mathcal{N}[[r_i,y_i]_\alpha,t]=\{0\} \text{ for all } y_i,t,r_1,...,r_n\in\mathcal{N}.$$

Since $\mathcal{Z}(\mathcal{N})=\{0\}$ and $\beta\neq0$ with using the 3-primeness of $\mathcal{N}$, we conclude that

$$D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)=0 \text{ or } [r_i,y_i]_\alpha=0 \text{ for all } y_i,t,r_1,...,r_n\in\mathcal{N}.$$

Then,

$$D(r_1,...,r_{i-1},[r_i,y_i]_\alpha,r_{i+1},...,r_n)=0 \text{ for all } y_i,r_1,r_2,...,r_n\in\mathcal{N}$$

forces that $\mathcal{N}$ is a commutative ring by the previous theorem, so that $\mathcal{N}=\mathcal{Z}(\mathcal{N})=\{0\}$, which is a contradiction. Hence $\mathcal{Z}(\mathcal{N})\neq\{0\}$. By our hypothesis, we have

$$D(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)\in\mathcal{Z}(\mathcal{N}) \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}. \tag{3.17}$$

Putting $z_i=tz_i$, where $t\in\mathcal{Z}(\mathcal{N})$, in (3.17), we obtain $D(r_1,...,r_{i-1},t,r_{i+1},...,r_n)[[r_i,y_i]_\alpha,z_i]_\beta+tD(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)\in\mathcal{Z}(\mathcal{N})$ for all $y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}$, which can be written as

$$\begin{aligned}&x\Big(D(r_1,...,r_{i-1},t,r_{i+1},...,r_n)[[r_i,y_i]_\alpha,z_i]+tD(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i],r_{i+1},...,r_n)\\&=\Big(D(r_1,...,r_{i-1},t,r_{i+1},...,r_n)[[r_i,y_i]_\alpha,z_i]_\beta+tD(r_1,...,r_{i-1},[[r_i,y_i]_\alpha,z_i]_\beta,r_{i+1},...,r_n)\Big)x\end{aligned}$$

for all $x,y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N},t\in\mathcal{Z}(\mathcal{N})$. From Lemma 2.4 and Lemma 2.3, the above equation gives $D(r_1,...,r_{i-1},t,r_{i+1},...,r_n)[[r_i,y_i]_\alpha,z_i]_\beta\in\mathcal{Z}(\mathcal{N})$ for all $y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N},t\in\mathcal{Z}(\mathcal{N})$. Also by Lemma 2.5 and Lemma 2.1, we obtain

$$D(r_1,...,r_{i-1},\mathcal{Z}(\mathcal{N}),r_{i+1},...,r_n)=\{0\} \text{ or } [[r_i,y_i]_\alpha,z_i]_\beta\in\mathcal{Z}(\mathcal{N}) \text{ for all } y_i,z_i,r_1,r_2,...,r_n\in\mathcal{N}. \tag{3.18}$$

We now distinguish two cases

- If $[[r_i, y_i]_\alpha, z_i]_\beta \in \mathcal{Z}(\mathcal{N})$ for all $y_i, z_i, r_i \in \mathcal{N}$, putting $\beta([r_i, y_i]_\alpha)z_i$ in place of z_i in the last equation, we arrive at $\beta([r_i, y_i]_\alpha)[[r_i, y_i]_\alpha, z_i]_\beta \in \mathcal{Z}(\mathcal{N})$ for all $y_i, z_i, r_i \in \mathcal{N}$. According to lemma 2.1 it follows $\beta([r_i, y_i]_\alpha) \in \mathcal{Z}(\mathcal{N})$ or $[[r_i, y_i]_\alpha, z_i]_\beta = 0$, for all $y_i, z_i, r_i \in \mathcal{N}$. Then $\beta([r_i, y_i]_\alpha)z_i = z_i\beta([r_i, y_i]_\alpha)$ for all $r_i, y_i, z_i \in \mathcal{N}$. Taking z_it in place of z_i in the last expression and using it again, we get $z_i[\beta([r_i, y_i]_\alpha), t] = 0$ for all $t, r_i, y_i, z_i \in \mathcal{N}$. Replacing z_i by z_is in the above expression and using Lemma 2.6, we get $\beta(z_i)s[[r_i, y_i]_\alpha, t] = 0$ for all $t, s, r_i, y_i, z_i \in \mathcal{N}$. Since $\beta \neq 0$ and by 3-primeness of $\mathcal{N}$, we get $[r_i, y_i]_\alpha \in \mathcal{Z}(\mathcal{N})$ for all $r_i, y_i \in \mathcal{N}$, now replacing and therefore by Lemma 2.2 $\mathcal{N}$ is a commutative ring.

- If $D(r_1, ..., r_{i-1}, \mathcal{Z}(\mathcal{N}), r_{i+1}, ..., r_n) = \{0\}$ for all $r_1, ..., r_{i-1}, r_{i+1}, ..., r_n \in \mathcal{N}$. Using (3.17) we obtain

$$D(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, [[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n), r_{i+1}, ..., r_n) = 0 \tag{3.19}$$

for all $z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}$. Substituting z_i by $\beta([r_i, t_i]_\alpha)z_i$ in (3.19), we infer that

$$\begin{aligned} 0 &= D(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, \beta([r_i, t_i]_\alpha)[[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n), r_{i+1}, ..., r_n) \\ &= D\big(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, \beta([r_i, t_i]_\alpha), r_{i+1}, ..., r_n), r_{i+1}, ..., r_n\big)[[r_i, t_i]_\alpha, z_i]_\beta \\ &\quad + 2D(r_1, ..., r_{i-1}, \beta([r_i, t_i]_\alpha), r_{i+1}, ..., r_n)D\big(r_1, ..., t_{i-1}, [[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) \\ &\quad + \beta([r_i, t_i]_\alpha)D\big(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, [[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n), r_{i+1}, ..., r_n\big) \\ &\quad \text{for all } z_i, t_i, r_{1,2}, ..., r_n \in \mathcal{N}. \end{aligned}$$

From (3.19), we conclude that

$$\begin{aligned} 0 &= D\big(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, \beta([r_i, t_i]_\alpha), r_{i+1}, ..., r_n), r_{i+1}, ..., r_n\big)[[r_i, t_i]_\alpha, z_i]_\beta \\ &\quad + 2D(r_1, ..., r_{i-1}, \beta([r_i, t_i]_\alpha), r_{i+1}, ..., r_n)D\big(r_1, ..., r_{i-1}, [[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) \\ &\quad \text{for all } z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= D\big(r_1, ..., r_{i-1}, D(r_1, ..., r_{i-1}, [\beta(r_i), t_i]_\alpha, r_{i+1}, ..., r_n), r_{i+1}, ..., r_n\big)[[r_i, t_i]_\alpha, z_i]_\beta \\ &\quad + 2D(r_1, ..., r_{i-1}, [\beta(r_i), t_i]_\alpha, r_{i+1}, ..., r_n)D\big(r_1, ..., r_{i-1}, [[r_i, t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) \\ &\quad \text{for all } z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}. \end{aligned} \tag{3.20}$$

Putting $r_i = [r_i, x_i]$ in (3.20) and applying Lemma 2.6 with the 2-torsion freeness of $\mathcal{N}$, and using (3.19), we obtain

$$D(r_1, ..., r_{i-1}, [[r_i, x_i]_\alpha, t_i]_\beta, r_{i+1}, ..., r_n)D\big(r_1, ..., r_{i-1}, [[[r_i, x_i], t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) = 0$$

for all $x_i, z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}$. Left multiplying by s, where $s \in \mathcal{N}$, together using the relation (3.17) with Lemma 2.5, the preceding expression gives, for all $x_i, z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}$

$$D(r_1, ..., r_{i-1}, [[r_i, x_i]_\alpha, t_i]_\beta, r_{i+1}, ..., r_n)\mathcal{N}D\big(r_1, ..., r_{i-1}, [[[r_i, x_i], t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) = \{0\}.$$

Using the 3-primeness of $\mathcal{N}$, we conclude that

$$\begin{aligned} &D(r_1, ..., r_{i-1}, [[r_i, x_i]_\alpha, t_i]_\beta, r_{i+1}, ..., r_n) = 0 \quad \textit{or} \quad D\big(r_1, ..., r_{i-1}, [[[r_i, x_i], t_i]_\alpha, z_i]_\beta, r_{i+1}, ..., r_n\big) = 0 \\ &\quad \text{for all } x_i, z_i, t_i, r_1, r_2, ..., r_n \in \mathcal{N}. \end{aligned} \tag{3.21}$$

Suppose that there exist $b_i, c_i, a_1, a_2, ..., a_n \in \mathcal{N}$ such that

$$D\big(a_1, ..., a_{i-1}, [[[a_i, b_i], c_i]_\alpha, z_i]_\beta, a_{i+1}, ..., a_n\big) = 0 \text{ for all } z_i \in \mathcal{N}.$$

Thus, $D\big(a_1,...,a_{i-1},[[[a_i,b_i]_\alpha,c_i]_\beta,z_i],a_{i+1},...,a_n\big)=0$ for all $z_i\in\mathcal{N}$, by Lemma 2.6. Now using Lemmas 2.3 and 2.4 together with (3.17), we obtain

$$D\big(a_1,...,a_{i-1},z_i,a_{i+1},...,a_n\big)[[a_i,b_i]_\alpha,c_i]_\beta = [[a_i,b_i]_\alpha,c_i]_\beta D\big(a_1,...,a_{i-1},z_i,a_{i+1},...,a_n\big) \text{ for all } z_i\in\mathcal{N}. \tag{3.22}$$

For $z_i=[[a_i,b_i]_\alpha,c_i]_\beta z_i$ in the left-hand side of the last equation, together using Lemma 2.4 and (3.22), we obtain

$$\begin{aligned}&D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta z_i,a_{i+1},...,a_n\big)[[a_i,b_i]_\alpha,c_i]_\beta\\&=D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta,a_{i+1},...,a_n\big)z_i[[a_i,b_i]_\alpha,c_i]_\beta\\&+[[a_i,b_i]_\alpha,c_i]_\beta^2 D\big(a_1,...,a_{i-1},z_i,a_{i+1},...,a_n\big)\quad\text{for all } z_i\in\mathcal{N}.\end{aligned}$$

On the other hand, putting $[[a_i,b_i]_\alpha,c_i]_\beta z_i$ instead of z_i in the right-hand side of equation (3.22) and using it together with (3.17), we get

$$\begin{aligned}&[[a_i,b_i]_\alpha,c_i]_\beta D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta z_i,a_{i+1},...,a_n\big)\\&=D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta z_i,a_{i+1},...,a_n\big)[[a_i,b_i]_\alpha,c_i]_\beta z_i\\&+[[a_i,b_i]_\alpha,c_i]_\beta^2 D\big(a_1,...,a_{i-1},z_i,a_{i+1},...,a_n\big)\quad\text{for all } z_i\in\mathcal{N}.\end{aligned}$$

From the last two equations, we get

$$\begin{aligned}&D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta,a_{i+1},...,a_n\big)z_i[[a_i,b_i]_\alpha,c_i]_\beta\\&=D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta z_i,a_{i+1},...,a_n\big)[[a_i,b_i]_\alpha,c_i]_\beta z_i\quad\text{for all } z_i\in\mathcal{N}.\end{aligned}\tag{3.23}$$

Replacing z_i by yz_i in (3.23) and using it, we infer that

$$D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta,a_{i+1},...,a_n\big)\mathcal{N}\big[[[a_i,b_i]_\alpha,c_i]_\beta,z_i\big]=\{0\}\text{ for all } z_i\in\mathcal{N}.$$

Using the 3-primeness of $\mathcal{N}$, we conclude that

$$D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta,a_{i+1},...,a_n\big)=0 \text{ or } [[a_i,b_i]_\alpha,c_i]_\beta\in\mathcal{Z}(\mathcal{N}).$$

By the hypothesis, the above expression implies that $D\big(a_1,...,a_{i-1},[[a_i,b_i]_\alpha,c_i]_\beta,a_{i+1},...,a_n\big)=0$. Hence, (3.21) becomes

$$D(r_1,...,r_{i-1},[[r_i,x_i]_\alpha,t_i]_\beta,r_{i+1},...,r_n)=0 \text{ for all } t_i,x_i,r_1,r_2,...,r_n\in\mathcal{N}. \tag{3.24}$$

Substituting $[r_i,x_i]_\alpha t_i$ for t_i in (3.24) and using it, we get

$$D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)[[r_i,x_i]_\alpha,t_i]_\beta=0 \text{ for all } x_i,t_i,r_1,r_2,...,r_n\in\mathcal{N}.$$

Which means that

$$\begin{aligned}&D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)[r_i,x_i]_\alpha\beta(t_i)\\&=D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)\beta(t_i)[r_i,x_i]_\alpha \text{ for all } x_i,t_i,r_1,r_2,...,r_n\in\mathcal{N}.\end{aligned}\tag{3.25}$$

Put $t_i=t_is$ in (3.25) and using it, we get $D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)\beta(t_i)[[r_i,x_i]_\alpha,s]=0$ for all $x_i,s,t_i,r_1,r_2,...,r_n\in\mathcal{N}$. Substituting ut_iv for t_i in the last equation, we obtain

$$D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)\mathcal{N}\beta(t_i)\mathcal{N}[[r_i,x_i]_\alpha,s]=\{0\}\text{ for all } x_i,t_i,s,r_1,r_2,...,r_n\in\mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime and $\beta\neq 0$, the last expression becomes $D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)=0$ or $[r_i,x_i]_\alpha\in\mathcal{Z}(\mathcal{N})$ for all $x_i,r_1,r_2,...,r_n\in\mathcal{N}$. Then, $D(r_1,...,r_{i-1},[r_i,x_i]_\alpha,r_{i+1},...,r_n)=0$ for all $x_i,r_1,r_2,...,r_n\in\mathcal{N}$. Otherwise, by Theorem 3.1, we conclude that $\mathcal{N}$ is a commutative ring.

□

We aim to prove in the example below that the condition of 3-primeness of $\mathcal{N}$ in Theorem 3.3 is essential.

Example 3.1 *Let $\mathcal{S}$ be a commutative left near-ring, and let β and α be any nonzero multipliers.*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid 0, x, y, z \in \mathcal{S} \right\}.$$

Let us define maps $D : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n\text{-times}} \longrightarrow \mathcal{N}$ *as follows:*

$$D\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & z_1 z_2 \dots z_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to verify that D is a nonzero n-derivation, and $D(R_1, ..., R_{i-1}, [[R_i, Y_i]_\alpha, Z_i]_\beta, R_{i+1}, ..., R_n) \in \mathcal{Z}(\mathcal{N})$ for all $Y_i, Z_i, R_1, R_2, ..., R_n \in \mathcal{N}$, but $\mathcal{N}$ is not a commutative ring.

Acknowledgments

The author is very grateful to the anonymous referee for his careful checking.

References

1. H. E. Bell, *On derivations in near-rings II, in Near-rings, Nearfields and K-Loops*, Mathematical of Applications, KluwerAcademic Publishers, Dordrecht, The Netherlands, 426, 191–197, (1997).
2. A. Boua and B. Davvaz, *Left multipliers and commutativity of 3-prime near-ring*, Mathematica, 65(2)(2023), 1222-9016.
3. M. Ashraf and A. Shakir, *On permuting n-derivations in near-rings*, Commun. Korean Math. Soc., 28(4), 697-707, (2013).
4. G. Maksa, *A remark on symmetric biadditive functions having nonnegative diagonalization*, Glas. Mat. Ser. III 15(35), 279–282, (1980).
5. M. A. Oztürk, *Permuting tri-derivations in prime and semi prime rings*, East. Asian. Math. J. 15(2), 177–190, (1999).
6. K. H. Park, *On prime and semi prime rings with symmetric n-derivations*, J.Chungcheong Math. Soc. 22(3), 451–458, (2009).
7. M. A. Ozt ürk and Y. B. Jun, *On trace of symmetric bi-derivations in near-rings*, Int.J. Pure Appl. Math. 17(1), 95–102, (2004).
8. K. H. Park and Y. S. Jung, *On permuting 3-derivations and commutativity in prime near-rings*, Commun. Korean Math. Soc. 25(1), 1–9, (2010).
9. A. Raji, A. Boua and A. Zerbane, *Results on quotient near-rings involving additive map.* J. Algebr. Syst., 13(3), 113-122, (2025).
10. A. Boua, A. Raji, A. Zerbane, *Study of quotient near-rings with additive maps*, Commun. Korean Math. Soc., 39(2), 353-361, (2024).
11. A. Zerbane, S. El Amrani, A. Raji, and A. Boua, *Study of the structure of quotient near-rings with additive maps*, Mathematica, 66(2),1222-9016, (2024).
12. A. Boua, A. Zerbane, Ö. Gölbaşi, *Lie right ideals in* 3*-prime near-rings with generalized homoderivations*, Afr. Mat., 36(2), 99, (2025).

Abdelilah Zerbane,
Department of Mathematics,
Polydisciplinary Faculty, Taza,
Morocco.
E-mail address: abdelilah.zerbane@usmba.ac.ma

and

Abdelkarim Boua,
Department of Mathematics,
Polydisciplinary Faculty, Taza,
Morocco.
E-mail address: abdelkarimboua@yahoo.fr

and

Abderrahmane Raji,
LMACS Laboratory,
Faculty of Sciences and Technology, Beni Mellal,
Morocco.
E-mail address: rajiabd2@gmail.com