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Qualitative Properties of Solutions to a PDE Problem Involving a Singular Potential Term

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ABSTRACT: This paper addresses some properties of solutions to a PDE problem characterized by a singular coefficient

$$\Delta_p U - \alpha x.\nabla U + \frac{|U|^{q-1}U}{|x|^2} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $p > 2$, $q \geq 1$, $N > 2$, and $\alpha > 0$.
The analysis deals with proving the existence of global solutions and characterizing the asymptotic properties of some solutions.

Key Words: Singular potential, global solutions, asymptotic properties, bounded solutions.

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1. Introduction

This research work is dedicated to the examination of the problem below involving a singular coefficient

$$\Delta_p U - \alpha x.\nabla U + \frac{|U|^{q-1}U}{|x|^2} = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

such that $p > 2$, $q \geq 1$, $N > 2$, and $\alpha > 0$.
The idea of studying the elliptic equation (1.1) has been inspired from the parabolic equation written

$$\upsilon_t = \Delta_p \upsilon + |x|^l |\upsilon|^{q-1}\upsilon \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty), \tag{1.2}$$

whose applicability extends to some important model of diffusion [12], and which is associated with a family of solutions [13] called radial self-similar solutions expressed as $\upsilon(x,t) = t^{-\gamma}\upsilon(t^{-\sigma}|x|)$ for which γ and σ are some explicit reals [4]. Due to this kind of solutions and by restricting our analysis to radial solutions, (i.e. functions satisfying $\upsilon(x) = \upsilon(|x|)$), the form of equation (1.2) becomes that of the ODE

$$\left(|\upsilon'|^{p-2}\upsilon'\right)' + \frac{N-1}{r}|\upsilon'|^{p-2}\upsilon' + \gamma\upsilon + \sigma r\upsilon' + r^l|\upsilon|^{q-1}\upsilon = 0 \quad r > 0, \tag{1.3}$$

Several cases of (1.3) have been previously studied in [1,3,5,6,7,8,9,11]. Relevant contributions include for example [14] when $p = 2$ and $l = 0$, and [10] when $-2 < l < 0$.

The case $p > 2$ and $l < 0$ combined with $\gamma < 0$, $\sigma < 0$ is explored in [4], and the case where $l = 0$ is adressed in [2].

Moreover, to the best of our knowledge, the case where $\gamma = 0$ has never been studied. Then by setting $\gamma = 0$, $l = -2$ and noting σ by $-\alpha$ with $\alpha > 0$, equation (1.2) reduces to (1.1).

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To rigorously investigate the radial solutions of (1.1), we restrict our attention to functions that are continuous at the origin and satisfying problem

$$(P)\begin{cases} \left(|\upsilon'|^{p-2}\upsilon'\right)' + \dfrac{N-1}{r}|\upsilon'|^{p-2}\upsilon' - \alpha r\upsilon' + \dfrac{|\upsilon|^{q-1}\upsilon}{r^2} = 0, \quad r>0, \\ \upsilon(0)=\xi, \quad \upsilon'(0)=0 \end{cases} \tag{1.4}$$

for which we denote $p>2$, $q\geq 1$, $N>2$, $\alpha>0$, and $\xi\in\mathbb{R}^*$.
Due to the symmetry $\upsilon(\cdot,\xi,\alpha)=-\upsilon(\cdot,-\xi,\alpha)$, it suffices to restrict our attention to the case $\xi>0$.

We aim to explore the classification of solutions to problem (P), with particular attention to their boundedness. Since $\lim_{r\to 0} r^2\left(|\upsilon'|^{p-2}\upsilon'\right)'(r) = -\xi^q$, we define an *entire solution*, as a function υ on $[0,+\infty[$ such that

$$\upsilon\in C^0([0,+\infty[)\cap C^1([0,+\infty[), \quad |\upsilon'|^{p-2}\upsilon'\in C^1(]0,+\infty[),$$

and which satisfies the problem (P).

The subsequent sections are detailed below. Section 2 addresses the existence of entire solutions. Section 3 shows that under a suitable initial data, these solutions remain strictly positive. Section 4 investigates their asymptotic behavior near infinity.

2. Existence of Entire Solutions

Next, we proceed by showing that problem (P) admits global solutions.

Theorem 2.1 *For each $\xi>0$, the problem (P) admits a unique entire solution $\upsilon=\upsilon_\xi$.*

Proof: Step 1: Local solutions.

Take υ a solution to problem (P), which is considered on the interval $[0,r_{\max}[$. Therefore, for each $r\in[0,r_{\max}[$, υ satisfies

$$\left[r^{N-1}|\upsilon'|^{p-2}\upsilon'(r) - \alpha r^N\upsilon(r)\right]' = -\alpha N r^{N-1}\upsilon(r) - r^{N-3}|\upsilon|^{q-1}\upsilon(r), \tag{2.1}$$

an integration over $(0,r)$ of (2.1) yields

$$|\upsilon'(r)|^{p-2}\upsilon'(r) = \alpha r\upsilon(r) - \frac{\alpha N}{r^{N-1}}\int_0^r s^{N-1}\upsilon(s)\,ds - \frac{1}{r^{N-1}}\int_0^r s^{N-3}|\upsilon(s)|^{q-1}\upsilon(s)\,ds. \tag{2.2}$$

Define

$$\Theta[f](s) = -\alpha s f(s) + s^{1-N}\int_0^s z^{N-1}\left[\alpha N + z^{-2}|f|^{q-1}(z)\right]f(z)\,dz \tag{2.3}$$

and

$$\Psi(s) = |s|^{(2-p)/(p-1)}s, \quad s\in\mathbb{R}. \tag{2.4}$$

Thus from (2.2), (2.3) and (2.4)

$$\upsilon' = -|\Theta[\upsilon]|^{\frac{2-p}{p-1}}\Theta[\upsilon] = -\Psi(\Theta[\upsilon]). \tag{2.5}$$

and then by integrating (2.2) over $(0,r)$ we derive

$$\upsilon(r) = \xi - \int_0^r \Psi(\Theta[\upsilon](s))\,ds. \tag{2.6}$$

Let $\delta>0, \xi>\eta>0$ and take the space

$$E_{\xi,\eta,\delta} = \left\{f\in C([0,\delta]) : \|f-\xi\|_0 \leq \eta\right\}, \tag{2.7}$$

which is metric and where $C([0,\delta])$ is the complete space of functions considered on $[0,\delta]$ endowed with the uniform norm $\|\cdot\|_0$.
Consider now the operator Λ on $E_{\xi,\eta,\delta}$ noted as

$$\Lambda[f](r) = \xi - \int_0^r \Psi(\Theta[f](s))ds. \tag{2.8}$$

In order to use fixed point theorem of Banach, we need to prove that Λ is a contraction from $E_{\xi,\eta,\delta}$ into itself for small δ.

Substep 1: Λ is a self-mapping on $E_{\xi,\eta,\delta}$ when η and δ are small enough.
Take $\alpha > 0$ and $f(r) \in [\xi-\eta, \xi+\eta]$, we infer directly that for $s \in]0,\delta]$,

$$\Theta[f](s) \leq \left[\frac{(\xi+\eta)^q}{N-2} + \alpha(\xi+\eta)s^2\right] s^{-1},$$

we choose

$$\delta \leq \left(\frac{(\xi+\eta)^{q-1}}{\alpha(N-2)}\right)^{1/2},$$

then, for any $s \in]0,\delta]$, we get

$$\Theta[f](s) \leq 2\frac{(\xi+\eta)^q}{N-2} s^{-1}. \tag{2.9}$$

Now, combining (2.4) and (2.8), we get

$$|\Lambda[f](r) - \xi| \leq \int_0^r |\Theta[f](s)|^{1/(p-1)} ds, \qquad \text{for every } r \in [0,\delta].$$

Considering (2.9), it yields that for $r \in [0,\delta]$

$$|\Lambda[f](r) - \xi| \leq \frac{p-1}{p-2}\left(\frac{2(\xi+\eta)^q}{N-2}\right)^{1/(p-1)} r^{(p-2)/(p-1)}.$$

Thus, by taking δ sufficiently small, it yields

$$|\Lambda[f](r) - \xi| \leq \eta \qquad \text{for } f \in E_{\xi,\eta,\delta}.$$

Substep 2: Now, we will show that Λ is a contraction from $E_{\xi,\eta,\delta}$ into itself for small δ.
To estimate $\Theta[f]$, we make use of equation (2.3), yielding

$$\Theta[f](s) \geq \left[\frac{(\xi-\eta)^q}{N-2} - 2\alpha\eta s^2\right] s^{-1},$$

then if we choose

$$\delta \leq \left(\frac{(\xi-\eta)^q}{4\alpha\eta(N-2)}\right)^{1/2},$$

we get for any $s \in]0,\delta]$

$$\Theta[f](s) \geq \frac{(\xi-\eta)^q}{2(N-2)} s^{-1}.$$

For any $r \in [0,\delta]$ and any $f, g \in E_{\xi,\eta,\delta}$, we have

$$|\Lambda[f](r) - \Lambda[g](r)| \leq \int_0^r |\Psi(\Theta[f](s)) - \Psi(\Theta[g](s))\,|\,ds.$$

Next, we put

$$\chi(s) = \min(|\Psi[f](s)|, |\,\Psi[g](s)]).$$

Then

$$|\Lambda[f](r)-\Lambda[g](r)| \leq \int_0^r (\chi(s))^{(2-p)/(p-1)} |\Theta[f](s)-\Theta[g](s)| ds.$$

We use (2.3) and (2.7) to get

$$|\Theta[f](s)-\Theta[g](s)| \leq \left[2\alpha s + \frac{q(\xi+\eta)^{q-1}}{N-2} s^{-1}\right] \|f-g\|_0. \tag{2.10}$$

Consequently,

$$|\Lambda[f](r)-\Lambda[g](r)| \leq \left(\frac{(\xi-\eta)^q}{2(N-2)}\right)^{\frac{2-p}{p-1}} \int_0^r s^{\frac{p-2}{p-1}} |\Theta[f](s)-\Theta[g](s)| ds,$$

hence, using (2.10), we get

$$|\Lambda[f](r)-\Lambda[g](r)| \leq \left[K_1 r^{\frac{3p-4}{p-1}} + K_2 r^{\frac{p-2}{p-1}}\right] \|f-g\|_0,$$

where

$$K_1 = \frac{2\alpha(p-1)}{3p-4} \left(\frac{(\xi-\eta)^q}{2(N-2)}\right)^{(2-p)/(p-1)},$$

and

$$K_2 = \frac{q(\xi+\eta)^{q-1}(p-1)}{(p-2)(N-2)} \left(\frac{(\xi-\eta)^q}{2(N-2)}\right)^{(2-p)/(p-1)}.$$

Then, by taking r sufficiently small, the operator Λ becomes a contraction.
Step 2: Existence of global solution.
We consider υ a solution of problem (P) over the interval $[0, r_{\max}[$. Let us admit, to reach a contradiction, that $r_{max} < +\infty$. Then,

$$\lim_{r\to r_{max}} |\upsilon(r)| = \lim_{r\to r_{max}} |\upsilon'(r)| = +\infty. \tag{2.11}$$

For any $r \in [0, r_{max}[$, introduce the function defined as

$$H(r) = \frac{p-1}{p}|\upsilon'(r)|^p + \frac{1}{q+1} r^{-2} |\upsilon(r)|^{q+1}. \tag{2.12}$$

Since $p > 2$ and $q \geq 1$, then $\lim_{r\to r_{max}} H(r) = +\infty$.
Multipliying (1.4) by υ', we get

$$\frac{p-1}{p}(|\upsilon'|^p)' + \frac{N-1}{r}|\upsilon'|^p - \alpha r \upsilon'^2 + \frac{(r^{-2}|\upsilon|^{q+1})'}{q+1} + \frac{2|\upsilon|^{q+1}}{(q+1)r^3} = 0.$$

and then derive from (2.12)

$$H'(r) = -\left(\frac{N-1}{r}|\upsilon'(r)|^p - \alpha r \upsilon'^2(r) + \frac{2}{q+1} r^{-3} |\upsilon(r)|^{q+1}\right). \tag{2.13}$$

Using (2.11) and taking into account $N > 2$, $p > 2$ and $\alpha > 0$, we get $\lim_{r\to r_{max}} H'(r) = -\infty$, wich is impossible. □

3. Positive Solutions

In this section, we highlight an important property, by establishing the strict positivity of solutions under appropriate conditions.

Theorem 3.1 *Assume that $q > 1$. Under appropriately small initial condition ξ, the solution of Problem (P) maintains strict positivity throughout $(0, +\infty)$.*

Proof: Take the function Q as

$$Q(r) = \frac{p-1}{p} |\upsilon'|^p + \frac{\alpha}{2}\upsilon^2 - \frac{1}{q+1} r^{-2}|\upsilon|^{q+1}, \tag{3.1}$$

using (1.4), we get

$$\left(\frac{p-1}{p}|\upsilon'|^p\right)' = -\frac{N-1}{r}|\upsilon'|^p + \alpha r \upsilon'^2 - r^{-2}|\upsilon|^{q-1}\upsilon\upsilon'.$$

Since

$$\left(\frac{r^{-2}|\upsilon|^{q+1}}{q+1}\right)' = -\frac{2r^{-3}|\upsilon|^{q+1}}{q+1} + r^{-2}|\upsilon|^{q-1}\upsilon\upsilon',$$

we obtain

$$Q'(r) = -\frac{N-1}{r}|\upsilon'(r)|^p + \alpha r\left(\upsilon'(r)\right)^2 - 2r^{-2}|\upsilon(r)|^{q-1}\upsilon(r)\upsilon'(r) + \alpha\upsilon(r)\upsilon'(r) + \frac{2}{q+1}r^{-3}|\upsilon(r)|^{q+1}. \tag{3.2}$$

Since $\upsilon(0) = \xi > 0$, then υ is either always positive on $(0, +\infty)$, or admits a first zero. Suppose there exists $r_0 > 0$ the first zero of υ. Since υ is continuous, then $\upsilon'(r_0) \le 0$ and there exists a left neighborhood $(r_0 - \varepsilon, r_0)$ with $\varepsilon > 0$ such that $\upsilon(r) > 0$ and $\upsilon'(r) < 0$ for all $r \in (r_0 - \varepsilon, r_0)$.
Aiming for a contradiction, suppose $\upsilon'(r_0) = 0$. Take the function

$$g_1(r) = r^{N-1}|\upsilon'|^{p-2}\upsilon'(r) - \alpha r^N \upsilon(r). \tag{3.3}$$

By virtue of (1.4) it yields that

$$g_1'(r) = -r^{N-1}\upsilon(r)\left(\alpha N + r^{-2}|\upsilon|^{q-1}\right). \tag{3.4}$$

Since, $\upsilon > 0$ on $(0, r_0)$, we have $g_1'(r) < 0$ on $(0, r_0)$. Therefore

$$g_1(r) > g_1(r_0) = 0 \quad \text{for every} \quad r \in (r_0 - \varepsilon, r_0).$$

It follows that $\upsilon'(r) > 0$ for any $r \in (r_0 - \varepsilon, r_0)$. However, thereby reaching a contradiction to $\upsilon'(r) \le 0$ for which $r \in (r_0 - \varepsilon, r_0)$. Thus, $\upsilon'(r_0) < 0$. Therefore, $Q(r_0) > 0$.

Now, we show that there exists $\rho \in (0, r_0)$ satisfying $Q(\rho) = 0$ and $Q'(\rho) \ge 0$. It's clear that Q can also take the form

$$Q(r) = r^{-2}|\upsilon|^{q+1}\left[\frac{-1}{q+1} + \frac{\alpha}{2}r^2|\upsilon|^{1-q} + \frac{p-1}{p}r^2|\upsilon|^{-q-1}|\upsilon'|^p\right]. \tag{3.5}$$

Hence $\lim_{r\to 0} r^2 Q(r) = \dfrac{-\xi^{q+1}}{q+1}$ and then $Q(r) < 0$ for r small enough. Combinning that with $Q(r_0) > 0$ allows the existence of $\rho \in (0, r_0)$ the first zero of Q, with the property that $Q(\rho) = 0$ and $Q'(\rho) \ge 0$.
By equation (3.2) we get

$$Q'(\rho) = \mu_1\rho^{-1}\upsilon^2(\rho) - \frac{\mu_2}{q+1}\rho^{-3}\upsilon^{q+1}(\rho) + \alpha\rho\upsilon'^2(\rho) + \alpha\upsilon(\rho)\upsilon'(\rho) - 2\rho^{-2}\upsilon^q(\rho)\upsilon'(\rho),$$

where

$$\mu_1 = \frac{\alpha p(N-1)}{2(p-1)} > 0 \quad \text{and} \quad \mu_2 = \frac{p(N-1)}{p-1} - 2 > 0.$$

Let's show that $Q'(\rho) < 0$ when ξ is sufficiently small. In fact, from $\upsilon(\rho) > 0$ and $\upsilon'(\rho) < 0$, it results that

$$Q'(\rho) < \rho^{-3}\upsilon^{q+1}(\rho)\left[\frac{-\mu_2}{q+1} + \mu_1\rho^2\upsilon^{1-q}(\rho) + \alpha\rho^4\frac{\upsilon'^2(\rho)}{\upsilon^{q+1}(\rho)} + 2\rho\frac{|\upsilon'(\rho)|}{\upsilon(\rho)}\right]. \tag{3.6}$$

As $Q(\rho) = 0,\ |\upsilon'(\rho)|^p > 0$ and due to (3.1) of $Q(\rho)$, it follows that

$$-\frac{\alpha}{2}\upsilon^2(\rho) + \frac{1}{q+1}\rho^{-2}|\upsilon|^{q+1}(\rho) > 0.$$

Since $0 < \upsilon(\rho) < \upsilon(0) = \xi$, then

$$\rho = \rho(\xi) < \left(\frac{2}{\alpha(q+1)}\right)^{1/2}\upsilon^{(q-1)/2}(\rho) < \left(\frac{2}{\alpha(q+1)}\right)^{1/2}\xi^{(q-1)/2}.$$

Therefore, $\lim\limits_{\xi\to 0}\rho(\xi) = 0$. The combination of this estimate with $\lim\limits_{r\to 0} r^2\upsilon^{1-q}(r) = 0$ and also using $\lim\limits_{r\to 0} r\upsilon'(r) = 0$, yields

$$\lim_{\xi\to 0}\rho^2\upsilon^{1-q}(\rho) = 0, \quad \lim_{\xi\to 0}\frac{\rho\,|\upsilon'(\rho)|}{\upsilon(\rho)} = 0,$$

and

$$\lim_{\xi\to 0}\rho^4\frac{\upsilon'^2(\rho)}{\upsilon^{q+1}(\rho)} = \lim_{\xi\to 0}\left(\frac{\rho\upsilon'(\rho)}{\upsilon(\rho)}\right)^2\rho^2\upsilon^{1-q}(\rho) = 0.$$

Moreover, combining these results with (3.6) and the positivity of μ_2, we conclude that for sufficiently small ξ we have $Q'(\rho) < 0$ which contradicts the inequality $Q'(\rho) \geq 0$. □

4. Asymptotic Behavior Near Infinity

We noticed that the positivity of the solution, together with its decrease, ensures the boundedness of solutions to (P), which is, in turn, of primary importance in the consideration of their asymptotic behavior.
We describe below the asymptotic properties of strictly positive solutions υ to problem (P).

Theorem 4.1 *Suppose $N > p$. Thus,*

$$\lim_{r\to+\infty} r^{\frac{N-p}{p-1}}\upsilon(r) = +\infty. \tag{4.1}$$

The proof of this result relies on the two propositions below.

Proposition 4.1 *The following statements hold*

i) $\upsilon'(r) < 0$ *for any* $r > 0$.

ii) $\lim\limits_{r\to+\infty}\upsilon(r) \in [0, +\infty[$ *and* $\lim\limits_{r\to+\infty}\upsilon'(r) = 0$.

Proof: i) Multipliying (1.4) by r^2 and using the fact that $\upsilon(0) = \xi$ and $\upsilon'(0) = 0$, we get $\lim\limits_{r\to 0} r^2\left(|\upsilon'|^{p-2}\upsilon'\right)'(r) = -\xi^q$, thus $\upsilon'(r) < 0$ for small r.
Hence, either υ stays strictly negative for all $r > 0$ or υ' vanishes in a certain point. Let's proceed by contradiction, assume that υ' has a first zero at r_1. It follows from the continuity of υ' that $(|\upsilon'|^{p-2}\upsilon')'(r_1) \geq 0$, which results in a contradiction with $(|\upsilon'|^{p-2}\upsilon')'(r_1) = -r_1^{-2}|\upsilon|^{q-1}\upsilon(r_1) < 0$ by equation (1.4). Thus, it results in $\upsilon'(r) < 0$ for every $r > 0$.
ii) Due to the positivity of υ and its decreasing nature, we have $\lim\limits_{r\to+\infty}\upsilon(r) \in [0, +\infty[$. Consequently, $\upsilon'(r)$ necessarily tends to 0 as $r \to +\infty$. □

Proposition 4.2 *For every $c > 0$, $r^c \upsilon(r)$ is strictly increasing for large r.*

Proof: To show that $r^c\upsilon(r)$ is strictly monotone, we will study the sign of a function denoted by $\Phi_c(r)$, and defined as

$$\Phi_c(r) = c\upsilon(r) + r\upsilon'(r), \qquad \text{for every } c \neq 0 \text{ and } \quad r > 0. \tag{4.2}$$

Thus

$$(r^c\upsilon(r))' = r^{c-1}\Phi_c(r), \quad r > 0, \tag{4.3}$$

and if $\upsilon'(r) \neq 0$, then

$$(p-1)|\upsilon'|^{p-2}(r)\Phi_c'(r) = (p - N + c(p-1))|\upsilon'|^{p-2}\upsilon'(r) + \alpha r^2\upsilon'(r) - r^{-1}\upsilon^q(r). \tag{4.4}$$

Thus, if $\Phi_c(r_0) = 0$ for some $r_0 > 0$, (1.4) results in

$$\begin{aligned}(p-1)|\upsilon'|^{p-2}(r_0)\Phi_c'(r_0) = -r_0\upsilon(r_0)\left[\alpha c + r_0^{-2}\upsilon^{q-1}(r_0)\right.\\ \left. + (p - N + c(p-1))\,|c|^{p-2}cr_0^{-p}\upsilon^{p-2}(r_0)\right].\end{aligned} \tag{4.5}$$

Suppose that $c \geq \dfrac{N-p}{p-1}$. By Proposition 4.1, we deduce $\upsilon'(r) < 0$ for every $r > 0$. Then, from equation (4.4), we obtain

$$(p-1)\frac{\Phi_c'(r)}{\upsilon'(r)} = (p - N + c(p-1)) + \frac{\alpha r^2}{|\upsilon'(r)|^{p-2}} + \frac{r^{-1}\upsilon^q(r)}{|\upsilon'(r)|^{p-1}}. \tag{4.6}$$

Subsequently $\Phi_c'(r) < 0$ for every $r > 0$.
Since $\Phi_c(0) = c\upsilon(0) > 0$, assume that there exists $r_0 > 0$ the first zero of Φ_c for a contradiction argument. In this case, $\Phi_c(r) < \Phi_c(r_0) = 0$ for all $r > r_0$. Consequently, by (4.3) the quantity $r^c\upsilon(r)$ converges to a finite value as $r \to +\infty$, and $\lim\limits_{r\to+\infty} \upsilon(r) = 0$. Moreover, the decrease of Φ_c allows that $\lim\limits_{r\to+\infty} \Phi_c(r) \in [-\infty, 0[$. Due to (4.2) it results that $\lim\limits_{r\to+\infty} r\upsilon'(r) \in [-\infty, 0[$, which is not possible because υ is positive. Then $\Phi_c(r) > 0$ for every $r > 0$ and $r^c\upsilon(r)$ increases strictly.

Suppose now that $c < \dfrac{N-p}{p-1}$. We show that $\Phi_c(r) \neq 0$ for large r. Let r_0 the first zero of Φ_c. As υ converges and $\alpha > 0$ then by (4.5), $\Phi_c'(r) < 0$ for large r. Hence $\Phi_c(r) \neq 0$ for large r.
Assume now for the sake of contradiction, that $\Phi_c(r) < 0$ for r large, yielding

$$\frac{\upsilon(r)}{|r\upsilon'(r)|} < \frac{1}{c} \quad \text{ for large } r. \tag{4.7}$$

Due to equation (1.4) we have

$$\left(|\upsilon'|^{p-2}\upsilon'\right)'(r) = r|\upsilon'(r)|\left[-\alpha + \frac{N-1}{r^2}|\upsilon'(r)|^{p-2} - \frac{r^{-3}|\upsilon(r)|^{q-1}\upsilon(r)}{|\upsilon'(r)|}\right].$$

Then, by (4.7) $\lim\limits_{r\to+\infty} \dfrac{|\upsilon(r)|^{q-1}\upsilon(r)}{r^3|\upsilon'(r)|} = 0$, and as $\upsilon > 0$ and $\upsilon'(r) < 0$, it yields that

$$\left(|\upsilon'|^{p-2}\upsilon'\right)'(r) < r|\upsilon'(r)|\left[-\alpha + \frac{N-1}{r^2}|\upsilon'|^{p-2}\right] < 0 \quad \text{for large } r.$$

Thus, $|\upsilon'(r)|^{p-2}\upsilon'(r)$ is a function that decreases, and that is also a negative function, then $\lim\limits_{r\to+\infty} \upsilon'(r) \in [-\infty, 0[$, which is contradictory to Proposition 4.1. Then, $\Phi_c(r) > 0$ with r large, and so $r^c\upsilon(r)$ is strictly increasing for large r. □

We hence proceed to prove Theorem 4.1.

Proof: By Proposition 4.2, it results that $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}}\upsilon(r) \in]0,+\infty]$.
Suppose for contradiction that $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}}\upsilon(r) = \ell > 0$. Introduce this logarithmic change

$$v_1(t) = r^{\frac{N-p}{p-1}}\upsilon(r), \quad \text{by taking } t = \ln(r). \tag{4.8}$$

Then, the function v_1 satisfies

$$w_1'(t) - \alpha e^{Kt}h_1(t) + e^{Mt}v_1^q(t) = 0, \tag{4.9}$$

such that

$$h_1(t) = v_1'(t) - \frac{N-p}{p-1}v_1(t) = r^{\frac{N-1}{p-1}}\upsilon'(r) \quad \text{and} \quad w_1(t) = |h_1|^{p-2}h_1(t), \tag{4.10}$$

$$K = \frac{N-p}{p-1}(p-2) + p \quad \text{and} \quad M = p-2-\frac{N-p}{p-1}(q+1-p).$$

It's obvious that from (4.10), we have $v_1'(t) = r^{\frac{N-p}{p-1}}\Phi_{\frac{N-p}{p-1}}(r)$ where $\Phi_{\frac{N-p}{p-1}}(r) = \frac{N-p}{p-1}\upsilon(r) + r\upsilon'(r)$. Then by Proposition (4.2), $v_1'(t) > 0$ for large t. The convergence of v_1 and its strictly increasing nature ensures that $\lim_{t\to+\infty} v_1'(t) = 0$. Thus, due to (4.10) we find

$$\lim_{t\to+\infty} h_1(t) = \frac{p-N}{p-1}\ell, \tag{4.11}$$

therefore

$$\lim_{t\to+\infty} w_1(t) = -\left(\frac{N-p}{p-1}\right)^{p-1}\ell^{p-1}. \tag{4.12}$$

We also have $w_1'(t) < \alpha e^{Kt}h_1(t)$ due to (4.11). However, since $K > 0$ and using (4.12), we have $\lim_{t\to+\infty} w_1'(t) = -\infty$. It follows that $\lim_{t\to+\infty} w_1(t) = -\infty$, which is impossible in view of (4.12). Hence $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}}\upsilon(r) = +\infty$. □

Theorem 4.2 *Assume that $p-1 < q$. Then, $\lim_{r\to+\infty} r^{\frac{p-2}{q+1-p}}\upsilon(r) = L > 0$ with $L > 0$.*

Proof: We begin by proving that $r^{\frac{p-2}{q+1-p}}\upsilon(r)$ remains bounded.
Using (2.1), we deduce that for every $r > 0$ it holds that

$$(r^{N-1}|\upsilon'(r)|^{p-2}\upsilon'(r))' < -r^{N-3}\upsilon^q(r). \tag{4.13}$$

Integrating (4.13) over $(0,r)$ for every $r > 0$, it gives

$$r^{N-1}|\upsilon'(r)|^{p-2}\upsilon'(r) < -\int_0^r s^{N-3}\upsilon^q(s)\,ds. \tag{4.14}$$

Given that $\upsilon'(r) < 0$, we derive

$$r^{N-1}|\upsilon'(r)|^{p-2}\upsilon'(r) < -\frac{1}{N-2}r^{N-2}\upsilon^q(r).$$

Hence,

$$\upsilon'(r) < -\left(\frac{1}{N-2}\right)^{\frac{1}{p-1}} r^{\frac{-1}{p-1}}\upsilon^{\frac{q}{p-1}}(r).$$

As $\upsilon(r) > 0$ and $p-1 < q$, then

$$\frac{p-1}{p-q-1}\left(\upsilon^{\frac{p-q-1}{p-1}}(r)\right)' < -\left(\frac{1}{N-2}\right)^{\frac{1}{p-1}} r^{\frac{-1}{p-1}}.$$

Thus,

$$\left(\upsilon^{\frac{p-q-1}{p-1}}(r)\right)' > \frac{q+1-p}{p-1}\left(\frac{1}{N-2}\right)^{\frac{1}{p-1}} r^{\frac{-1}{p-1}}.$$

By an integration over $(0,r)$ for $r>0$, we get

$$\upsilon^{\frac{p-q-1}{p-1}}(r) > \frac{q+1-p}{p-2}\left(\frac{1}{N-2}\right)^{\frac{1}{p-1}} r^{\frac{p-2}{p-1}}.$$

Consequently,

$$\upsilon(r) \leq M\, r^{-\frac{p-2}{q+1-p}}, \tag{4.15}$$

where $M = \left[\frac{q+1-p}{p-2}\left(\frac{1}{N-2}\right)^{\frac{1}{p-1}}\right]^{-\frac{(p-1)}{q+1-p}}$.

Hence, $r^{\frac{p-2}{q+1-p}}\upsilon(r)$ remains bounded. Due to Proposition 4.2, we know that $r^{\frac{p-2}{q+1-p}}\upsilon(r)$ is strictly increasing. This allows the existence of $L>0$ satisfying $\lim_{r\to+\infty} r^{\frac{p-2}{q+1-p}}\upsilon(r) = L$. □

5. Discussion

In this work, beyond establishing the existence of solutions to problem (P), an important feature of our analysis is the existence of $\xi_0 > 0$ such that for any $\xi \in (0,\xi_0)$, the solution $\upsilon = \upsilon_\xi$ becomes strictly positive. We were also able to derive the asymptotic behavior of the function υ, in particular, in the presence of the condition $q > p-1$, where we established an equivalent of υ for large r, providing further insight into its long-term dynamics.

Moreover, the nature of the equivalent of υ for large r in the case $q \leq p-1$ remains an open question, and this issue will be addressed in future research directions.

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