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Variations on Quasi Cauchy Double Sequences

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Abstract: The notion of a p quasi Cauchy double sequence is introduced and investigated. A double sequence $\{x_{k,l}\}$ is called p-quasi-Cauchy if given an $\epsilon > 0$ there exists an $n_0 \in \mathbf{N}$ such that

$$\max_{r,s=1 \text{ and/or } 0} \{|x_{k,l} - x_{k+p+r,l+p+s}|\} < \epsilon$$

whenever $k, l > n_0$. We study compactness types of theorems of a double subset $A \times A$ of $\mathbf{R}^2$, and continuity type properties of factorable double functions defined on a double subset $A \times A$ of $\mathbf{R}^2$ into $\mathbf{R}$, and obtain interesting results related to uniform continuity, sequential continuity, continuity, compactness, and a newly introduced type of continuity of factorable double functions defined on a double subset $A \times A$ of $\mathbf{R}^2$ into $\mathbf{R}$.

Keywords: Double sequences, P -convergent, compactness, continuity.

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1. Introduction

Pringsheim ([25]) introduced the concept of convergence of real double sequences. Four years later, Hardy ([19]) improved the convergence of real double sequences, introducing the notion of regular convergence for double sequences in the sense that double sequence has a limit in Pringsheim's sense and has one sided limits which guaranties limit of a double sequence in the first variable when the second is fixed, vice versa (see also [26,18]). A considerable number of papers which appeared in recent years study double sequences from various points of view (see [14,16,20,22,23,24]). Some results in the investigation are generalizations of known results concerning simple sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (e.g., the fact that a double sequence may converge without being bounded).

Using the idea of continuity of a real function in terms of single and double sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: double slowly oscillating continuity ([14]), double ward continuity ([23]), slowly oscillating continuity ([3]), quasi-slowly oscillating continuity, Δ-quasi-slowly oscillating continuity ([15], [5], and [4]), ward continuity, ([9]), δ-ward continuity, ([6]), p-ward continuity ([13]), statistical ward continuity, ρ statistical ward continuity ([11]), lacunary statistical ward continuity, ([7] and [8]). Investigation of some of these kinds of continuities lead some authors to find certain characterizations of uniform continuity of a real function in terms of sequences in the above manner ([23, Theorem 3.7], [14, Theorem 3.5], [27, Theorem 8], [7, Theorem 6], [2, Theorem 1], [10, Theorem 3.8], [13, Corollary 3.10, and Corollary 3.13]). Recently, two concepts of continuity in terms of double sequences, namely functions that preserve quasi Cauchy double sequences and functions that preserve slowly oscillating double sequences, are introduced and studied in [23] and [14], respectively.

The aim of this paper is to introduce p-quasi-Cauchy double sequences, and investigate newly defined types of compactness, and types of continuities for factorable double functions.

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2. Preliminaries

In this section, we recall main definitions.

Definition 2.1 ([25]) A double sequence $\mathbf{x} = \{x_{k,l}\}$ is Cauchy provided that, given an $\epsilon > 0$ there exists an $n_0 \in \mathbf{N}$ such that $|x_{k,l} - x_{s,t}| < \epsilon$ whenever $k, l, s, t > n_0$.

Definition 2.2 ([25]) A double sequence $\mathbf{x} = \{x_{k,l}\}$ has a **Pringsheim limit** L (denoted by P-$\lim x = L$) provided that, given an $\epsilon > 0$ there exists an $n_0 \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n_0$. Such an $\mathbf{x}$ is described more briefly as "P-convergent".

If $\lim |\mathbf{x}| = \infty$, (equivalently, for every $\varepsilon > 0$ there are $n_1, n_2 \in N$ such that $|x_{m,n}| > M$ whenever $m > n_1$, $n > n_2$), then $\mathbf{x} = \{x_{m,n}\}$ is said to be definitely divergent. A double sequence $\mathbf{x} = \{x_{m,n}\}$ is bounded if there is an $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in \mathbf{N}$. Notice that a P-convergent double sequence need not be bounded.

Definition 2.3 ([21]) A double sequence $\mathbf{y}$ is a **double subsequence** of $\mathbf{x} = (\mathbf{x_{nk}})$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then $\mathbf{y}$ is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

Definition 2.4 ([23]) A factorable double function f defined on a double subset $A \times A$ of $\mathbf{R}^2$ into $\mathbf{R}$ is double sequentially continuous at a point L of $A \times A$ if $f(\mathbf{x})$ is P-convergent to $f(L)$ whenever $\mathbf{x} = \{x_{k,l}\}$ is a P-convergent double sequence of points in $A \times A$ with P-limit L. If f is double sequentially continuous at every point of $A \times A$, we say f is double sequentially continuous on $A \times A$.

3. Main Results

In the definition of a quasi double sequence, considering p-th forward difference instead of 1th-forward difference we introduce the following definition.

Definition 3.1 Let p be a constant positive integer. A double sequence $\mathbf{x} = \{x_{k,l}\}$ is called p quasi-Cauchy if each $\epsilon > 0$ there exists an $n_0 \in \mathbf{N}$ such that

$$\max_{r,s=1 \text{ and/or } 0} \{|x_{k,l} - x_{k+p+r,l+p+s}|\} < \epsilon$$

whenever $k, l > n_0$.

Using the following equality
$x_{k,l} - x_{k+p+r,l+p+s} = (x_{k,l} - x_{k+1+r,l+1+s}) + (x_{k+1+r,l+1+s} - x_{k+2+r,l+2+s}) + (x_{k+2+r,l+2+s} - x_{k+3+r,l+3+s}) + -\ldots - x_{k+p-1+r,l+p-1+s} + (x_{k+p-1+r,l+p-1+s} - x_{k+p+r,l+p+s})$
we obtain the following inequality
$|x_{k,l} - x_{k+p+r,l+p+s}| \leq |x_{k,l} - x_{k+1+r,l+1+s}| + |x_{k+1+r,l+1+s} - x_{k+2+r,l+2+s}| + \cdots + |x_{k+p-1+r,l+p-1+s} - x_{k+p+r,l+p+s}|$.

Hence
$\max_{r,s=1 \text{ and/or } 0}\{|x_{k,l} - x_{k+p+r,l+p+s}|\} \leq \max_{r,s=1 \text{ and/or } 0}\{|x_{k,l} - x_{k+1+r,l+1+s}|\}$
$+ \max_{r,s=1 \text{ and/or } 0}\{|x_{k+1+r,l+1+s} - x_{k+2+r,l+2+s}|\}$
$+ \ldots + \max_{r,s=1 \text{ and/or } 0}\{|x_{k+p-1+r,l+p-1+s} - x_{k+p+r,l+p+s}|\}\}$.

We see from this inequality that any quasi Cauchy double sequence is p quasi Cauchy double sequence for any positive integer p. Furthermore if there exist positive integers m and q such that $p = m.q$, either m quasi Cauchyness or q quasi Cauchness implies p quasi Cauchyness. Any P-convergent double sequence is p-quasi-Cauchy, so any regularly convergent double sequence is p-quasi-Cauchy. Any Cauchy double sequence is p-quasi-Cauchy. Recalling that a double sequence $\{x_{k,l}\}$ of real numbers is called slowly oscillating if for any given $\varepsilon > 0$, there exist $\alpha = \alpha(\varepsilon)$, $\delta = \delta(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon)$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$ if $k, l \geq n_0(\varepsilon)$ and $k \leq s \leq (1+\alpha)k$, $l \leq t \leq (1+\delta)l$, we see that any slowly oscillating double sequence is p-quasi-Cauchy for any constant positive integer p. Any subsequence of a P-convergent double sequence is P-convergent. Any subsequence of a Cauchy double sequence is Cauchy, and subsequence of a slowly oscillating double sequence is slowly oscillating double sequence. But situation is different for p-quasi-Cauchy double sequences. There are subsequences of a p-quasi-Cauchy double sequence which are not p-quasi-Cauchy.

Example 3.1 *Write $s_n = log(n+p)$ for each positive integer n. Then the double sequence defined by*

$$\begin{array}{ccccc}
s_1 & s_2 & s_3 & s_4 & \cdots \\
s_2 & s_2 & s_3 & s_4 & \cdots \\
s_3 & s_3 & s_3 & s_4 & \cdots \\
s_4 & s_4 & s_4 & s_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is not P-convergent nor Cauchy, however it is a p-quasi-Cauchy double sequence for any constant $p \in \mathbf{N}$. In addition this double sequence has subsequences that are not p quasi-Cauchy at all.

Theorem 3.1 *If a factorable double function f defined on a double subset $A \times A$ of $\mathbf{R}^2$ preserves factorable double p quasi-Cauchy sequences from $A \times A$, then it is continuous.*

Proof: Suppose that f preserves factorable double p quasi-Cauchy sequences from $A \times A$. Let $\boldsymbol{\alpha} = \{a_{i,j}\}$ be a double sequence defined by

$$\begin{array}{ccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & & & \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & & & \\
a_{3,1} & a_{3,2} & a_{3,3} & \cdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

be any P-convergent factorable double sequence with P-limit L. Then the sequence

$$\begin{array}{ccccccc}
a_{1,1}\ L\ L\ ...L & a_{1,2}\ L\ L\ ...L & a_{1,3}\ L\ L\ ...L & ... & & & \\
L & L & L & L & L & L & ... \\
L & L & L & L & L & L & ... \\
... & & & & & & \\
L & L & L & L & L & L & ... \\
a_{2,1}\ L\ L\ ...L & a_{2,2}\ L & a_{2,3}\ L\ L\ ...L & \cdots & & & \\
L & L & L & L & L & L & ... \\
a_{3,1}\ L\ L\ ...L & a_{3,2}\ L\ L\ ...L & a_{3,3}\ L\ L\ ...L & \cdots & & & \\
L & L & L & L & L & L & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is also P-convergent with P-limit L, where L repeats p times. Since any convergent double sequence is p quasi-Cauchy this sequence is p quasi-Cauchy. So the transformed sequence $f(\boldsymbol{\alpha}) = \{f(a_{i,j})\}$ of the sequence $\boldsymbol{\alpha}$ is p quasi-Cauchy. Thus it follows that

$$\begin{array}{ccccccc}
f(a_{1,1})\ f(L)\ f(L)\ ...f(L) & f(a_{1,2})\ f(L)\ f(L)\ ...f(L) & f(a_{1,3})\ f(L)\ f(L)\ ...f(L) & ... & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(a_{2,1})\ f(L)\ f(L)\ ...f(L) & f(a_{2,2})\ f(L)\ f(L)\ ...f(L) & f(a_{2,3})\ f(L)\ f(L)\ ...f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
...f(L) & f(L) & f(L) & f(L) & f(L) & f(L) & ... \\
f(a_{3,1})\ f(L)\ f(L)\ ...f(L) & f(a_{3,2})\ f(L)\ f(L)\ ...f(L) & f(a_{3,3})\ f(L)\ f(L)\ ...f(L) & \cdots & & & \\
f(L) & f(L) & f(L) & L & f(L) & f(L) & ... \\
f(L) & f(L) & f(L) & L & f(L) & f(L) & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is factorable p quasi-Cauchy double sequence, where $f(L)$ repeats p times. Now it follows that $\{f(a_{i,j})\}$ is a P-convergent factorable double sequence with P-limit $f(L)$. Since double sequential continuity implies continuity, we get that the function f is continuous. This completes the proof of the theorem. $\square$

Theorem 3.2 *Suppose that $I \times I$ is a two dimensional interval and*

$$\begin{array}{ccccccc}
a_{1,1} & b_{1,1} & a_{1,2} & b_{1,2} & a_{1,3} & b_{1,3} & \cdots \\
d_{1,1} & c_{1,1} & d_{1,2} & c_{1,2} & d_{1,3} & c_{1,3} & \cdots \\
a_{2,1} & b_{2,1} & a_{2,2} & b_{2,2} & a_{2,3} & b_{2,3} & \cdots \\
d_{2,1} & c_{2,1} & d_{2,2} & c_{2,2} & d_{2,3} & c_{2,3} & \cdots \\
a_{3,1} & b_{3,1} & a_{3,2} & b_{3,2} & a_{3,3} & b_{3,3} & \cdots \\
d_{3,1} & c_{3,1} & d_{3,2} & c_{3,2} & d_{3,3} & c_{1,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is a double sequence of ordered pairs in $I \times I$ with

$$\lim_i |a_{i,i} - b_{i,i}| = \lim_i |a_{i,i} - c_{i,i}| = \lim_i |a_{i,i} - d_{i,i}| = 0.$$

Then there exists a p quasi-Cauchy double sequence $\{x_{i,j}\}$ with the property that for any ordered pair of integers (i,j); $i,j > 1$ there exists an ordered pair $(\bar{i},\bar{j})$; $\bar{i},\bar{j} > 1$ such that

$$(a_{i,j}, b_{i,j}) = (x_{\bar{i},\bar{j}}, x_{\bar{i},\bar{j}+p})$$

$$(a_{i,j}, c_{i,j}) = (x_{\bar{i},\bar{j}}, x_{\bar{i}+p,\bar{j}+p})$$

and

$$(a_{i,j}, d_{i,j}) = (x_{\bar{i},\bar{j}}, x_{\bar{i}+p,\bar{j}}).$$

Proof: For every (k,l); $k,l \geq 1$, fix

$$\begin{array}{cccc}
y^{k,l}_{0,0} & y^{k,l}_{0,1} & \cdots & y^{k,l}_{0,n_l} \\
y^{k,l}_{1,0} & y^{k,l}_{1,1} & \cdots & y^{k,l}_{1,n_l} \\
\vdots & \vdots & \vdots & \vdots \\
y^{k,l}_{m_k,0} & y^{k,l}_{m_k,1} & \cdots & y^{k,l}_{m_k,n_l}
\end{array}$$

in $I \times I$ with

$$y^{k,l}_{m_k,0} = y^{k,l}_{0,n_l} = y^{k,l}_{m_k,n_l} = a_{k+p,l+p},$$

$$y^{k,l}_{m_k,2} = y^{k+p,l+p}_{0,0} = y^{k,l}_{m_k,1} = b_{k+p,l+p},$$

$$y^{k+p,l+p}_{0,1} = y^{k+p,l+p}_{1,0} = y^{k+p,l+p}_{0,0} = c_{k+p,l+p},$$

and

$$y^{k,l}_{1,n_l} = y^{k+p,l+p}_{0,0} = y^{k,l}_{2,n_l} = d_{k+p,l+p}.$$

for $1 \leq i \leq m_k$ and $1 \leq j \leq n_l$ with

$$|y^{k,l}_{i,j} - y^{k,l}_{i-1,j}| < \frac{1}{kl},$$

$$|y^{k,l}_{i,j} - y^{k,l}_{i,j-1}| < \frac{1}{kl},$$

and

$$|y^{k,l}_{i,j} - y^{k,l}_{i-1,j-1}| < \frac{1}{kl}.$$

Now the double sequence

$$\begin{array}{cccccccccc}
a_{1,1} & b_{1,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} & a_{1,1} & b_{1,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} \\
d_{1,1} & c_{1,1} & y^{1,1}_{1,0} & \cdots & y^{1,1}_{1,n_1} & a_{1,1} & b_{1,1} & y^{1,1}_{1,0} & \cdots & y^{1,1}_{1,n_1} \\
y^{1,1}_{0,0} & y^{1,1}_{0,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} & y^{1,1}_{0,0} & y^{1,1}_{0,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y^{1,1}_{m_1,0} & y^{1,1}_{m_1,1} & y^{1,1}_{m_1,0} & \cdots & y^{1,1}_{m_1,n_1} & y^{1,1}_{m_1,0} & y^{1,1}_{m_1,1} & y^{1,1}_{m_1,0} & \cdots & y^{1,1}_{m_1,n_1} \\
a_{1,1} & b_{1,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} & a_{2,2} & b_{2,2} & y^{2,2}_{0,0} & \cdots & y^{2}_{0,n_1} \\
d_{1,1} & c_{1,1} & y^{1,1}_{1,0} & \cdots & y^{1,1}_{1,n_1} & d_{2,2} & c_{2,2} & y^{2,2}_{1,0} & \cdots & y^{2,2}_{1,n_2} \\
y^{1,1}_{0,0} & y^{1,1}_{0,1} & y^{1,1}_{0,0} & \cdots & y^{1,1}_{0,n_1} & y^{2,2}_{0,0} & y^{2,2}_{0,1} & y^{2,2}_{0,0} & \cdots & y^{2,2}_{0,n_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y^{1,1}_{m_1,0} & y^{1,1}_{m_1,1} & y^{1,1}_{m_1,0} & \cdots & y_{m_1,1} & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & y^{2,2}_{m_2,0} & y^{2,2}_{m_2,1} & y^{2,2}_{m_2,0} & \cdots & y^{2,2}_{m_2,n_2}
\end{array} .$$

is clearly a double sequence that has the desired property. □

Theorem 3.3 *Suppose that $I \times I$ is any two dimensional interval. Then a two dimensional factorable real-valued function defined on $I \times I$ is uniformly continuous if and only if it preserves factorable double p-quasi-Cauchy sequences from $I \times I$ for any constant $p \in \boldsymbol{N}$.*

Proof: It is clear that two dimensional uniformly continuous functions preserve p quasi-Cauchy double sequences.

Conversely, suppose that f defined on $I \times I$ is not uniformly continuous. Then there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exist $(a, b), (\bar{a}, \bar{b}) \in I \times I$ with $\sqrt{(a-\bar{a})^2 + (b-\bar{b})^2} < \delta$ but $|f(a, b) - f(\bar{a}, b)| \geq \epsilon$, $|f(a, b) - f(a, \bar{b})| \geq \epsilon$, and $|f(a, b) - f(\bar{a}, \bar{b})| \geq \epsilon$, respectively. Then by Theorem 3.2 there exists a factorable quasi-Cauchy double sequence $\mathbf{x}= \{x_k x_l\}$ such that for any ordered pair (i, j) with $i \geq 1$ and $j \geq 1$, there exist ordered pairs integers $(\bar{i}, \bar{j})$ with $a_{i,j} = x_{\bar{i},\bar{j}}$ and $b_{i,j} = x_{\bar{i}+p,\bar{j}+p}$. This implies that

$$|f(x_{\bar{i}}, x_{\bar{j}}) - f(x_{\bar{i}+p}, x_{\bar{j}})| \geq \epsilon,$$

$$|f(x_{\bar{i}}, x_{\bar{j}}) - f(x_{\bar{i}}, x_{\bar{j}+p})| \geq \epsilon,$$

and

$$|f(x_{\bar{i}}, x_{\bar{j}}) - f(x_{\bar{i}+p}, x_{\bar{j}+p})| \geq \epsilon.$$

Thus $\{f(x_i, x_j)\}$ is not a p quasi-Cauchy double sequence. Thus f does not preserve p quasi-Cauchy double sequence. This contradiction completes the proof. □

Theorem 3.4 *Suppose that f is a factorable double function defined on the bounded double interval $I \times I$. Then f is uniformly continuous on $I \times I$ if and only if the image under f of any Cauchy double sequence in $I \times I$ is p-quasi-Cauchy.*

Proof: Since factorable double function preserves quasi Cauchy sequences , then by Theorem 3.3 if f is a factorable uniformly continuous on $I \times I$ then the image of any p quasi-Cauchy double sequence in $I \times I$ is p quasi-Cauchy. Therefore the image of any double Cauchy under factorable function is p quasi-Cauchy. Now let us establish the converse, to that end, suppose that the image of every Cauchy double sequence is p quasi-Cauchy but the factorable to be uniformly continuous. Then there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exist $(x, y), (\bar{x}, \bar{y}) \in I \times I$ with $\sqrt{(x-\bar{x})^2 + (y-\bar{y})^2} < \delta$ but $|f(x, y) - f(\bar{x}, y)| \geq \epsilon$, $|f(x, y) - f(x, \bar{y})| \geq \epsilon$, and $|f(x, y) - f(\bar{x}, \bar{y})| \geq \epsilon$, respectively. For each (m, n); $m, n \geq 1$, for fix double sequence (x_m, y_n) and $(\bar{x}_m, \bar{y}_n)$ in $I \times I$ with $\sqrt{(x_m-\bar{x}_m)^2 + (y_n-\bar{y}_n)^2} < \frac{1}{mn}$ but

$$|f(x_m, y_n) - f(\bar{x}_m, y_n)| \geq \epsilon,$$

$$|f(x_m, y_n) - f(x_m, \bar{y}_n)| \geq \epsilon,$$

and

$$|f(x_m, y_n) - f(\bar{x}_m, \bar{y}_n)| \geq \epsilon,$$

respectively. Since $I \times I$ is bounded there exists a P-convergent subsequence by a simple extension of Bolzano-Weierstrass theorem, say $\{x_{k,l}\}$. The following double sequence

$$\begin{array}{ccccccc}
x_{1,1} & y_{1,2} & x_{1,3} & y_{1,4} & x_{1,5} & y_{1,6} & \cdots \\
y_{2,1} & x_{2,2} & y_{2,3} & x_{2,4} & y_{2,5} & x_{2,6} & \cdots \\
x_{3,1} & y_{3,2} & x_{3,3} & y_{3,4} & x_{3,5} & y_{3,6} & \cdots \\
y_{4,1} & x_{4,2} & y_{4,3} & x_{4,4} & y_{4,5} & x_{4,6} & \cdots \\
x_{5,1} & y_{5,2} & x_{5,3} & y_{5,4} & x_{5,5} & x_{5,6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is P-convergent. Thus Cauchy, however the image

$$\begin{array}{ccccccc}
f(x_1, x_1) & f(y_1, y_2) & f(x_1, x_3) & f(y_1, y_4) & f(x_1, x_5) & f(y_1, y_6) & \cdots \\
f(y_2, y_1) & f(x_2, x_2) & f(y_2, y_3) & f(x_2, f_4) & f(y_2, y_5) & f(x_2, x_6) & \cdots \\
f(x_3, x_1) & f(y_3, y_2) & f(x_3, x_3) & f(y_3, y_4) & f(x_3, x_5) & f(y_3, y_6) & \cdots \\
f(y_4, y_1) & f(x_4, x_2) & f(y_4, y_3) & f(x_4, x_4) & f(y_4, y_5) & f(x_4, x_6) & \cdots \\
f(x_5, x_1) & f(y_5, x_2) & f(x_5, x_3) & f(y_5, y_4) & f(x_5, x_5) & f(x_5, x_6) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

is not p quasi-Cauchy. Thus we have a contradiction. □

4. Conclusion

It is not difficult to see that a Cauchy double sequence is p quasi Cauchy double for any positive integer p. The converse is easily seen to be false as in the single dimensional case ([9], [3], [15], [27], [28], [29]). Furthermore if there exist positive integers m and q such that $p = m.q$, then either m quasi Cauchyness or q quasi Cauchness implies p quasi Cauchyness. One should also note that there are nice connections between p quasi Cauchy double sequences and uniform continuity of two-dimensional real valued functions. A two dimensional factorable function is uniformly continuous on a subset $E \times E$ of R^2 if and only if it preserves factorable double p quasi Cauchy sequences from $E \times E$ for any constant positive integer p. Extensions and variations are also presented. For a further study, we suggest to investigate p quasi-Cauchy double sequences of fuzzy points, and ward continuity for the factorable fuzzy functions (see [16], [17] , and [12] for the definitions and related concepts in fuzzy setting). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work. (see [1], and [23] for the definitions and related concepts in the double case).

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