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The Second Hankel determinant and the Fekete-Szego Functional for a Subclass of Analytic Functions by Using q-Al-Oboudi Differential Operator

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ABSTRACT: In this article,by generalizing the q-difference operator, we develop a new family of analytic functions in the open unit disk $\mathcal{U}$.. Our research provides fundamental insights into the behavior of these functions. Sharp inequalities for the initial Taylor coefficients are established. For all functions in this class, we determine the second-order Hankel determinant and obtain optimal estimates for the Fekete–Szegö problem. Furthermore, the results of this study indicate meaningful connections with earlier research in the field.

Keywords: Analytic function, Fekete-Szego functional, Hankel determinant q-Al-Oboudi differential operator.

Contents

1. Introduction

The theory of q-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, as well as geometric function theory of complex analysis. The fractional q-calculus is the q-extension of the ordinary fractional calculus and dates back to early 20-th century [8] and [2].

The geometrical interpretation of q-analysis involves studies of different q-analouge differential operators. The q-analouge of the well-know Ruscheweyh differential operator was defined in [9] and following this idea, the q-analouge of Salagean differential operator was defined in [5]. Those operators provided interesting results when they were used to introduce new sets of univalent functions as seen in [10] - [11].

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$.

A function $f(z) \in \mathcal{S}^*$ is said to be starlike if and only if $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ z \in \mathcal{U}$.

A function $f(z) \in \mathcal{K}^*$ is said to be convex if and only if $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ z \in \mathcal{U}$.

We observe that $f(z) \in \mathcal{K} \Longleftrightarrow zf'(z) \in \mathcal{S}^*,$ describes the relationship between the classes $\mathcal{S}^*$ and $\mathcal{K}$.

Using the concept of convolution, Dileep L and Mallige Rajeev [3] define the following differential operator $D^n_{\lambda,q} : \mathcal{A} \longrightarrow \mathcal{A},\ n \in \mathbb{N},\ \lambda > 0\ q \in (0,1)$.

$$D^n_{\lambda,q} f(z) = f(z) * G_{q,n}(z) \tag{1.2}$$

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where

$$G_{q,n}(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]_q^n z^k \tag{1.3}$$

Making use of (1.2) and (1.3), the power series of $D^n_{\lambda,q} f(z)$ for f of the form (1.1) is given by

$$D^n_{\lambda,q} f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]_q^n a_j z^j \tag{1.4}$$

If $q \to 1^-$, the operator $D^n_{\lambda,q}$ reduces to the Al-Oboudi operator [1].

If $\lambda = 1$, the operator $D^n_{\lambda,q}$ reduces to the Sălăgean q differential operator defined in [5].

For a parametric values $q \to 1^-$, $\lambda = 1$, the operator $D^n_{\lambda,q}$ reduces to Salagean differential oprator [13].

The new subclasses of analytic functions are now defined using the q- Al-Oboudi differential operator [1].

Let $\mathcal{C}_n(\beta, \gamma, \lambda; q)$ be the subclass of $\mathcal{A}$, having functions of the form (1.1)if it satifies the conditions

$$\Re \left\{ e^{i\gamma} \left(1 - e^{-2i\gamma} \beta^2 z^2\right) \frac{D^{n+1}_{\lambda,q} f(z)}{z} \right\} > 0, \tag{1.5}$$

where $\beta \in [0, 1]$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $n \in \mathbb{N}_0$.

Remark:

1. For $\lambda = 1$, we get the classes studied in [4].
2. For $q \to 1^-$, $\lambda = 1$, we get the classes studied by Ayinla and Opoola [12].

2. Preliminaries

To validate our main results, we need the following lemmas.

Let $\mathcal{P}$ denote the class of Caratheodary functions

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots z \in \mathcal{U}$$

which are analytic and satisfy $p(0) = 1$ and $\Re p(z) > 0$.

Lemma 2.1 *Let* $p \in \mathcal{P}$. *Then* $|c_k| \leq 2$ $(k \in \mathbb{N})$.

Lemma 2.2 *Let* $p \in \mathcal{P}$, *then for any real* λ

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1-\lambda), & if\ \lambda \leq 0 \\ 2, & if\ 0 \leq \lambda \leq 2 \\ 2(\lambda - 1), & if\ \lambda \geq 2. \end{cases}$$

Lemma 2.3 *Let* $p \in \mathcal{P}$, *then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

$$4c_3 = c_1^2 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some value of x, z *such that* $|x| \leq 1$ *and* $|z| \leq 1$.

The upper bounds of the initial Taylor coefficients $|a_2|$, $|a_3|$ and $|a_4|$ as well as the Feketo-Szego functional $|a_3 - \eta a_2^2|$ for subclasses of analytic functions $\mathcal{C}_n(\beta, \gamma, \lambda; q)$ in the open unit disk are now examined using the q-Al-Oboudi differential operator.

3. Main Results

Theorem 3.1 *Let* $f(z) = \mathcal{C}_n(\beta, \gamma, \lambda; q)$, *then*

$$|a_2| \leq \frac{2\cos\gamma}{[2\lambda]_q^{n+1}}. \tag{3.1}$$

$$|a_3| \leq \frac{2\cos\gamma + \beta^2}{[3\lambda]_q^{n+1}}. \tag{3.2}$$

$$|a_4| \leq \frac{2\cos\gamma + 2\beta^2\cos\gamma}{[4\lambda]_q^{n+1}}. \tag{3.3}$$

Proof: Let $f(z) = \mathcal{C}_n(\beta, \gamma, \lambda; q)$. Then the inequality (1.5) is satisfied. Now, if we have

$$e^{i\gamma}(1 - e^{-2i\gamma\beta^2z^2})\frac{D_{\lambda,q}^{n+1}f(z)}{z} = e^{i\gamma} + \sum_{k=2}^{\infty} t_k z^k = (cos\gamma + isin\gamma) + \sum_{k=1}^{\infty} t_k z^k, \tag{3.4}$$

then

$$e^{i\gamma}(1 - e^{-2i\gamma\beta^2z^2})\frac{D_{\lambda,q}^{n+1}f(z)}{z} = p(z)cos\gamma + isin\gamma \tag{3.5}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$. That is

$$cos\gamma + c_1cos\gamma z + c_2cos\gamma z^2 + c_3cos\gamma z^3 + \cdots = cos\gamma + t_1z + t_2z^2 + t_3z^3 + \cdots. \tag{3.6}$$

Comparing the coefficients of (3.4) and (3.6) we get

$$|a_2| = \frac{c_1e^{-i\gamma}\cos\gamma}{[2\lambda]_q^{n+1}}. \tag{3.7}$$

$$|a_3| = \frac{c_2e^{-i\gamma}\cos\gamma + \beta^2e^{-2i\gamma}}{[3\lambda]_q^{n+1}}. \tag{3.8}$$

$$|a_4| = \frac{c_3e^{-i\gamma}\cos\gamma + c_1\beta^2e^{-3i\gamma}\cos\gamma}{[4\lambda]_q^{n+1}}. \tag{3.9}$$

Solving for the bounds of (3.7), (3.8), (3.9) and by using Lemma 2.1, we obtain the inequalities (3.1),(3.2) and (3.3). □

For a parametric value $\lambda = 1$ the above theorem acquire the outcome of E E Ali et. al [4].

Theorem 3.2 *Let* $f(z) = \mathcal{C}_n(\beta, \gamma, \lambda; q)$, *then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{\beta^2 + 2cos\gamma}{[3\lambda]_q^{n+1}} - \dfrac{4\eta e^{-i\gamma}cos^2\gamma}{([2\lambda]_q^{n+1})^2}, & \eta \leq 0 \\ \dfrac{\beta^2 + 2cos\gamma}{[3\lambda]_q^{n+1}}, & if\ 0 \leq \eta \leq \dfrac{[2\lambda]_q^{2n+2}e^{i\gamma}}{[3\lambda]_q^{n+1}cos\gamma} \\ \dfrac{\beta^2 - 2cos\gamma}{[3\lambda]_q^{n+1}} + \dfrac{4\eta e^{-i\gamma}cos^2\gamma}{([2\lambda]_q^{n+1})^2}, & if\ \eta \geq \dfrac{[2\lambda]_q^{2n+2}e^{i\gamma}}{[3\lambda]_q^{n+1}cos\gamma}. \end{cases} \tag{3.10}$$

Proof: Using (3.7) and (3.8) we obtain that

$$\left|a_3-\eta a_2^2\right| = \left|\frac{e^{-i\gamma}cos\gamma}{[3\lambda]_q^{n+1}}c_2 + \frac{e^{-2i\gamma}\beta^2}{[3\lambda]_q^{n+1}} - \frac{\eta e^{-2i\gamma}cos^2\gamma}{([2\lambda]_q^{n+1})^2}c_1^2\right|$$

$$\left|a_3-\eta a_2^2\right| \leq \frac{\beta^2}{[3\lambda]_q^{n+1}} + \frac{cos\gamma}{[3\lambda]_q^{n+1}}\left|c_2 - \frac{2\beta e^{-i\gamma}cos\gamma[3\lambda]_q^{n+1}}{([2\lambda]_q^{n+1})^2}\frac{c_1^2}{2}\right|. \tag{3.11}$$

Using Lemma 2.2, in (3.11), we thus obtain

$$\left|a_3-\eta a_2^2\right| \leq \frac{\beta^2+2cos\gamma}{[3\lambda]_q^{n+1}} - \frac{4\eta e^{-i\gamma}cos^2\gamma}{([2\lambda]_q^{n+1})^2} \qquad (\beta \leq 0).$$

Now if $0 \leq \dfrac{2\beta e^{-i\gamma}cos\gamma[3\lambda]_q^{n+1}}{([2\lambda]_q^{n+1})^2} \leq 2,$ then by, Lemma 2.2, we find that

$$\left|a_3-\eta a_2^2\right| \leq \frac{\beta^2+2cos\gamma}{[3\lambda]_q^{n+1}}.$$

We now suppose that $\dfrac{2\beta e^{-i\gamma}cos\gamma[3\lambda]_q^{n+1}}{([2\lambda]_q^{n+1})^2} \geq 2,$ then by using Lemma 2.2, we get

$$\left|a_3-\eta a_2^2\right| \leq \frac{\beta^2-2cos\gamma}{[3\lambda]_q^{n+1}} + \frac{4\eta e^{-i\gamma}cos^2\gamma}{([2\lambda]_q^{n+1})^2}$$

□

Theorem 3.3 *Let* $f(z) = \mathcal{C}_n(\beta,\gamma,\lambda;q),$ *then*

$$H_2(2) = |a_2a_4 - a_3^2| \leq \frac{\beta^4+4\beta^2cos\gamma+4cos^2\gamma}{[3\lambda]_q^{2n+2}} + \frac{(\beta^4+6\beta^2+9)cos\gamma}{[2\lambda]_q^{3n+4}}.$$

Proof: Using (3.7), (3.8) and (3.9) gives

$$|a_2a_4-a_3^2| = \left|\frac{c_1e^{-i\gamma}cos\gamma}{[2\lambda]_q^{n+1}}\left(\frac{c_3e^{-i\gamma}cos\gamma+\beta^2c_1e^{-3i\gamma}cos\gamma}{[4\lambda]_q^{n+1}}\right) - \left(\frac{c_2e^{-i\gamma}cos\gamma+\beta^2e^{-2i\gamma}}{[3\lambda]_q^{n+1}}\right)^2\right| \tag{3.12}$$

$$\begin{aligned}|a_2a_4-a_3^2| = \Bigg| & \frac{c_1^4e^{-2i\gamma}cos^2\gamma}{[2\lambda]_q^{3n+5}} + \frac{c_1^2(4-c_1^2)e^{-2i\gamma}\chi cos^2\gamma}{[2\lambda]_q^{3n+4}} - \frac{c_1^2(4-c_1^2)e^{-2i\gamma}\chi^2cos^2\gamma}{[2\lambda]_q^{3n+5}} \\ & + \frac{c_1(4-c_1^2)(1-|\chi^2|)e^{-2i\gamma}cos^2\gamma z}{[2\lambda]_q^{3n+4}} + \frac{c_1^2\beta^2e^{-4i\gamma}cos^2\gamma z}{[2\lambda]_q^{3n+3}} - \frac{c_1^4e^{-2i\gamma}cos^2\gamma z}{4[3\lambda]_q^{2n+2}} \\ & - \frac{\chi c_1(4-c_1^2)e^{-2i\gamma}cos^2\gamma}{2[3\lambda]_q^{2n+2}} - \frac{c_1^2\alpha^2e^{-3i\gamma}cos\gamma}{[3\lambda]_q^{2n+2}} - \frac{\chi^2(4-c_1^2)^2e^{-2i\gamma}cos^2\gamma}{4[3\lambda]_q^{2n+2}} \\ & - \frac{\beta^2\chi(4-c_1^2)e^{-3i\gamma}cos\gamma}{[3\lambda]_q^{2n+2}} - \frac{\beta^4e^{-4i\gamma}}{[3\lambda]_q^{2n+2}}\Bigg|\end{aligned} \tag{3.13}$$

Suppose $c_1 = c$, and recall that $|c_1| \leq 2$, and assuming without restriction that $c \in [0,2]$ and put $\Psi = |\chi| \leq 1$ Then, using triangle inequality (3.13) becomes

$$|a_2a_4-a_3^2| = \left\{\frac{c^4cos^2\gamma}{[2\lambda]_q^{3n+5}} + \frac{c(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}} + \frac{c\beta^2cos^2\gamma}{[2\lambda]_q^{3n+3}} + \frac{c^4cos^2\gamma}{4[3\lambda]_q^{2n+2}} + \frac{c^2\beta^2cos\gamma}{[3\lambda]_q^{2n+2}} + \frac{\beta^4}{[3\lambda]^{2n+2}}\right\}$$

$$+\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c)^2c^2cos^2\gamma}{(2)[3\lambda]_q^{2n+2}}-\frac{\beta^2(4-c^2)cos\gamma}{[3\lambda]_q^{2n+2}}\Psi \tag{3.14}$$

$$+\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+5}}-\frac{c(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c^2)^2cos^2\gamma}{(4)[3\lambda]_q^{2n+2}}\Psi^2=\mathcal{F}(c,\Psi).$$

Differentiating $\mathcal{F}(c,\Psi)$ partially with respect to Ψ in the closed interval $0\le\Psi\le1$

$$\frac{\partial\mathcal{F}(c,\Psi)}{\partial\Psi}=\left\{\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c)^2c^2cos^2\gamma z}{(2)[3\lambda]_q^{2n+2}}-\frac{\beta^2(4-c^2)cos\gamma z}{[3\lambda]_q^{2n+2}}\right\}$$

$$+\left\{\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+5}}-\frac{c(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c^2)^2cos^2\gamma}{(4)[3\lambda]_q^{2n+2}}\Psi\right\} \tag{3.15}$$

for $0\le\Psi\le1$, therefore $\mathcal{F}(c,\Psi)$ is an increasing function. Hence, it attains maximum point at $\Psi=1$. Thus,

$$max_{0\le\Psi\le1}\mathcal{F}(c,\Psi)=\mathcal{F}(c,1)\le\left\{\frac{c^4cos^2\gamma}{[2\lambda]_q^{3n+5}}+\frac{c(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{c\beta^2cos^2\gamma}{[2\lambda]_q^{3n+3}}+\frac{c^4cos^2\gamma}{4[3\lambda]_q^{2n+2}}\right.$$

$$+\frac{c^2\beta^2cos\gamma}{[3\lambda]_q^{2n+2}}+\frac{\beta^4}{[3\lambda]^{2n+2}}+\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c)^2c^2cos^2\gamma}{(2)[3\lambda]_q^{2n+2}}-\frac{\beta^2(4-c^2)cos\gamma}{[3\lambda]_q^{2n+2}} \tag{3.16}$$

$$+\frac{c^2(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+5}}-\frac{c(4-c^2)cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{(4-c^2)^2cos^2\gamma}{(4)[3\lambda]_q^{2n+2}}=\mathcal{G}(c).$$

$$\mathcal{G}'(c)=\frac{(3+\beta^2)\cos^2\gamma}{[2\lambda]_q^{3n+2}}c-\frac{cos^2\gamma}{[2\lambda]_q^{3n+2}}c^3.$$

Now, the critical points occur at

$$c_0=0,\ c_1=\sqrt{\alpha^2+3}\text{ and }c_2=-\sqrt{\alpha^2+3}$$

but the maximum point occurring at $\sqrt{\alpha^2+3}$ (3.16) becomes

$$\mathcal{G}(c)=\frac{\beta^4+4\beta^2cos\gamma+4cos^2\gamma}{[3\lambda]_q^{2n+2}}+\frac{\beta^4cos^2\gamma}{[2\lambda]_q^{3n+4}}+\frac{6\beta^2cos^2\gamma+9cos^2\gamma}{[2\lambda]_q^{3n+4}}. \tag{3.17}$$

Therefore,

$$|a_2a_4-a_3^2|\le\frac{\beta^4+4\beta^2cos\gamma+4cos^2\gamma}{[3\lambda]_q^{2n+2}}+\frac{(\beta^4+6\beta^2+9)cos^2\gamma}{[2\lambda]_q^{3n+4}}.$$

□

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