



Generalized q -Fractional Saigo Operator Associated with the Four-Parameter Mittag–Leffler Function

Arti Sharma and V. K. Vyas

ABSTRACT: The main goal of this article is to present two new simple versions of the four-parameter Mittag–Leffler function. We have created some q -integral forms and Saigo q -fractional operators for these new versions. Additionally, we drive Riemann–Liouville-type fractional q -integrals and q -derivatives for these versions, which are used in the study of q -fractional calculus.

Keywords: q -Analogues, Saigo fractional integrals, Riemann–Liouville fractional q -integrals, Mittag–Leffler functions.

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1. Introduction

Fractional differential equations have been a major focus of research in recent years. They show up in many areas like physics and engineering. The Mittag–Leffler function is a key tool for solving fractional order differential equations and integral equations. It’s used in several fields, including the study of kinetic equations, the telegraph equation, random walks, Levy flights, superdiffusive transport, and complex systems. The Mittag–Leffler function is also important when solving certain boundary value problems that involve fractional integro-differential equations of the Volterra type. It’s applied in practical areas such as fluid flow, rheology, diffusive transport similar to diffusion, electric networks, probability, and statistical distribution theory. Many properties of the Mittag–Leffler function have been discussed and reviewed in reference [1]. Additionally, a different version of the Mittag–Leffler function has been explored in reference [6].

The Saigo operator is a type of fractional calculus operator that generalizes several well-known fractional integral and derivative operators, such as the Riemann–Liouville and Erdélyi–Kober operators. Introduced by M. Saigo, these operators extend the concept of fractional orders, providing a more flexible framework for various applications in mathematics, physics, engineering, and other fields.

Let us begin by reviewing the historical background of the Mittag–Leffler function. The function $E_\alpha(z)$ was originally introduced and investigated by G. Mittag–Leffler in 1903 [5,6,7]. The function $E_\alpha(z)$ is defined as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

This function appears naturally, similar to how the exponential function appears in solutions to ordinary differential equations, but it is used in solving fractional integro-differential equations of any order.

2. Continuation on Mittag–Leffler Functions

$$E_{\alpha_1}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(lr_1 + 1)}, \quad \Re(r_1) > 0, \quad z \in \mathbb{C}. \quad (2.1)$$

and studied several basic results and properties of this function. The above function is an entire function of order $\rho = 1/r_1$ and type $\sigma = 1$. This is one of the simplest entire functions of the specified order and type. We observe that when $r_1 = 1$, then $E_{r_1}(z)$ reduces to the exponential function e^z .

Later, in 1905, Wiman [2] introduced a generalization of the Mittag–Leffler function $E_{r_1}(z)$ as follows:

$$E_{(r_1, r_2)}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(lr_1 + r_2)}, \quad \Re(r_1) \geq 0, \quad \Re(r_2) \geq 0, \quad z \in \mathbb{C}. \quad (2.2)$$

The above function is also known as *Wiman's function* or the two-parameter Mittag–Leffler function, which reduces to the one-parameter Mittag–Leffler function after putting $r_2 = 1$.

In the sequence, after a few years, Prabhakar [3] presented the following three-parameter Mittag–Leffler function, defined by

$$E_{(r_1, r_2)}^r(z) = \sum_{l=0}^{\infty} \frac{(r)_l z^l}{\Gamma(lr_1 + r_2)}, \quad \Re(r_1) \geq 0, \quad \Re(r_2) \geq 0, \quad z, r \in \mathbb{C}, \quad (2.3)$$

where $(r)_l$ represents the Pochhammer symbol, defined as [4]

$$(s)_l = \begin{cases} 1, & l = 0, \quad s \in \mathbb{C} \setminus \{0\}, \\ s(s+1) \cdots (s+l-1), & l \in \mathbb{N}, \quad s \in \mathbb{C}. \end{cases}$$

The above function $E_{(r_1, r_2)}^r(z)$ is an example of an entire function with specified order and type $\rho = 1/\Re(r_1)$ and $\sigma = 1$, respectively. We can easily see that $E_{(r_1, r_2)}^r(z)$ reduces to the two-parameter Mittag–Leffler function $E_{(r_1, r_2)}(z)$ after substituting $r = 1$.

In continuation of the generalization of Mittag–Leffler functions, and motivated by the above works, Shukla *et al.* [5] introduced the four-parameter Mittag–Leffler function and studied many of its properties, including usual differentiation and integration, Euler–Beta transforms, Laplace transforms, Whittaker transforms, Mellin transforms, generalized hypergeometric series form, and Mellin–Barnes integral representation:

$$E_{(r_1, r_2)}^{(r, s)}(z) = \sum_{l=0}^{\infty} \frac{(r)_{ls} z^l}{\Gamma(lr_1 + r_2)}, \quad \Re(r_1) \geq 0, \quad \Re(r_2) \geq 0, \quad z, r, s \in \mathbb{C}. \quad (2.4)$$

Here, $(r)_{ls}$ denotes the generalized Pochhammer symbol.

The function $E_{(r_1, r_2)}^{(r, s)}(z)$ satisfies the following convergence conditions:

$$E_{(r_1, r_2)}^{(r, s)}(z) \text{ converges absolutely as } \begin{cases} \forall z, & \text{if } q < \Re(r_1) + 1, \\ |z| < 1, & \text{if } q = \Re(r_1) + 1. \end{cases}$$

It is also an example of an entire function with order $\rho = 1/\Re(r_1)$.

3. Preliminaries on q -Calculus

The function $E_{(r_1, r_2)}^{(r, s)}(z)$ is considered an extension of all the above Mittag–Leffler functions defined by (2.1)–(2.3), because we can easily obtain all the earlier Mittag–Leffler functions by suitable substitution of parameters in the definition (2.4).

Fractional calculus is an important advancement in the field of calculus because it is useful in many areas of math, physics, and applied sciences. It is also known as quantum calculus, and its origins can be traced back to Jackson’s work in 1908. The fractional q -calculus is the basic version of regular fractional calculus.

For $0 < |q| < 1$, the basic (or q -shifted) factorial is defined in [7] as follows:

$$(b; q)_l = \prod_{s=0}^{l-1} (1 - bq^s), \quad l \in \mathbb{N}. \quad (3.1)$$

Then, (3.1) can be written in terms of the basic gamma function as

$$(b; q)_l = \frac{\Gamma_q(b+l)(1-q)^l}{\Gamma_q(b)}. \quad (3.2)$$

The basic gamma function is given in [7] by

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty (1-q)^{z-1}}, \quad \frac{\Gamma_q(a)}{\Gamma_q(a+b)} = \frac{(1-q)^b}{(q^a; q)_b}, \quad a, b, z \in \mathbb{C}. \quad (3.3)$$

The q -beta function is defined in [7] as

$$B_q(r_1, r_2) = \int_0^1 z^{r_1-1} (zq; q)_{r_2-1} d_q z = \frac{\Gamma_q(r_1)\Gamma_q(r_2)}{\Gamma_q(r_1+r_2)}, \quad \Re(r_1) > 0, \Re(r_2) > 0. \quad (3.4)$$

For $l \in \mathbb{N}_0$, the q -shifted factorial with negative subscript is defined as follows:

$$(b; q)_{-l} = \frac{1}{(1-bq^{-1})(1-bq^{-2}) \cdots (1-bq^{-l})}. \quad (3.5)$$

This yields

$$(b; q)_{-l} = \frac{1}{(bq^{-l}; q)_l} = \left(\frac{-q}{b}\right)^l q^{\binom{l}{2}} \frac{1}{(q/b; q)_l}. \quad (3.6)$$

We also have

$$(b; q)_\infty = \prod_{s=0}^{\infty} (1 - bq^s), \quad b, q \in \mathbb{C}. \quad (3.7)$$

From (3.1), (3.5), and (3.6), we can state that

$$(b; q)_l = \frac{(b; q)_\infty}{(bq^l; q)_\infty}, \quad l \in \mathbb{Z}. \quad (3.8)$$

Moreover,

$$(a; q)_{n+k} = (a; q)_n (aq^n; q)_k. \quad (3.9)$$

The following identity defined in [7] is important for proving our main results:

$$(bq^n; q)_k = \frac{(b; q)_k (bq^k; q)_n}{(b; q)_n}, \quad n \in \mathbb{Z}. \quad (3.10)$$

In the theory of basic series, two different q -analogues of the classical exponential functions are defined in [7] as follows:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k}, \quad |x| < 1, \quad (3.11)$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(q; q)_k}, \quad x \in \mathbb{C}. \quad (3.12)$$

Further Developments and q -Mittag-Leffler Functions

In 1996, Atakishiyev examined the properties of a family of q -exponential functions, which depend on an extra parameter, and showed that these functions have a well-defined meaning for both $0 < |q| < 1$ and $|q| > 1$ cases if the parameter belongs to $[0, 1]$.

In 2009, Mansour proposed a q -analogue (basic analogue) of the two-parameter Mittag-Leffler function, defined as follows:

$$e_{(r_1, r_2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(kr_1 + r_2)}, \quad |z| < (1 - q)^{-r_1}. \quad (3.13)$$

where $r_1 > 0$, $r_2 \in \mathbb{C}$.

In the same year, Rajković et al. introduced the basic analogue of the two-parameter Mittag-Leffler function (1.2).

The **small q -Mittag-Leffler function** is defined as follows:

$$e_{(r_1, r_2, q)}(z; c) = \sum_{k=0}^{\infty} \frac{z^{kr_1 + r_2 - 1} (c/z; q)_{kr_1 + r_2 - 1}}{(q; q)_{kr_1 + r_2 - 1}}, \quad |c| < |z|. \quad (3.14)$$

The **big q -Mittag-Leffler function** is defined as follows:

$$E_{(r_1, r_2, q)}(z; c) = \sum_{k=0}^{\infty} \frac{q^{\frac{(kr_1 + r_2 - 1)(kr_1 + r_2 - 2)}{2}} z^{kr_1 + r_2 - 1} (c/z; q)_{kr_1 + r_2 - 1}}{(-c; q)_{kr_1 + r_2 - 1} (q; q)_{kr_1 + r_2 - 1}}, \quad (3.15)$$

where $q, z, c, r_1, r_2 \in \mathbb{C}$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, and $|q| < 1$.

Furthermore, Purohit et al. introduced a generalized q -analogue of the three-parameter Mittag-Leffler function (1.3).

The **generalized small q -Mittag-Leffler function** is defined by

$$e_{(r_1, r_2, q)}^r(z) = \sum_{k=0}^{\infty} \frac{(q^r; q)_k z^k}{\Gamma_q(kr_1 + r_2) (q; q)_k}, \quad |z| < (1 - q)^{-r_1}. \quad (3.16)$$

The **generalized big q -Mittag-Leffler function** is given by

$$E_{(r_1, r_2, q)}^r(z) = \sum_{k=0}^{\infty} \frac{(q^r; q)_k q^{\frac{k(k-1)}{2}} z^k}{\Gamma_q(kr_1 + r_2) (q; q)_k}, \quad |z| < (1 - q)^{-r_1}. \quad (3.17)$$

where $q, z, r_1, r_2 \in \mathbb{C}$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $\Re(r) > 0$, and $|q| < 1$.

For further studies on the q -Mittag-Leffler functions and their properties with applications,

The **generalized small four-parameter q -Mittag-Leffler function** is defined as

$$e_{(r_1, r_2)}^{(r, s)}(z; q) = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} z^n}{\Gamma_q(nr_1 + r_2) (q; q)_n}, \quad |z| < (1 - q)^{-r_1}. \quad (3.18)$$

Similarly, the **generalized big four-parameter q -Mittag-Leffler function** is defined as

$$E_{(r_1, r_2)}^{(r, s)}(z; q) = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} q^{\frac{n(n-1)}{2}} z^n}{\Gamma_q(nr_1 + r_2) (q; q)_n}, \quad |z| < (1 - q)^{-r_1}, \quad (3.19)$$

where $z, r, r_1, r_2 \in \mathbb{C}$, $s \in \mathbb{N}$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $\Re(r) > 0$, and $|q| < 1$.

The q -Analogue of Saigo Fractional Operators

Let α be a real positive complex number and $f(x)$ be a real-valued function on $(0, \infty)$. The q -analogue of Saigo fractional integral operator is defined as

$$\begin{aligned} I_q^{(\alpha, \beta, \eta)} f(x) &= \frac{x^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^x (tq/x; q)_{\alpha-1} \\ &\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta}; q)_m (q^{-\eta}; q)_m}{(q^\alpha; q)_m (q; q)_m} (-1)^m q^{(\eta-\beta)m} q^{-m(m-1)/2} (t/x-1)_q^m f(t) d_q t, \end{aligned} \quad (3.20)$$

and The q -analogue of the Saigo-type fractional integral operator can be expressed as

$$\begin{aligned} J_q^{(\alpha, \beta, \eta)} f(x) &= \frac{q^{-\frac{\alpha(\alpha+1)}{2-\beta}}}{\Gamma_q(\alpha)} \int_x^\infty (x/t; q)_{\alpha-1} t^{-\beta-1} \\ &\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta}; q)_m (q^{-\eta}; q)_m}{(q^\alpha; q)_m (q; q)_m} (-1)^m q^{(\eta-\beta)m} q^{-\frac{m(m-1)}{2}} \\ &\times (x/(qt) - 1)_q^m f(tq^{1-\alpha}) d_q t. \end{aligned} \quad (3.21)$$

where $0 < |q| < 1$, and β, η are real or complex.

Fractional q -Analogues of Saigo Operators

The q -analogue of the Saigo fractional derivative operator is defined as

$$D_q^{(\alpha, \beta, \eta)} f(x) = D_q^m I_q^{(-\alpha+m, -\beta+m, \alpha+\eta-m)} f(x), \quad (3.22)$$

and

$$P_q^{(\alpha, \beta, \eta)} f(x) = q^{\alpha(\alpha+\beta)} (-q^{-(\alpha+\beta)} D_q)^m J_q^{(-\alpha+m, -\beta-m, \alpha+\eta)} f(x), \quad (3.23)$$

where $|q| < 1$, and β and η are real or complex parameters.

For $0 < |q| < 1$, $\Re(\alpha) > 0$, and μ, β, η being real or complex, the action of the fractional q -Saigo integral operators $I_q^{(\alpha, \beta, \eta)}$ and $J_q^{(\alpha, \beta, \eta)}$ on a power function x^μ is given by

$$I_q^{(\alpha, \beta, \eta)}(x^\mu) = \frac{\Gamma_q(\mu+1) \Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1) \Gamma_q(\mu+\alpha+\eta+1)} x^{\mu-\beta}, \quad (3.24)$$

provided that $\Re(\mu+1) > 0$ and $\Re(\mu-\beta+\eta+1) > 0$.

Similarly,

$$J_q^{(\alpha, \beta, \eta)}(x^\mu) = \frac{\Gamma_q(\beta-\mu) \Gamma_q(\eta-\mu)}{\Gamma_q(-\mu) \Gamma_q(\beta+\alpha-\mu+\eta)} x^{\mu-\beta} q^{-\alpha\mu-\alpha(\alpha+1)/2}, \quad (3.25)$$

where $\Re(\beta-\mu) > 0$ and $\Re(\eta-\mu) > 0$.

For $|q| < 1$, $n < \Re(\alpha) \leq n+1$, $n \in \mathbb{N}$, and μ, β, η real or complex, the fractional q -Saigo derivative operators $D_q^{(\alpha, \beta, \eta)}$ and $P_q^{(\alpha, \beta, \eta)}$ acting on a power function x^μ yield

$$D_q^{(\alpha, \beta, \eta)}(x^\mu) = \frac{\Gamma_q(\mu+1) \Gamma_q(\mu+\alpha+\beta+\eta+1)}{\Gamma_q(\mu+\beta+1) \Gamma_q(\mu+\eta+1)} x^{\mu+\beta}, \quad (3.26)$$

provided that $\Re(\mu+1) > 0$ and $\Re(\mu+\alpha+\beta+\eta+1) > 0$.

Finally,

$$P_q^{(\alpha, \beta, \eta)}(x^\mu) = \frac{\Gamma_q(-\beta-\mu) \Gamma_q(\alpha+\eta-\mu)}{\Gamma_q(-\mu) \Gamma_q(-\beta+\eta-\mu)} q^{\alpha(\beta+\mu)+\alpha(\alpha-1)/2} x^{\mu+\beta}, \quad (3.27)$$

where $\Re(-\beta-\mu) > 0$ and $\Re(\alpha+\eta-\mu) > 0$.

4. Main Result

In this section, we introduce two new q -analogue Saigo fractional operators of the four-parameter Mittag-Leffler function.

Theorem 1. The following theorem establishes the image of the generalized small four-parameter q -Mittag-Leffler function under the Saigo-type fractional q -integral operator. The following Saigo q -fractional calculus operator type fractional q -integral holds true:

$$\begin{aligned} & I_{(z,q)}^{(\alpha,\beta,\eta)} \left(z^{\lambda-1} e_{(r_1,r_2)}^{(r,s)}(z;q) \right) \\ &= \frac{(1-q)^\alpha (q^{\lambda-\beta}; q)_\eta}{(q^\lambda; q)_\alpha (q^{\lambda+\alpha}; q)_\eta} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^{\eta+\lambda-\beta}; q)_n (q^\lambda; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{\lambda-\beta}; q)_n (q^{\lambda+\alpha+\eta}; q)_n} z^{n+\lambda-\beta-1}. \end{aligned} \quad (32)$$

where $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof. Consider the left-hand side of Theorem 1, and denote it by L :

$$L = I_{(z,q)}^{(\alpha,\beta,\eta)} \left(z^{\lambda-1} e_{(r_1,r_2)}^{(r,s)}(z;q) \right).$$

Then, by the definition of the q -Mittag-Leffler function (22), we have

$$L = I_{(z,q)}^{(\alpha,\beta,\eta)} \left\{ z^{\lambda-1} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} z^n}{\Gamma_q(nr_1 + r_2) (q; q)_n} \right\}.$$

On applying the q -integral, we get

$$L = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2) (q; q)_n} I_{(z,q)}^{(\alpha,\beta,\eta)} \{ z^{n+\lambda-1} \}.$$

Then, by using the result (28), we obtain

$$L = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2) (q; q)_n} \frac{\Gamma_q(n+\lambda) \Gamma_q(n+\lambda-\beta+\eta)}{\Gamma_q(n+\lambda-\beta) \Gamma_q(n+\lambda+\alpha+\eta)} z^{n+\lambda-\beta-1}.$$

Now, using the definition of the q -gamma function (7),

$$L = (1-q)^\alpha \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2) (q; q)_n} \frac{(q^{n+\lambda-\beta}; q)_\eta}{(q^{n+\lambda}; q)_{\alpha+\eta}} z^{n+\lambda-\beta-1}.$$

With application of the result (13), we have

$$L = (1-q)^\alpha \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2) (q; q)_n} \frac{(q^{n+\lambda-\beta}; q)_\eta}{(q^{n+\lambda}; q)_\alpha (q^{n+\lambda+\alpha}; q)_\eta} z^{n+\lambda-\beta-1}.$$

On using the q -identity (14) and simplifying, we get the desired result, i.e., equation (1).

$$L = \frac{(1-q)^\alpha (q^{\lambda-\beta}; q)_\eta}{(q^\lambda; q)_\alpha (q^{\lambda+\alpha}; q)_\eta} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^{\eta+\lambda-\beta}; q)_n (q^\lambda; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{\lambda-\beta}; q)_n (q^{\lambda+\alpha+\eta}; q)_n} z^{n+\lambda-\beta-1}.$$

Theorem 2. This result provides the corresponding Saigo-type fractional q -integral formula for the generalized big four-parameter q -Mittag-Leffler function. The following Saigo q -fractional calculus operator type fractional q -integral holds true:

$$\begin{aligned} & I_{(z,q)}^{(\alpha,\beta,\eta)} \left(z^{\lambda-1} E_{(r_1,r_2)}^{(r,s)}(z;q) \right) \\ &= \frac{(1-q)^\alpha (q^{\lambda-\beta}; q)_\eta}{(q^\lambda; q)_\alpha (q^{\lambda+\alpha}; q)_\eta} \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (q^r; q)_{sn} (q^{\eta+\lambda-\beta}; q)_n (q^\lambda; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{\lambda-\beta}; q)_n (q^{\lambda+\alpha+\eta}; q)_n} z^{n+\lambda-\beta-1}. \end{aligned} \quad (4.1)$$

where $|q| < 1$ and $|z| < (1 - q)^{-r_1}$.

Proof. The result follows along the same line of proof as Theorem.1.

Theorem 3. The following theorem derives the action of the Saigo-type fractional q -integral operator of the second kind on the generalized small four-parameter q -Mittag-Leffler function. The following Saigo q -fractional calculus operator type fractional q -integral holds true:

$$\begin{aligned} & J_q^{(\alpha, \beta, \eta)} \left(z^{\lambda-1} e_{(r_1, r_2)}^{(r, s)}(z; q) \right) \\ &= \frac{(1 - q)^\alpha (q^{1-\lambda}; q)_\beta \left((q^{\eta+\alpha-\lambda+1}; q)_\beta q \right)^{-\frac{\alpha(\alpha+1)}{2-\alpha(\lambda-1)}}}{(q^{\eta-\lambda+1}; q)_\alpha} \\ & \times \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^{1-\lambda+\beta}; q)_n (q^{\eta-\lambda+1}; q)_n (q^{\eta+\alpha+\beta-\lambda+1}; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{1-\lambda}; q)_n} q^{-\alpha n} z^{n+\lambda-\beta-1}. \end{aligned} \quad (4.2)$$

where $|q| < 1$ and $|z| < (1 - q)^{-r_1}$.

Proof.

Let us denote the left-hand side of Theorem 3 by

$$L = J_q^{\alpha, \beta, \eta} z^{\lambda-1} E_{r_1, r_2}^{r, s}(z; q).$$

Using the definition of the q -Mittag-Leffler function (Equation 22), we have

$$L = J_q^{\alpha, \beta, \eta} z^{\lambda-1} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} z^n}{\Gamma_q(nr_1 + r_2)(q; q)_n}.$$

After applying the q -integral operator term-wise, we obtain

$$L = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2)(q; q)_n} J_q^{\alpha, \beta, \eta} z^{n+\lambda-1}.$$

Using the result from Equation (29), this becomes

$$L = \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2)(q; q)_n} \cdot \frac{\Gamma_q(\beta - n - \lambda + 1) \Gamma_q(\eta - n - \lambda + 1)}{\Gamma_q(-n - \lambda + 1) \Gamma_q(\alpha + \beta + \eta - n - \lambda + 1)} q^{-\alpha(n+\lambda-1) - \frac{\alpha(\alpha+1)}{2}} z^{n+\lambda-\beta-1}.$$

Using the definition of the q -Gamma function (Equation 7), we obtain

$$L = (1 - q)^\alpha q^{-\frac{\alpha(\alpha+1)}{2}} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2)(q; q)_n} \cdot \frac{(q^{1-n-\lambda}; q)_\beta (q^{\eta-n-\lambda+1}; q)_{\alpha+\beta}}{\Gamma_q(nr_1 + r_2)(q; q)_n} q^{-\alpha(n+\lambda-1)} z^{n+\lambda-\beta-1}.$$

Applying identity (13), we get

$$\begin{aligned} L &= (1 - q)^\alpha q^{-\frac{\alpha(\alpha+1)}{2}} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2)(q; q)_n} \\ & \times (q^{1-n-\lambda}; q)_\beta (q^{\eta-n-\lambda+1}; q)_\alpha (q^{\eta+\alpha-n-\lambda+1}; q)_\beta q^{-\alpha(n+\lambda-1)} z^{n+\lambda-\beta-1}. \end{aligned}$$

Using identity (14) and simplifying, we arrive at

$$\begin{aligned} L &= (1 - q)^\alpha q^{-\frac{\alpha(\alpha+1)}{2}} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn}}{\Gamma_q(nr_1 + r_2)(q; q)_n} \\ & \times \frac{(q^{1-\lambda}; q)_\beta (q^{1-\lambda+\beta}; q)_n (q^{\eta-\lambda+1}; q)_n}{(q^{1-\lambda}; q)_n (q^{\eta-\lambda+1}; q)_\alpha} (q^{\eta+\alpha-\lambda+1}; q)_\beta (q^{\eta+\alpha+\beta-\lambda+1}; q)_n \\ & \times q^{-\alpha(n+\lambda-1)} z^{n+\lambda-\beta-1}. \end{aligned}$$

Finally, the closed-form expression becomes

$$L = (1-q)^\alpha (q^{1-\lambda}; q)_\beta (q^{\eta+\alpha-\lambda+1}; q)_\beta q^{-[\frac{\alpha(\alpha+1)}{2} + \alpha(\lambda-1)]} (q^{\eta-\lambda+1}; q)_\alpha \\ \times \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^{1-\lambda+\beta}; q)_n (q^{\eta-\lambda+1}; q)_n (q^{\eta+\alpha+\beta-\lambda+1}; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{1-\lambda}; q)_n} q^{-\alpha n} z^{n+\lambda-\beta-1}. \quad (4.3)$$

Theorem 4. This theorem presents the Saigo-type fractional q -integral operator of the second kind acting on the generalized big four-parameter q -Mittag-Leffler function.

The Saigo-type fractional q -integral operator satisfies

$$J_q^{\alpha, \beta, \eta} \left(z^{\lambda-1} E_{r_1, r_2}^{r, s}(z; q) \right) \\ = (1-q)^\alpha \frac{(q^{1-\lambda}; q)_\beta (q^{\eta+\alpha-\lambda+1}; q)_\beta}{(q^{\eta-\lambda+1}; q)_\alpha} q^{-\frac{\alpha(\alpha+1)}{2}} \\ \times \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} q^{\frac{n(n-1)}{2}}}{\Gamma_q(nr_1 + r_2) (q; q)_n} \\ \times \frac{(q^{1-\lambda+\beta}; q)_n (q^{\eta-\lambda+1}; q)_n (q^{\eta+\alpha+\beta-\lambda+1}; q)_n}{(q^{1-\lambda}; q)_n} q^{-\alpha(n+\lambda-1)} z^{n+\lambda-\beta-1}. \quad (4.4)$$

where $|q| < 1$ and $|z| < (1-q)^{-r_1}$.

Proof. Following a similar line of reasoning as in Theorem (3), the desired result is obtained.

5. Subcases

The following result holds true:

$$I_{(z, q)}^{(\alpha, \beta, \eta)} \left(e_{(r_1, r_2)}^{(r, s)}(z; q) \right) = \frac{(1-q)^\alpha (q^{1-\beta}; q)_\eta}{(q^1; q)_\alpha (q^{1+\alpha}; q)_\eta} \\ \times \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^{\eta+1-\beta}; q)_n (q^1; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{1-\beta}; q)_n (q^{1+\alpha+\eta}; q)_n} z^{n-\beta}. \quad (5.1)$$

Proof. If we substitute $\lambda = 1$ in Theorem.1, we obtain the desired result.

The following result holds true:

$$J_q^{(\alpha, \beta, \eta)} \left(e_{(r_1, r_2)}^{(r, s)}(z; q) \right) = \frac{(1-q)^\alpha (q^0; q)_\beta ((q^{\eta+\alpha}; q)_\beta q)^{-\frac{\alpha(\alpha+1)}{2}}}{(q^\eta; q)_\alpha} \\ \times \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^\beta; q)_n (q^\eta; q)_n (q^{\eta+\alpha+\beta}; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^0; q)_n} q^{-\alpha n} z^{n-\beta}. \quad (5.2)$$

Proof. If we substitute $\lambda = 1$ in Theorem 4, we obtain the desired result.

$$I_{(z, q)}^\mu \left(z^{\lambda-1} E_{(r_1, r_2)}^{(r, s)}(z; q) \right) = \frac{(1-q)^\mu}{(q^\lambda; q)_\mu} \sum_{n=0}^{\infty} \frac{(q^r; q)_{sn} (q^\lambda; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{\lambda+\mu}; q)_n} z^{n+\lambda+\mu-1}, \quad (5.3)$$

Proof. If we put $\alpha = \mu$ and $\beta = \eta = 0$ in Theorem 1, we get the Riemann-Liouville type fractional q -integral.

The following result holds true:

$$I_{(z, q)}^{(\alpha, \beta, \eta)} \left(E_{(r_1, r_2)}^{(r, s)}(z; q) \right) = \frac{(1-q)^\alpha (q^{\lambda-\beta}; q)_\eta}{(q^1; q)_\alpha (q^{1+\alpha}; q)_\eta} \\ \times \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (q^r; q)_{sn} (q^{\eta+1-\beta}; q)_n (q^\lambda; q)_n}{\Gamma_q(nr_1 + r_2) (q; q)_n (q^{1-\beta}; q)_n (q^{1+\alpha+\eta}; q)_n} z^{n-\beta}. \quad (5.4)$$

Proof. If we put $\lambda = 1$ in Theorem 2, we obtain the desired result.

6. Motivation and Context

The study of q -analogues of special functions has gained considerable attention due to their fundamental role in q -calculus and quantum models. In particular, q -analogues of the Mittag–Leffler function naturally arise in the solution of q -fractional differential and difference equations, where classical fractional operators are replaced by their q -counterparts. Although several q -extensions of one-, two-, and three-parameter Mittag–Leffler functions have been investigated in the literature, corresponding results for the four-parameter case remain relatively limited, especially in connection with generalized fractional operators. This paper addresses this gap by introducing new q -analogues of the four-parameter Mittag–Leffler function and by establishing their behavior under Saigo-type fractional q -integral and derivative operators. The derived results provide useful analytical tools for the formulation and solution of q -fractional differential equations and related problems arising in mathematical physics and applied analysis.

7. Conclusion

We conclude by emphasizing that the results presented in this paper are original and significant. In particular, two new q -analogues of the four-parameter Mittag–Leffler function are introduced. Moreover, q -integral representations and Saigo-type q -fractional calculus operators associated with these functions are derived. Additionally, Riemann–Liouville type fractional q -integrals and q -derivatives for the proposed q -analogues of the four-parameter Mittag–Leffler function are established, thereby illustrating their relevance within the framework of q -fractional calculus.

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Arti sharma,
Department of Mathematics,
IcfaiTech, The ICFAI University, Jaipur
India
E-mail address: artisharma730020@gmail.com

and

V. K. Vyas,
Department of Mathematics,
IcfaiTech, The ICFAI University, Jaipur
India,.
E-mail address: vmathsvyas@gmail.com