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Radial Large Solutions to a Nonlinear Elliptic Problem in $\mathbb{R}^N$: on the Existence and Asymptotic Analysis

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ABSTRACT: We consider the following elliptic equation

$$\Delta_p u = g(x)h(u) \quad \text{in } \mathbb{R}^N,$$

where $\Delta_p u$ is the $p-$Laplacian with $N > p > 2$, and the functions h and g satisfy appropriate assumptions. We establish existence results for large solutions and describe their asymptotic behavior as $|x| \to \infty$.

Key Words: Nonlinear elliptic equation, p-Laplacian, radial solutions, large solutions, existence of solutions, asymptotic behavior.

Contents

1. Introduction and the Main Results

This work establishes the existence of positive entire large solutions to the p-Laplacian equation

$$\Delta_p u = g(x)h(u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

We assume $N > p > 2$, $g : \mathbb{R}^N \to \mathbb{R}^{+*}$ is radial function and continuous, and that the nonlinearity f satisfies appropriate structural conditions.
The analysis developed in this paper relies on the following structural assumptions, which are fundamental for the formulation and proof of our main results,

(G_1) : $\displaystyle\int_0^\infty r \min_{|x|=r} g(x)dr = \infty.$

(G_2) : $g(r) \sim r^{-\eta}$ at ∞ for $\eta \leq 2$.

(F_1) : $h \in C^1(0,\infty),\ h' \geq 0$ and $\begin{cases} h(0) = 0 \\ h > 0 \quad \text{in}\,(0,\infty). \end{cases}$

(F_2) : $\displaystyle\int_1^{+\infty} h^{-\frac{1}{p-1}}(t)dt = +\infty.$

(F_3) : $h(t) \sim t^\gamma$ at ∞ for $\gamma \in (0,1)$.

Our objective is to extend the analysis of solutions to (1.1) by first examining the radial Ordinary Differential Equation associated with the problem

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' = g(r)h(u(r)), \tag{1.2}$$

$$u(0) = a, \quad u'(0) = 0. \tag{1.3}$$

Extensive research has been devoted to studying (1.1) in the case $p = 2$. Models of the form

$$\Delta u = G(x)u^\beta \quad \text{and} \quad \Delta u = G(x)e^{2u}$$

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were initially examined by Cheng and Li [5], establishing foundational results on solution structures. These were later extended by Cheng and Ni [6,7], who studied conformal curvature equations on $\mathbb{R}^N$, deepening the connection between geometry and nonlinear Partial Differential Equations.
A key focus has been large solutions that blow up at infinity. Bandle and Marcus [3] provided sharp criteria for their existence and uniqueness, while Lair [9] established necessary and sufficient conditions in the semilinear case. Lair and Wood [10] further explored sublinear elliptic equations, contrasting them with the superlinear case.

The asymptotic behavior of solutions has also been extensively investigated, particularly for sublinear nonlinearities ($\gamma \in (0,1)$). Lazer and Mckenna [11] pioneered the study of blow-up phenomena in boundary value problems, while Cirstea and Radulescu [8] classified blow-up rates. Yang [16] generalized these results to unbounded domains, assuming $h \in C^1(0,\infty)$, $h' \geq 0$, and $\begin{cases} h(0) = 0 \\ h > 0 \quad \text{in } (0,\infty) \end{cases}$ and $\displaystyle\int_1^\infty h^{-1}(t)dt = \infty$.
For critical nonlinearities, Ni [13] analyzed $\Delta u + G(x)u^{(N+2)/(N-2)} = 0$, linking it to geometric problems. Li and Ni [12] further developed this connection, studying conformal scalar curvature equations. Bachar and Zeddini [2] derived existence results for positive solutions under the condition

$$(F_0) \quad \forall \epsilon > 0, \exists \alpha > 0 \text{ such that } |h(s_1) - h(s_2)| \leq \alpha |s_1 - s_2| \quad \forall s_1, s_2 \in [\epsilon, \infty).$$

Pucci, Serrin, and Zou [15] established strong maximum principles for singular elliptic inequalities, while Olofsson [14] studied decay properties of solutions to Osserman-Keller-type equations. Recent contributions by Bouzelmate, El Baghouri and Sennouni [4] and Alaa, Bouzelmate, Charkaoui and Hathout [1] have expanded the understanding of radial solutions for p-Laplacian problems in exterior domains and variable-growth settings, respectively.
A significant open problem originating from [10] concerns the study of the existence of large solutions to (1.2) When g is not radial but still satisfies the condition

$$\int_0^\infty r \min_{|x|=r} g(x) dr = +\infty.$$

This question arises naturally in various contexts within Riemannian geometry and mathematical physics. It is of particular relevance when considering nonlinearities in the superlinear regime ($\gamma > 1$) or when the nonlinearity satisfies the condition of Keller-Osserman, $\displaystyle\int_1^{+\infty} \frac{dt}{\sqrt{H(t)}} = +\infty$, where $H(t) = \displaystyle\int_0^t h(s)\, ds$ denotes the primitive of f. The resolution of this problem remains an active area of research. (see, e.g., [3,5,8,9,11,14]).

The following fundamental theorems contain the principal results of this work.

Theorem 1.1 *Assume that the nonlinearity h satisfies (F_1) - (F_2) and that the function g satisfies (G_1). Then for every $a > 0$, the problem (1.2) - (1.3) admits a positive solution $u(r)$ on $[0,\infty)$ such that* $\lim_{r\to\infty} u(r) = \infty$.

Theorem 1.2 *Consider a large positive solution u of problem (1.2).*
Assume that h satisfies (F_1)-(F_3) and g satisfies (G_2). Then the following asymptotic behavior holds

$$u(r) \underset{\infty}{\sim} \begin{cases} r^{\frac{p-\eta}{p-1-\gamma}} & \text{if } \eta < 2. \\ (\ln(r))^{\frac{p-1}{p-1-\gamma}} & \text{if } \eta = 2. \end{cases} \tag{1.4}$$

Moreover,

$$u'(r) \underset{\infty}{\sim} \begin{cases} r^{\frac{p-\eta}{p-1-\gamma}-1} & \text{if } \eta < 2. \\ r^{-1}(\ln(r))^{\frac{\gamma}{p-1-\gamma}} & \text{if } \eta = 2. \end{cases} \tag{1.5}$$

$$u''(r) \leq \begin{cases} C\, r^{-1-\frac{1-\gamma}{p-1-\gamma}(\eta-\gamma p)} & \text{at } \infty \quad \text{if } \eta < 2. \\ C\, r^{-2}(\ln r)^{\frac{\gamma}{1-\gamma}} & \text{at } \infty \quad \text{if } \eta = 2. \end{cases} \tag{1.6}$$

The research work as follows. Section 2 presents the demonstration of Theorem 1.1 and addresses the nonexistence of both large solutions in bounded domains. Section 3 focuses on proving the asymptotic analysis of large solutions. Specifically, we examine the case where the nonlinearity satisfies the asymptotic condition

$$h(t) \sim t^{\gamma} \quad \text{as } t \to \infty, \quad \text{with } \gamma \in (0,1),$$

and the function behaves as

$$g(r) \sim r^{-\eta} \quad \text{as } r \to \infty, \quad \text{with } \eta \leq 2.$$

This paper establishes the exact growth rate of solutions and analyzes the role of the parameters γ and η in determining their asymptotic properties. Finally, in the last section, we add a conclusion that provides an overview of the main results and some open questions, such as the existence of non radial large solutions.

2. Existence and Nonexistence

The present analysis establishes the existence result of Theorem 1.1. It also addresses two distinct nonexistence phenomena: the absence of large solutions in bounded domains and the absence of bounded entire solutions in $\mathbb{R}^N$. These results extend and complement earlier work found in [2,10]. We begin by stating the following proposition, which plays a central role in our analysis and is of independent interest.

Proposition 2.1 *Under the assumptions that Ω is a bounded domain in $\mathbb{R}^N$ $(N > p > 2)$, $g \in \mathcal{C}(\bar{\Omega})$, and h fulfills conditions (F_1) and (F_2), we establish the non-existence of a positive solution for the problem*

$$\begin{aligned} &\Delta_p u = g(x)h(u) \quad \text{in } \Omega, \\ &u(x) \longrightarrow \infty \quad \text{as } x \to \partial\Omega. \end{aligned} \tag{2.1}$$

Proof: We proceed by contradiction, assuming a positive solution $u(r)$ exists. Define the function

$$\psi(r) = \int_0^{u(r)} h(s)^{-\frac{1}{p-1}} ds.$$

Differentiating gives

$$\psi'(r) = h(u)^{-\frac{1}{p-1}} u',$$

and consequently computing the p-Laplacian of ψ,

$$\Delta_p \psi = \left(|\psi'|^{p-2}\psi'\right)' + \frac{N-1}{r}|\psi'|^{p-2}\psi' \tag{2.2}$$

$$= h(u)^{-1}\left(|u'|^{p-2}u'\right)' - \frac{h'(u)u'}{h(u)^2}|u'|^{p-2}u' + \frac{N-1}{r}h(u)^{-1}|u'|^{p-2}u'. \tag{2.3}$$

Using the equation (1.2), which allows us to simplify

$$\Delta_p \psi = h(u)^{-1}g(r)h(u) - \frac{h'(u)}{h(u)^2}|u'|^p = g(r) - \frac{h'(u)}{h(u)^2}|u'|^p.$$

By assumption, we obtain

$$\Delta_p \psi \leq g(r) \leq \max_{\overline{\Omega}} g =: M.$$

Now consider the function

$$\phi(r) = \psi(r) - \frac{M}{N(p-1)}r^p,$$

then $\Delta_p(r^p) > 0$. Since $\Delta_p \psi \leq -g(r)h(u)^{1/(p-1)} \leq 0$, it follows that

$$\Delta_p \phi \leq -\frac{M}{N(p-1)}p^{p-1}(N+p-2)r^{(p-1)(p-2)} < 0 \quad \text{for all } r \in (0,R).$$

However, ϕ is continuous on $[0, R]$ and attains its maximum at some $r^* \in [0, R]$. By the maximum principle for p-superharmonic functions, we must have $\Delta_p \phi(r^*) \geq 0$, which contradicts the strict inequality $\Delta_p \phi < 0$ everywhere. This contradiction shows that no such solution u can exist, completing the proof. □

The demonstration of Theorem 1.1 is founded upon the preceding proposition, which yields a fundamental estimate governing the asymptotic properties of solutions. By applying this estimate in conjunction with supplementary analytical techniques, we rigorously establish the existence of solutions adhering to the prescribed asymptotic criteria under the stated hypotheses.

Proof: (of Theorem 1.1) We begin with the radial formulation of the p-Laplacian equation

$$\left(|u^{'}|^{p-2}u^{'}\right)^{'} + \frac{N-1}{r}|u^{'}|^{p-2}u^{'} = g(r)h(u(r))$$

This expression leads to

$$\left(r^{N-1}|u^{'}|^{p-2}u^{'}\right)^{'} = r^{N-1}g(r)h(u(r)) \tag{2.4}$$

Integrating equation (2.4) from 0 to s, we obtain

$$s^{N-1}|u^{'}|^{p-2}u^{'} = \int_0^s t^{N-1}g(t)h(u(t))dt \tag{2.5}$$

$$|u^{'}|^{p-2}u^{'} = s^{1-N}\int_0^s t^{N-1}g(t)h(u(t))dt.$$

From this identity, we deduce that $|u^{'}|^{p-2}u^{'} > 0$, which implies that u is strictly increasing and

$$(u')^{p-1}(s) = s^{1-N}\int_0^s t^{N-1}g(t)h(u(t))\,dt.$$

$$u(r) = a + \int_0^r \left(s^{1-N}\int_0^s t^{N-1}g(t)h(u(t))\,dt\right)^{\frac{1}{p-1}} ds. \tag{2.6}$$

Let $a > 0$. Then there exists $r_0 > 0$ such that the integral equation (2.6) admits a positive solution $u \in C([0, r_0])$.

We construct a solution iteratively. Set

$$u_0(r) = a, \quad u_{k+1}(r) = a + \int_0^r \left(s^{1-N}\int_0^s t^{N-1}g(t)h(u_k(t))\,dt\right)^{\frac{1}{p-1}} ds, \quad k \geq 0.$$

Let $r_0 > 0$ be taken sufficiently small such that

$$\int_0^{r_0} \left(s^{1-N}\int_0^s t^{N-1}g(t)h(u_k(t))\,dt\right)^{\frac{1}{p-1}} ds < \frac{a}{h^{\frac{1}{p-1}}(a)}. \tag{2.7}$$

By applying the method of mathematical induction, we obtain

$$a \leq u_k(r) \leq 2a, \quad r \in [0, r_0], \quad k \in \mathbb{N}, \tag{2.8}$$

and

$$0 \leq (u_{k+1}^{'})^{p-1}(r) = r^{1-N}\int_0^r t^{N-1}g(t)h(u_k(t))dt \leq h(2a)\int_0^{r_0} g(t)dt < \infty, r \in [0, r_0]. \tag{2.9}$$

Consequently, the sequence (u_k) is uniformly bounded and equicontinuous on the interval $[0, r_0]$. An application of the Arzela-Ascoli theorem guarantees the existence of a uniformly convergent subsequence on $[0, r_0]$, whose limit u is a continuous function satisfying the integral equation (2.6).

Note that for $r \in [0, r_0]$,

$$(u')^{p-1}(r) = r^{1-N} \int_0^r t^{N-1} g(t) h(u(t))\, dt < \infty.$$

This ensures that u remains finite for finite r. Otherwise, if $u(r) \to \infty$ as $r \to r^*$ for some $r^* < \infty$, then u would be a large positive solution of (2.1) in $\Omega = \mathcal{B}_{r^*}$, which contradicts Proposition 1. Consequently, u admits a global extension to $[0, \infty)$ and remains positive.
Integrating (2.4) from 0 to r yields

$$r^{N-1} |u'(r)|^{p-2} u'(r) = \int_0^r s^{N-1} g(s) h(u(s))\, ds. \tag{2.10}$$

Now, suppose that u is bounded and strictly increasing, with $\lim\limits_{r\to\infty} u(r) = L \in (0, +\infty)$. Then, by the properties of h, there exists $c > 0$ and $R_0 > 0$ such that $h(u(r)) \geq c$ for all $r \geq R_0$. For $r \geq R_0$, the right-hand side of (2.10) satisfies the estimate

$$\int_0^r s^{N-1} g(s) h(u(s))\, ds \geq c \int_{R_0}^r s^{N-1} g(s)\, ds.$$

Given $N > p > 2$ and $s \geq 1$, hypothesis (G_1) implies that

$$\int_{R_0}^r s^{N-1} g(s)\, ds \geq \int_{\max(R_0,1)}^r s g(s)\, ds \to +\infty \quad \text{as} \quad r \to +\infty.$$

From (2.10), it follows that

$$u'(r) \geq \left(\frac{c}{r^{N-1}} \int_{R_0}^r s^{N-1} g(s)\, ds \right)^{1/(p-1)} \to +\infty \quad \text{as} \quad r \to +\infty.$$

since $N > p$, the numerator grows faster than the denominator. Integrating $u'(r)$ from R_0 to $+\infty$ then leads to a contradiction with the fact that u is bounded. We conclude that $\lim\limits_{r\to\infty} u(r) = +\infty$. □

Under more restrictive conditions, Theorem 1.1 leads to the following corollary.

Corollary 2.1 *If the nonlinearity h satisfies assumptions (F_1) - (F_3) and the function g satisfies (G_2), then problem (1.2) admits infinitely many positive large solutions.*

Proof: For $g(r) = r^{-\eta}$ with $r \geq R_0$, condition (G_1) holds for all $\eta \leq 2$.
The function h clearly satisfies conditions (F_1) and (F_2).
By the Theorem 1.1, for each $a > 0$, there exists a solution $u_a(r)$ with initial value $u(0) = a$ that grows unboundedly as $r \to \infty$. Since a is arbitrary, there are infinitely many such solutions. Under the assumptions, the problem admits infinitely many positive solutions $u(r; a)$ that grow unboundedly as $r \to \infty$. □

3. Asymptotic Behavior

We now establish Theorem 1.2 characterizing the asymptotic behavior of solutions when the nonlinearity h has asymptotic growth $h(t) \sim t^\gamma$ ($\gamma \in (0, 1)$) and the function g decays as $g(r) \sim r^{-\eta}$ ($\eta \leq 2$).

Lemma 3.1 *let u be a solution of the (1.2).Define*

$$v(r) = r^{-\frac{p-\eta}{p-1-\gamma}} u(r) \quad \textit{for } \eta < 2, \qquad w(r) = (\ln r)^{-\frac{p-1}{p-1-\gamma}} u(r) \quad \textit{for } \eta = 2. \tag{3.1}$$

Then under assumptions (F_3) on h and (G_2) on g , we have *(i)* $\liminf_{r\to\infty} v(r) = 0 \;\Rightarrow\; \lim_{r\to\infty} v(r) = 0.$

(ii) $\liminf_{r\to\infty} w(r) = 0 \;\Rightarrow\; \lim_{r\to\infty} w(r) = 0.$

Proof: We study the behavior of the solutions for both cases.

Case (i): $\eta < 2$.

Assume $\liminf_{r\to\infty} v(r) = 0$. Suppose that the conclusion is false, then there exist sequences $r_n \to \infty$ with $v(r_n) \to 0$ and $s_n \to \infty$ with $v(s_n) \geq \epsilon > 0$. For each n, select $\rho_n \in (r_n, s_n)$ with $v(\rho_n) = \epsilon/2$. There exists $t_n \in (r_n, \rho_n)$ such that

$$v'(t_n) \geq \frac{v(\rho_n) - v(r_n)}{\rho_n - r_n} \geq -\frac{v(r_n)}{\rho_n - r_n}.$$

Then,

$$|v'(t_n)|^{p-2} v''(t_n) \approx C t_n^{\frac{p-\eta}{p-1-\gamma}(p-1)-\eta} v^{\gamma}(t_n).$$

where $C = (p-1)\left(\frac{p-\eta}{p-1-\gamma}\right)^{p-2}\left(\frac{p-\eta}{p-1-\gamma} - \frac{N-1}{p-1}\right) > 0$. Integrating between t_n and ρ_n gives

$$\int_{v(t_n)}^{\epsilon/2} \frac{dv}{v^{\gamma/(p-1)}} \leq C' \int_{t_n}^{\rho_n} t^{\frac{\frac{p-\eta}{p-1-\gamma}(p-1)-\eta}{p-1}} dt.$$

Since $\frac{p-\eta}{p-1-\gamma}(p-1) - \eta = \frac{(p-1)(p-\eta-\eta)+\gamma\eta}{p-1-\gamma} < 0$, the right-hand side tends to 0 while the left-hand side satisfies

$$\liminf_{n\to\infty} \int_{v(t_n)}^{\epsilon/2} \frac{dv}{v^{\gamma/(p-1)}} \geq \int_{0}^{\epsilon/2} \frac{dv}{v^{\gamma/(p-1)}} = +\infty.$$

This contradiction establishes that $v(r) \to 0$ as $r \to \infty$.

Case (ii): $\eta = 2$.

Suppose that $\liminf_{r\to\infty} w(r) = 0$, but $w(r)$ does not converge to zero globally. Then there exist sequences (r_n),(s_n) with $r_n \to \infty$, $s_n \to \infty$, and a constant $\epsilon > 0$ such that

$$w(r_n) \to 0 \quad \text{and} \quad w(s_n) \geq \epsilon \quad \text{for all } n.$$

For each n, we can find $\rho_n \in (r_n, s_n)$ satisfying

$$w(\rho_n) = \frac{\epsilon}{2} \quad \text{and} \quad w'(t_n) \geq 0 \text{ for some } t_n \in (r_n, \rho_n).$$

Moreover, exploiting the rescaling argument, we obtain $t_n \to \infty$, where

$$C = \frac{(p-1-\gamma)^{p-1}}{(p-1)^{p-2}} > 0$$

is a positive constant.

Integrating between t_n and ρ_n gives

$$\int_{w(t_n)}^{\epsilon/2} \frac{dw}{w^{(\gamma-(p-1))/(p-1)}} \leq C' \int_{t_n}^{\rho_n} \frac{dt}{t^2 \ln t}. \tag{3.2}$$

As $n \to \infty$, the right-hand side tends to 0 because

$$\int_{t_n}^{\rho_n} \frac{dt}{t^2 \ln t} \sim \frac{1}{t_n \ln t_n} \to 0.$$

However, since $\gamma < p-1$, the exponent $\frac{\gamma-(p-1)}{p-1} < 0$, and the left-hand side is bounded below by a positive constant. Specifically,

$$\int_{w(t_n)}^{\epsilon/2} \frac{dw}{w^{\frac{\gamma-(p-1)}{p-1}}} \geq \int_0^{\epsilon/2} \frac{dw}{w^{\frac{\gamma-(p-1)}{p-1}}} = \infty,$$

which contradicts the convergence to 0.

Therefore, the assumption that $w(r)$ does not converge to 0 is false. We conclude that $w(r) \to 0$ as $r \to \infty$.

□

Lemma 3.2 *Provided the hypotheses of Lemma 3.1 are satisfied, there exists a constant $A > 0$ such that for $\eta < 2$,*

$$u(r) \geq A\, r^{\frac{p-\eta}{p-1-\gamma}} \quad \textit{for all } r \textit{ sufficiently large.}$$

Proof: Let $u(r)$ be a positive radial solution to equation (1.2) under the conditions $0 < \eta < 2$ and $p > 1+\gamma$. To analyze its asymptotic behavior, we use the normalized function v given by (3.1).

Substituting $v(r)$ into equation (1.2) gives

$$\left(r^{N-1} \left| v' + \frac{p-\eta}{p-1-\gamma} r^{-1} v \right|^{p-2} \left(v' + \frac{p-\eta}{p-1-\gamma} r^{-1} v \right) \right)' = r^{N-1-\eta+\gamma\frac{p-\eta}{p-1-\gamma}} v^{\gamma}. \tag{3.3}$$

Differentiating the left-hand side leads to

$$r^{N-1} \left| v' + \frac{p-\eta}{p-1-\gamma} r^{-1} v \right|^{p-2} \left[v'' + \left(\frac{p-\eta}{p-1-\gamma} + \frac{N-1}{r} \right) v' \right.$$
$$\left. + \frac{p-\eta}{p-1-\gamma} \left(\frac{p-\eta}{p-1-\gamma} - 1 \right) r^{-2} v \right] = r^{N-1-\eta+\gamma\frac{p-\eta}{p-1-\gamma}} v^{\gamma}. \tag{3.4}$$

Now, assume that for large r, the function $v(r)$ has the asymptotic behavior

$$v(r) \to A > 0, \quad v'(r) = o(r^{-1}), \quad \text{and} \quad v''(r) = o(r^{-2}).$$

This implies that asymptotically, (3.4) becomes

$$\left(\frac{p-\eta}{p-1-\gamma} \right)^{p-1} \left[(p-1) \frac{p-\eta}{p-1-\gamma} + (N-1) \right] r^{N-1-p+\frac{(p-\eta)(p-1)}{p-1-\gamma}} A^{p-1} \sim r^{N-1-\eta+\gamma\frac{p-\eta}{p-1-\gamma}} A^{\gamma}. \tag{3.5}$$

For the asymptotic balance to hold, the exponents of r on both sides must match. This condition determines the constant A, yielding

$$A = \left((p-1) \left(\frac{p-\eta}{p-1-\gamma} \right)^2 + (N-1) \frac{p-\eta}{p-1-\gamma} \right)^{\frac{1}{p-1-\gamma}}.$$

By the definition of $\lim_{r\to\infty} v(r) = A$, for any $\epsilon > 0$, there exists a radius R_ϵ such that for all $r \geq R_\epsilon$, we have $v(r) \geq A - \epsilon$. Consequently, from the definition of v, we derive the lower bound for the solution

$$u(r) = v(r) r^{\frac{p-\eta}{p-1-\gamma}} \geq (A-\epsilon) r^{\frac{p-\eta}{p-1-\gamma}}.$$

Since $\epsilon > 0$ was arbitrary, this inequality holds in the limit as $\epsilon \to 0$. In particular, by choosing $\epsilon = A/2$, we conclude that there exists a sufficiently large R_0 and a positive constant $A_0 = A/2$ such that

$$u(r) \geq A_0 r^{\frac{p-\eta}{p-1-\gamma}} \quad \text{for all } r \geq R_0. \tag{3.6}$$

This establishes the desired lower asymptotic bound for the solution. □

Lemma 3.3 *Under the hypotheses of Lemma 3.1, there exists a constant $A > 0$ such that for $\eta = 2$,*

$$u(r) \geq A(\ln r)^{\frac{p-1}{p-1-\gamma}} \quad \textit{for all sufficiently large } r.$$

Proof: Let $t = \ln r$ and define

$$y(t) = u(e^t). \tag{3.7}$$

The original differential equation then transforms into

$$\left(e^{(N-1)t}|y'|^{p-2}y'\right)' = e^{(N-3)t}y^{\gamma}. \tag{3.8}$$

Expanding the derivative on the left-hand side (and noting $y'(t) > 0$ for $t > 0$) yields

$$e^{(N-1)t}|y'|^{p-2}\left[(p-1)y'' + (N-1)y'\right] = e^{(N-3)t}y^{\gamma}.$$

For large t, we seek an asymptotic subsolution of the form

$$z(t) = c\left(t^{\frac{p-1}{p-1-\gamma}} - t^{\frac{p-1}{p-1-\gamma}-\epsilon}\right), \tag{3.9}$$

where $c > 0$ and $\epsilon > 0$ is small. The dominant terms in the derivatives are

$$z'(t) \sim c\frac{p-1}{p-1-\gamma}t^{\frac{p-1}{p-1-\gamma}-1}, \quad z''(t) \sim c\frac{p-1}{p-1-\gamma}\left(\frac{p-1}{p-1-\gamma}-1\right)t^{\frac{p-1}{p-1-\gamma}-2}.$$

Substituting $z(t)$ into the differential equation and matching the leading-order terms, we obtain the balance condition

$$\left(c\frac{p-1}{p-1-\gamma}\right)^{p-1}\left[(p-1)\left(\frac{p-1}{p-1-\gamma}-1\right)+(N-1)\right] = c^{\gamma}.$$

Solving for c gives

$$c = \left(\frac{\left(\frac{p-1}{p-1-\gamma}\right)^{p-1}}{(p-1)\left(\frac{p-1}{p-1-\gamma}-1\right)+(N-1)}\right)^{\frac{1}{p-1-\gamma}}.$$

By comparison, $y(t) \geq z(t)$ for large t. Thus, for large r

$$u(r) \geq c(\ln r)^{\beta}\left(1-(\ln r)^{-\epsilon}\right),$$

where $\beta = \frac{p-1}{p-1-\gamma}$. For sufficiently large r, the term $1-(\ln r)^{-\epsilon} \geq \frac{1}{2}$, hence

$$u(r) \geq \frac{c}{2}(\ln r)^{\frac{p-1}{p-1-\gamma}}.$$

Taking $A = \frac{c}{2}$ completes the proof. □

Lemma 3.4 *Assuming the hypotheses of Lemma 3.1 are valid. There exists a constant $A > 0$ such that*

$$\begin{array}{llll} u(r) \leq & Ar^{\frac{p-\eta}{p-1-\gamma}} & \textit{if } \eta < 2 & \textit{for sufficiently large } r. \\ u(r) \leq & A(\ln(r))^{\frac{p-1}{p-1-\gamma}} & \textit{if } \eta = 2 & \textit{for sufficiently large } r. \end{array}$$

Proof: Suppose $u(r)$ is a positive radial solution of (1.2) with $p > 2$, $0 < \gamma < 1$, and $\eta \leq 2$. The proof proceeds by considering the two cases $\eta < 2$ and $\eta = 2$ separately.

First, assume $\eta < 2$. We start by writing the integrated version of the equation

$$r^{N-1}|u'(r)|^{p-1} = \int_{R_0}^{r} s^{N-1-\eta}u(s)^{\gamma}ds.$$

Substituting into the integral yields

$$r^{N-1}|u'(r)|^{p-1} \le A^\gamma \int_{R_0}^r s^{N-1-\eta+\gamma\frac{p-\eta}{p-1-\gamma}}\,ds \approx \frac{A^\gamma}{N-\eta+\gamma\frac{p-\eta}{p-1-\gamma}} r^{N-\eta+\gamma\frac{p-\eta}{p-1-\gamma}}. \tag{3.10}$$

$$|u'(r)| \le A^{\gamma/(p-1)} r^{(1-\eta+\gamma\frac{p-\eta}{p-1-\gamma})/(p-1)}.$$

The optimal A must satisfy $A^{p-1-\gamma} = \dfrac{(p-1)\left(\frac{p-\eta}{p-1-\gamma}-1\right)+(N-1)}{\left(\frac{p-\eta}{p-1-\gamma}\right)^{p-1}}$.

Using the comparison principle, we get

$$u(r) \le A r^{\frac{p-\eta}{p-1-\gamma}} \quad \text{for all } r \ge R. \tag{3.11}$$

Now consider the case $\eta = 2$. Let $t = \ln r$, by (3.7) The equation (1.2) becomes

$$(e^{(N-p)t}|y'|^{p-1})' = e^{(N-p-2)t} y^\gamma.$$

Assume $y(t) \sim A t^{\frac{p-1}{p-1-\gamma}}$ at ∞ . This gives

$$(N-p)A^{p-1}\left(\frac{p-1}{p-1-\gamma}\right)^{p-1} e^{(N-p)t} t^{\left(\frac{p-1}{p-1-\gamma}-1\right)(p-1)} \approx e^{(N-p-2)t} A^\gamma t^{\gamma\frac{p-1}{p-1-\gamma}}.$$

The optimal A satisfies $A^{p-1-\gamma} = (N-p)\left(\dfrac{p-1}{p-1-\gamma}\right)^{p-1}$, then $u(r) \le A(\ln r)^{\frac{p-1}{p-1-\gamma}}$ for all $r \ge R$. □

We establish a comparison lemma ensuring that, under appropriate conditions on the nonlinearities and weight functions, the positive large radial solutions of p-Laplacian equations can be rigorously ordered.

Lemma 3.5 *Assume v_1 and v_2 are positive large solutions satisfying the following initial value problems*

$$\left(|v_1'|^{p-2}v_1'\right)' + \frac{N-1}{r}|v_1'|^{p-2}v_1' = g_1(r)h_1(v_1(r)),$$
$$\left(|v_2'|^{p-2}v_2'\right)' + \frac{N-1}{r}|v_2'|^{p-2}v_2' = g_2(r)h_2(v_2(r)),$$

with $v_1(0) = \alpha_1 > 0$, $v_2(0) = \alpha_2 \ge \alpha_1$, and $v_1'(0) = v_2'(0) = 0$. Suppose h_1 satisfies (F_1), and assume further that

$$h_2(t) \ge h_1(t) \quad \textit{for } t \ge \alpha_1, \qquad \textit{and} \qquad g_2(r) \ge g_1(r) \ge 0 \quad \textit{for } r \ge 0.$$

Then,

$$v_2(r) \ge v_1(r) \quad \textit{for all } r \in [0,\infty).$$

Proof: Set $v := v_2 - v_1$, from Theorem 1.1, $v_2(r) \ge \alpha_2 \ge \alpha_1 \ge 0$.

Hence,

$$\Delta_p v = g_2(r)h_2(v_2) - g_1(r)h_1(v_1) \;\ge\; g_1(r)\big(h_1(v_2) - h_1(v_1)\big).$$

Since $v_2 \ne v_1$, define $c(r) := g_1(r)\,\frac{h_1(v_2)-h_1(v_1)}{v_2-v_1} \;\ge\; 0$, so that $\Delta_p v \;\ge\; c(r)\,v$ in $[0,\infty)$.
By the maximum principle applied on each ball $\mathcal{B}_r$, it follows that $v(r) \ge v(0) \ge 0$, $r \ge 0$. □

We now aim to characterize the asymptotic behavior of solutions to our problem in the case where the nonlinearity satisfies $h(t) \sim t^\gamma$ as $t \to \infty$ with $\gamma \in (0,1)$, and $g(r) \sim r^{-\eta}$ as $r \to \infty$ with $\eta \le 2$.

Proof: (of Theorem 1.2) Since $g(r) \sim r^{-\eta}$ and $h(t) \sim t^\gamma$ as $r, t \to \infty$, there exist radial functions $\hat{g}_1(r)$ and $\hat{g}_2(r)$ such that

$$\hat{g}_1(r) \le g(r) \le \hat{g}_2(r) \quad \text{for all } r \ge 0,$$

and

$$\hat{g}_i(r) = A_i\, r^{-\eta} \quad \text{for } r \geq r_0 > 0, \quad i = 1, 2,$$

with constants $A_1, A_2 > 0$.

Similarly, there exist constants $B_1, B_2 > 0$ such that

$$B_1 t^{\gamma} \leq h(t) \leq B_2 t^{\gamma} \quad \text{for } t \geq \frac{1}{2}u(0),$$

with B_1 sufficiently small and B_2 sufficiently large.

By Lemmas 3.1–3.5, we can construct radial solutions u_1 and u_2 satisfying

$$u_1(r) \leq u(r) \leq u_2(r) \quad \text{for all } r > 0,$$

and for $i = 1, 2$,

$$u_i(r) \underset{\infty}{\sim} \begin{cases} r^{\frac{p-\eta}{p-1-\gamma}}, & \text{if } \eta < 2, \\ (\ln r)^{\frac{p-1}{p-1-\gamma}}, & \text{if } \eta = 2. \end{cases}$$

Hence $u(r)$ satisfies the asymptotic behavior stated in the theorem.

We infer by (2.5) ,

$$r^{N-1}(u_i')^{p-1}(r) \sim \begin{cases} \displaystyle\int_{r_0}^{r} s^{-\eta} s^{\frac{p-\eta}{p-1-\gamma}\gamma} s^{N-1} ds \sim r^{N-\eta+\gamma\frac{p-\eta}{p-1-\gamma}}, & \text{if } \eta < 2, \\ \displaystyle\int_{r_0}^{r} s^{-2} (\ln s)^{\frac{p-1}{p-1-\gamma}\gamma} s^{N-1} ds \sim r^{N-2} (\ln r)^{\frac{p-1}{p-1-\gamma}\gamma}, & \text{if } \eta = 2. \end{cases}$$

Consequently, the asymptotic behavior (1.5) is established.

Furthermore, employing the original equation and the fact that $u'(r) > 0$, we derive the second-order estimate (1.6).

This completes the proof.

□

Under an alternative set of hypotheses on g adapted to a specific case, the conclusion follows from the argument established in Theorem 1.2.

Theorem 3.1 *Assume that h satisfies (F_1) - (F_3), and that g satisfies $g(r) = g_0(r) + O(r^{-\alpha})$ as $r \to \infty$, with $\alpha > \eta + \frac{p-\eta}{1-\gamma}$, where g_0 satisfies (G_1) - (G_2). Then, problem (1.2) admits infinitely many positive radial large solutions u satisfying (1.4). Moreover, the solution u satisfies (1.5) and (1.6).*

Proof: We begin by considering the family of radial solutions $(v_n)_n$ to the equation

$$\Delta_p v = g_0(r)h(v) \tag{3.12}$$

with initial values $v_n(0) = a_n$, where (a_n) is an increasing sequence such that $a_n \to \infty$ as $n \to \infty$. By Theorem 1.1 and Corollary 2.1, these solutions exist and satisfy the asymptotic behavior

$$v_n(r) \underset{\infty}{\sim} \begin{cases} r^{\frac{p-\eta}{p-1-\gamma}} & \text{for } \eta < 2, \\ (\ln(r))^{\frac{p-1}{p-1-\gamma}} & \text{for } \eta = 2. \end{cases} \tag{3.13}$$

For the full problem $\Delta_p u = g(r)h(u)$, we construct solutions u_n via comparison. Since $g(r) \geq g_0(r)$ and h is increasing, Lemma 3.5 guarantees the existence of minimal solutions $u_n \geq v_n$ with the same initial data. The condition $\alpha > \eta + \frac{p-\eta}{1-\gamma}$ ensures the perturbation $g(r) - g_0(r)$ is negligible in the asymptotic regime.

The asymptotic bounds follow from Lemmas 3.1-3.5.

For $\eta < 2$,

$$A_1 r^{\frac{p-\eta}{p-1-\gamma}} \leq u_n(r) \leq A_2 r^{\frac{p-\eta}{p-1-\gamma}}, \tag{3.14}$$

And for $\eta = 2$,

$$A_1(\ln(r))^{\frac{p-1}{p-1-\gamma}} \leq u_n(r) \leq A_2(\ln(r))^{\frac{p-1}{p-1-\gamma}}, \tag{3.15}$$

for $\eta < 2$,

$$u_n'(r) \underset{\infty}{\sim} \frac{p-\eta}{p-1-\gamma} r^{\frac{p-\eta}{p-1-\gamma}-1},$$

and for $\eta = 2$,

$$u_n'(r) \underset{\infty}{\sim} \frac{p-1}{p-1-\gamma} r^{-1} \ln^{\frac{\gamma}{p-1-\gamma}}(r).$$

The second derivative bounds follow from the equation structure.

For sufficiently large r assuming $\eta < 2$,

$$u_n''(r) \leq A r^{-1-\frac{1-\gamma}{p-1-\gamma}(\eta-\gamma p)}, \tag{3.16}$$

and for sufficiently large r assuming $\eta = 2$,

$$u_n''(r) \leq A r^{-2}(\ln(r))^{\frac{\gamma}{1-\gamma}}. \tag{3.17}$$

The infinite family $(u_n)_n$ is distinguished by their initial values $(a_n)_n$ and ordered by the comparison principle. Each solution has distinct asymptotics, proving the existence of infinitely many solutions with the claimed properties. The proof is completed □

4. Conclusion

We have studied the nonlinear elliptic problem

$$\Delta_p u = g(x)h(u), \quad u \geq 0 \quad \text{in } \mathbb{R}^N.$$

subject to $u(x) \to \infty$ as $|x| \to \infty$. Theorem 1.1 guarantees the existence of large radial positive solutions under appropriate conditions on h and g, while Theorem 1.2 provides their precise asymptotic behavior, illustrating the influence of the parameters γ and η on the growth of $u(r)$ and its derivatives.

An open problem remains: whether large solutions exist when g is nonradial but satisfies

$$\int_0^{+\infty} r \min_{|x|=r} g(x)\, dr = +\infty.$$

and more generally, whether there exist nonradial large solutions of the problem. Resolving these questions would provide insight into the effect of anisotropic weights and symmetry-breaking phenomena on the existence and asymptotic behavior of large solutions, representing a natural direction for future research.

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