Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–26.
ISSN-0037-8712
doi:10.5269/bspm.79745

Weak Sharpness and Minimum Principle Sufficiency in Variational Inequalities with Fractional Curvilinear Integral Functional Constraints

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ABSTRACT: This paper develops a comprehensive theoretical framework for analyzing weak sharp solutions and minimum principle sufficiency conditions in variational inequalities with fractional curvilinear integral functional constraints. By extending classical variational analysis to the fractional calculus setting, we establish novel characterizations of solution sets through dual gap functionals and geometric properties of feasible regions. We introduce enhanced sufficiency conditions that bridge the gap between convexity assumptions and weak sharpness, providing refined estimates for the distance between arbitrary feasible points and the solution set. The main contributions include: (i) generalized dual gap analysis with explicit modulus of weak sharpness; (ii) equivalence results connecting minimum principle sufficiency to geometric properties of normal and tangent cones; (iii) stability analysis of solution sets under perturbations of the integral kernel; (iv) neutrosophic extension incorporating indeterminacy components for uncertain systems; (v) comprehensive discussion of theoretical and practical implications.

Keywords: Fractional calculus, variational inequalities, weak sharpness, minimum principle sufficiency, dual gap functionals, curvilinear integrals, neutrosophic theory.

Contents

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2020 *Mathematics Subject Classification:* 60H10, 26A33, 60G22.
Submitted October 26, 2025. Published June 02, 2026.

1. Introduction

Variational inequalities provide a unified mathematical framework for modeling equilibrium problems, constrained optimization, and complementarity systems across diverse scientific domains [9,10]. The classical theory, extensively developed for standard integral functionals, has demonstrated remarkable success in mechanics, economics, control theory, and engineering applications. However, numerous physical and biological phenomena—including viscoelastic material behavior, anomalous diffusion in heterogeneous media, and systems with hereditary properties—exhibit memory effects and non-local interactions that transcend the modeling capabilities of integer-order calculus [14,17,12,20]. Fractional calculus addresses this limitation by generalizing differentiation and integration to arbitrary real or complex orders, thereby providing essential mathematical tools for capturing intricate temporal and spatial dependencies. The Riemann-Liouville fractional integral, through its weighted averaging mechanism over historical function values, naturally incorporates memory effects and non-local influences. When combined with curvilinear integral functionals—which arise naturally in mechanical work calculations, electromagnetism, and path-dependent thermodynamic systems—fractional operators enable the formulation of variational problems with unprecedented modeling flexibility and physical fidelity [21,22].

The concept of weak sharp minima, introduced by Polyak and further developed by Marcotte and Zhu [15], represents a significant refinement of classical optimality conditions in optimization theory. A solution set exhibits weak sharpness when the objective function grows at least linearly with distance from the solution set, characterized by a positive sharpness modulus. This geometric property carries profound implications: it ensures linear convergence rates for projection-based algorithms, provides computable error bounds for iterative methods, and enables sharp sensitivity analysis in parametric optimization [11]. Recent investigations have extended weak sharpness concepts to variational inequalities, where it characterizes solution set stability and problem well-posedness [4]. Complementing weak sharpness, the minimum principle sufficiency property—developed by Ferris and Mangasarian [18]—asserts that maximizers of auxiliary dual problems coincide precisely with the original solution set. This duality-based characterization connects naturally to saddle-point theory and provides alternative computational approaches through dual formulations.

The intersection of fractional calculus, variational inequalities, and uncertainty quantification remains largely unexplored in existing literature. While fuzzy approaches have been applied to variational problems [1], and intuitionistic fuzzy methods have addressed multi-criteria decision-making [19], these frameworks cannot simultaneously capture truth, indeterminacy, and falsity—three independent components essential for modeling complex uncertain systems. Neutrosophic theory, introduced by Smarandache [16], overcomes this limitation by incorporating three membership functions representing truth (T), indeterminacy (I), and falsity (F) with values in $[0,1]$ satisfying $0 \leq T + I + F \leq 3$. Recent advances have demonstrated neutrosophic approaches in optimization [3], decision-making under uncertainty [2], metric space theory [5], and variational problems with incomplete information [6]. However, the extension of weak sharpness theory and minimum principle sufficiency to fractional variational inequalities within a neutrosophic framework represents uncharted theoretical territory.

This paper contributes to the literature through several original theoretical advancements. First, we establish enhanced primal and dual gap functionals incorporating fractional differential operators and prove their differentiability under weaker regularity assumptions than standard convex analysis requires, specifically accounting for the non-local nature of fractional integrals. Second, we derive explicit lower bounds for the weak sharpness modulus expressed in terms of the fractional order parameters α_λ, the strong convexity modulus μ, the domain measure $|\Delta|$, and the Gamma function, revealing how fractional order influences solution stability. Third, we extend the minimum principle sufficiency framework beyond classical convex functionals to quasi-convex settings, identifying precise conditions under which weak sharpness becomes both necessary and sufficient for minimum principle sufficiency. Fourth, we conduct rigorous stability and perturbation analysis, establishing quantitative Hausdorff distance bounds between original and perturbed solution sets as functions of perturbation magnitude, sharpness modulus, and fractional parameters. Fifth, we develop a comprehensive neutrosophic extension that incorporates truth, indeterminacy, and falsity components to model uncertain and imprecise variational systems [16,5,3], achieving complete parallel theoretical results including neutrosophic weak sharpness characterization, neutrosophic minimum principle sufficiency equivalence, and neutrosophic stability estimates. Sixth, we provide explicit computational guidance through gap functional evaluations and dual formulations relevant to recent advances in fractional variational methods [1,8,7]. Finally, we synthesize these contributions in a comprehensive discussion section analyzing theoretical implications, computational considerations, and applications to fractional mechanics, uncertain control systems, and anomalous transport phenomena.

The paper organization systematically develops this theoretical framework. Section 2 establishes mathematical preliminaries, including fractional operators based on Riemann-Liouville theory [14,12], function space topology, convexity definitions adapted to fractional settings, and variational derivatives for curvilinear integral functionals. Section 3 formulates the variational inequality problem, introduces primal and dual gap functionals with explicit fractional differential structure, and proves fundamental properties connecting gap functionals to solution sets. Section 4 presents the complete characterization of weak sharpness through geometric frameworks involving normal and tangent cones, establishes the main equivalence between weak sharpness and linear gap functional growth, and derives quantitative bounds for the sharpness modulus incorporating fractional parameters. Section 5 analyzes minimum principle sufficiency, proves equivalence with weak sharpness under convexity and constancy assumptions, and establishes bidirectional implications through rigorous set-theoretic arguments. Section 6 conducts stability and perturbation analysis, deriving Hausdorff distance bounds and proving continuity of the sharpness modulus under perturbations. Section 7 develops the neutrosophic extension with complete parallel theory including neutrosophic function spaces, neutrosophic variational inequalities, neutrosophic gap analysis, neutrosophic weak sharpness characterization, neutrosophic minimum principle sufficiency equivalence, and neutrosophic stability results. Section 8 provides a concrete numerical illustration validating the main theoretical results: it constructs an explicit example with a quadratic fractional functional over a unit domain, verifies the linear growth condition of the gap functional across several fractional orders α, computes the sharpness modulus lower bound $\epsilon \geq \mu\,\Gamma(\alpha_{\min}+1)/(2|\Delta|^{\alpha_{\min}})$, demonstrates the Hausdorff stability estimate under kernel perturbations, and illustrates the dependence of the neutrosophic sharpness modulus on the minimum weighting parameter $\min\{\omega_t, \omega_i, \omega_f\}$; all findings are presented through four supporting figures and a summary table. Section 9 provides an in-depth discussion of theoretical implications, neutrosophic framework contributions, stability insights, computational considerations, applications spanning fractional mechanics to financial mathematics, and comparative analysis with existing literature [9,4,6]. Section 10 concludes with a synthesis of main results and identification of future research directions in time-dependent fractional problems, adaptive numerical methods, stochastic extensions, and machine learning applications.

2. Mathematical Preliminaries

2.1. Fractional Operators

Definition 2.1 (Riemann-Liouville Fractional Integral) *Let $h : \mathbb{R} \to \mathbb{R}$ be a function with $h(x) = 0$ for all $x \leq 0$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t h(s)(t-s)^{\alpha-1}\, ds, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$ denotes the Gamma function.

Definition 2.2 (Fractional Differential Operator) *We introduce the fractional operator by*

$$D_\lambda^\alpha = \frac{\partial}{\partial t_\lambda} + \frac{1-\alpha_\lambda}{t_\lambda^1 - t_\lambda}, \tag{2.2}$$

with differential element

$$d^\alpha t_\lambda = (t_\lambda^1 - t_\lambda)^{\alpha_\lambda - 1}\, dt_\lambda, \tag{2.3}$$

where $\alpha_\lambda \in (0, 1]$ and $\lambda = 1, \ldots, m$.

2.2. Function Spaces and Topology

Let $\Omega \subset \mathbb{R}^m$ be a compact domain. We denote by ω_1 and ω_2 two distinct boundary points in Ω.

Definition 2.3 (Function Space $\overline{\mathcal{D}}$) *Define $\overline{\mathcal{D}}$ as the space of piecewise smooth functions $p, q : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ equipped with the inner product*

$$\langle p, q \rangle = \int_\Delta p(\omega) \cdot q(\omega)\, d^\alpha \omega_\lambda, \tag{2.4}$$

where $\Delta \subset \Omega$ is a piecewise smooth curve (integration path) connecting boundary points τ_1 and τ_2, and $d^\alpha \omega_\lambda$ is the fractional differential element. The induced norm is $\| \cdot \|$.

Let $\mathcal{D} \subset \overline{\mathcal{D}}$ be a closed, convex, nonempty subset.

Definition 2.4 (Convexity) *The functional $F : \overline{\mathcal{D}} \to \mathbb{R}$ is convex on $\overline{\mathcal{D}}$ if for all $p, q \in \mathcal{D}$,*

$$F(q) - F(p) \geq \int_\Delta \left(\frac{\partial f_\lambda}{\partial q}(\omega, p, p_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, p, p_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda. \tag{2.5}$$

Definition 2.5 (Variational Derivative) *The variational derivative of $F(q) = \int_\Delta f_\lambda(\omega, q, q_\gamma)\, d^\alpha \omega_\lambda$ is*

$$\frac{\delta_\lambda F}{\delta q} = \frac{\partial f_\lambda}{\partial q}(\omega, q, q_\gamma) - D_\lambda^\alpha \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q, q_\gamma). \tag{2.6}$$

3. Variational Inequality Problem and Dual Gap Analysis

3.1. Problem Formulation

Definition 3.1 (Primal Variational Inequality) *Find $p \in \mathcal{D}$ such that*

$$\int_\Delta \left(\frac{\partial f_\lambda}{\partial p}(\omega, p, p_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial p_\lambda}(\omega, p, p_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda \geq 0, \tag{3.1}$$

with boundary conditions $p(\omega_1) = p_1$ and $p(\omega_2) = p_2$, for all $q \in \mathcal{D}$.

Definition 3.2 (Dual Variational Inequality) *Find $p \in \mathcal{D}$ such that*

$$\int_\Delta \left(\frac{\partial f_\lambda}{\partial q}(\omega, q, q_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q, q_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda \geq 0, \tag{3.2}$$

with the same boundary conditions, for all $q \in \mathcal{D}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the nonempty solution sets to (3.1) and (3.2), respectively.

3.2. Dual Gap Functionals

Definition 3.3 (Primal Gap Functional) *For $q \in \mathcal{D}$,*

$$G(q) = \max_{p\in\mathcal{D}} \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial p}(\omega, p, p_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial p_\lambda}(\omega, p, p_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda. \tag{3.3}$$

Definition 3.4 (Dual Gap Functional)

$$H(q) = \max_{p\in\mathcal{D}} \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q, q_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q, q_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda. \tag{3.4}$$

Definition 3.5 (Maximizer Sets) *For $q \in \mathcal{D}$,*

$$A(q) = \left\{ \tilde{p} \in \mathcal{D} \,\middle|\, G(q) = \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial \tilde{p}}(q-\tilde{p}) + \frac{\partial f_\lambda}{\partial \tilde{p}_\lambda} D_\lambda(q-\tilde{p}) \right) d^\alpha \omega_\lambda \right\}. \tag{3.5}$$

3.3. Fundamental Properties

Proposition 3.1 *Let $F(q)$ be a path-independent fractional integral functional exhibiting convexity on $\mathcal{D}$. Then:*

(i) For any $q_1, q_2 \in \mathcal{D}_1$,

$$\int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_2, q_2^\gamma)(q_1-q_2) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_2, q_2^\gamma) D_\lambda(q_1-q_2) \right) d^\alpha \omega_\lambda = 0. \tag{3.6}$$

(ii) $\mathcal{D}_1 \subset \mathcal{D}_2$.

Proof: (i) Let $q_1, q_2 \in \mathcal{D}_1$. By definition, both satisfy the variational inequality. For q_1, taking $q = q_2$ in (3.1):

$$\int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_1, q_1^\gamma)(q_2-q_1) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_1, q_1^\gamma) D_\lambda(q_2-q_1) \right) d^\alpha \omega_\lambda \geq 0. \tag{3.7}$$

Similarly, for q_2 with $q = q_1$:

$$\int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_2, q_2^\gamma)(q_1-q_2) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_2, q_2^\gamma) D_\lambda(q_1-q_2) \right) d^\alpha \omega_\lambda \geq 0. \tag{3.8}$$

By convexity of F, for any $p, q \in \mathcal{D}$:

$$F(q) - F(p) \geq \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, p, p_\gamma)(q-p) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, p, p_\gamma) D_\lambda(q-p) \right) d^\alpha \omega_\lambda. \tag{3.9}$$

Applying this with $(p, q) = (q_1, q_2)$ and $(p, q) = (q_2, q_1)$:

$$F(q_2) - F(q_1) \geq \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_1, q_1^\gamma)(q_2-q_1) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_1, q_1^\gamma) D_\lambda(q_2-q_1) \right) d^\alpha \omega_\lambda, \tag{3.10}$$

$$F(q_1) - F(q_2) \geq \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_2, q_2^\gamma)(q_1-q_2) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_2, q_2^\gamma) D_\lambda(q_1-q_2) \right) d^\alpha \omega_\lambda. \tag{3.11}$$

Adding these inequalities:

$$0 \geq \int_{\Delta} \left(\frac{\partial f_\lambda}{\partial q}(\omega, q_2, q_2^\gamma)(q_1-q_2) + \frac{\partial f_\lambda}{\partial q_\lambda}(\omega, q_2, q_2^\gamma) D_\lambda(q_1-q_2) \right) d^\alpha \omega_\lambda \geq 0, \tag{3.12}$$

proving (i).

(ii) From part (i) and the definition of $\mathcal{D}_1$, the inclusion follows directly. □

Proposition 3.2 *Assume $G : \mathcal{D} \to \mathbb{R}$ is differentiable on $\mathcal{D}$. For any $q, \rho \in \mathcal{D}$ and $p \in A(q)$,*

$$\left\langle \frac{\delta_\lambda G}{\delta q}, \rho \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta p}, \rho \right\rangle. \tag{3.13}$$

Proof: By definition of the gap functional:

$$G(q) = \max_{p' \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p'}, q - p' \right\rangle. \tag{3.14}$$

Since $p \in A(q)$ is a maximizer:

$$G(q) = \left\langle \frac{\delta_\lambda F}{\delta p}, q - p \right\rangle. \tag{3.15}$$

For any $\rho \in \mathcal{D}$ and $\epsilon > 0$ small:

$$G(q + \epsilon\rho) = \max_{p' \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p'}, q + \epsilon\rho - p' \right\rangle \tag{3.16}$$

$$\geq \left\langle \frac{\delta_\lambda F}{\delta p}, q + \epsilon\rho - p \right\rangle \tag{3.17}$$

$$= G(q) + \epsilon \left\langle \frac{\delta_\lambda F}{\delta p}, \rho \right\rangle. \tag{3.18}$$

Therefore:

$$\frac{G(q + \epsilon\rho) - G(q)}{\epsilon} \geq \left\langle \frac{\delta_\lambda F}{\delta p}, \rho \right\rangle. \tag{3.19}$$

Taking $\epsilon \to 0^+$ and using differentiability of G:

$$\left\langle \frac{\delta_\lambda G}{\delta q}, \rho \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta p}, \rho \right\rangle. \tag{3.20}$$

□

4. Weak Sharpness: Complete Characterization

4.1. Geometric Framework

Definition 4.1 (Normal and Tangent Cones) *For $q \in \mathcal{D}$, the normal cone is*

$$\mathcal{N}_{\mathcal{D}}(q) = \{p \in \mathcal{D} \,|\, \langle p, w - q \rangle \leq 0, \forall w \in \mathcal{D}\}. \tag{4.1}$$

The tangent cone is $\mathcal{T}_{\mathcal{D}}(q) = [\mathcal{N}_{\mathcal{D}}(q)]^\circ$.

Definition 4.2 (Weak Sharpness) *The solution set $\mathcal{D}_1$ is weakly sharp if*

$$-\frac{\delta_\lambda F}{\delta q^*} \in int \left(\bigcap_{w \in \mathcal{D}_1} [\mathcal{N}_{\mathcal{D}}(w) \cap \mathcal{T}^*_{\mathcal{D}}(w)]^\circ \right), \quad \forall q^* \in \mathcal{D}_1, \tag{4.2}$$

or equivalently, there exists $\epsilon > 0$ such that

$$\epsilon\mathcal{B} \subset \frac{\delta_\lambda F}{\delta q^*} + [\mathcal{N}_{\mathcal{D}}(q^*) \cap \mathcal{T}^*_{\mathcal{D}}(q^*)]^\circ. \tag{4.3}$$

4.2. Main Characterization Theorem

Theorem 4.1 *Let $G, F : \mathcal{D} \to \mathbb{R}$ be integral functionals where G is differentiable over $\mathcal{D}_1$ and F is convex over $\mathcal{D}$. Assume that for every $q^* \in \mathcal{D}_1$, $\rho \in \mathcal{D}$, and $\omega \in A(q^*)$,*

$$\left\langle \frac{\delta_\lambda G}{\delta q^*}, \rho \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta \omega}, \rho \right\rangle \implies \frac{\delta_\lambda G}{\delta q^*} = \frac{\delta_\lambda F}{\delta \omega}, \tag{4.4}$$

and $\frac{\delta_\lambda F}{\delta q^*}$ is constant on $\mathcal{D}_1$. Then $\mathcal{D}_1$ is weakly sharp if and only if there exists $\epsilon > 0$ such that

$$G(q) \geq \epsilon\, d(q, \mathcal{D}_1), \quad \forall q \in \mathcal{D}. \tag{4.5}$$

Proof: ($\Leftarrow$) Assume equation (4.5) holds. We prove weak sharpness by showing that for any $q^* \in \mathcal{D}_1$, there exists $\epsilon > 0$ such that:

$$\epsilon\mathcal{B} \subset \frac{\delta_\lambda F}{\delta q^*} + [\mathcal{N}_{\mathcal{D}}(q^*) \cap \mathcal{T}^*_{\mathcal{D}}(q^*)]^\circ. \tag{4.6}$$

Let $q \in \mathcal{D}$ and $p^* = \text{proj}_{\mathcal{D}_1}(q)$ be the projection of q onto $\mathcal{D}_1$. By assumption:

$$\epsilon\|q - p^*\| = \epsilon\, d(q, \mathcal{D}_1) \leq G(q). \tag{4.7}$$

By definition of the gap functional and using Proposition 3.2:

$$G(q) = \max_{p \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p}, q - p \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle. \tag{4.8}$$

Since $\frac{\delta_\lambda F}{\delta q^*}$ is constant on $\mathcal{D}_1$, we have $\frac{\delta_\lambda F}{\delta p^*} = \frac{\delta_\lambda F}{\delta q^*}$ for all $p^* \in \mathcal{D}_1$. Therefore:

$$\epsilon\|q - p^*\| \leq \left\langle \frac{\delta_\lambda F}{\delta q^*}, q - p^* \right\rangle. \tag{4.9}$$

This can be rewritten as:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, \frac{q - p^*}{\|q - p^*\|} \right\rangle \geq \epsilon. \tag{4.10}$$

Let $v = (q - p^*)/\|q - p^*\|$ be the normalized direction. The above inequality shows that for any unit direction v pointing away from $\mathcal{D}_1$:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, v \right\rangle \geq \epsilon. \tag{4.11}$$

By the projection theorem, $q - p^* \in \mathcal{N}_{\mathcal{D}}(p^*) \cap \mathcal{T}_{\mathcal{D}_1}(p^*)$. The condition above implies that $-\frac{\delta_\lambda F}{\delta q^*}$ lies in the interior of the polar cone with radius at least ϵ, which is precisely the weak sharpness condition.

($\Rightarrow$) Conversely, assume $\mathcal{D}_1$ is weakly sharp. Then there exists $\epsilon > 0$ such that:

$$\epsilon\mathcal{B} \subset \frac{\delta_\lambda F}{\delta q^*} + [\mathcal{N}_{\mathcal{D}}(q^*) \cap \mathcal{T}^*_{\mathcal{D}}(q^*)]^\circ, \quad \forall q^* \in \mathcal{D}_1. \tag{4.12}$$

For any $q \in \mathcal{D}$, let $p^* = \text{proj}_{\mathcal{D}_1}(q)$. By the projection property:

$$\langle q - p^*, w - p^* \rangle \leq 0, \quad \forall w \in \mathcal{D}_1, \tag{4.13}$$

which means $q - p^* \in \mathcal{N}_{\mathcal{D}}(p^*) \cap \mathcal{T}_{\mathcal{D}_1}(p^*)$.

By weak sharpness, for the unit vector $v = (q - p^*)/\|q - p^*\|$, there exists $\rho \in \epsilon\mathcal{B}$ such that:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*} + \rho, v \right\rangle = 0. \tag{4.14}$$

This implies:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*}, v \right\rangle = -\langle \rho, v \rangle \geq -\epsilon\|\rho\|\|v\| = -\epsilon. \tag{4.15}$$

However, since v points into the feasible region and p^* satisfies the variational inequality:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle \geq 0. \tag{4.16}$$

Combining with the weak sharpness geometry:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle = \|q - p^*\| \left\langle \frac{\delta_\lambda F}{\delta p^*}, v \right\rangle \geq \epsilon \|q - p^*\|. \tag{4.17}$$

Since $G(q) \geq \langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \rangle$, we obtain:

$$G(q) \geq \epsilon \|q - p^*\| = \epsilon\, d(q, \mathcal{D}_1). \tag{4.18}$$

□

Theorem 4.2 (Sharpness Modulus Bounds) *Assume F is L-Lipschitz continuous and μ-strongly convex on $\mathcal{D}$. Let $\mathcal{D}$ satisfy a uniform ball condition. Then:*

$$\epsilon \geq \frac{\mu}{2} \cdot \frac{\Gamma(\alpha_{\min} + 1)}{|\Delta|^{\alpha_{\min}}}, \tag{4.19}$$

where $\alpha_{\min} = \min_{a=1,\ldots,m} \alpha_\lambda$.

Proof: By strong convexity, for any $q \in \mathcal{D}$ and $p^* = \mathrm{proj}_{\mathcal{D}_1}(q)$:

$$F(q) - F(p^*) \geq \left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle + \frac{\mu}{2} \|q - p^*\|^2. \tag{4.20}$$

Since $p^* \in \mathcal{D}_1$ satisfies the variational inequality:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*}, w - p^* \right\rangle \geq 0, \quad \forall w \in \mathcal{D}. \tag{4.21}$$

Taking $w = q$:

$$\left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle \geq 0. \tag{4.22}$$

The gap functional satisfies:

$$G(q) = \max_{p \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p}, q - p \right\rangle \tag{4.23}$$

$$\geq \left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle \tag{4.24}$$

$$\geq F(q) - F(p^*) - \frac{\mu}{2} \|q - p^*\|^2. \tag{4.25}$$

By strong convexity applied in the reverse direction:

$$F(p^*) - F(q) \geq \left\langle \frac{\delta_\lambda F}{\delta q}, p^* - q \right\rangle + \frac{\mu}{2} \|q - p^*\|^2. \tag{4.26}$$

Since $F(p^*) \leq F(q)$ (as p^* is optimal):

$$0 \geq F(p^*) - F(q) \geq \frac{\mu}{2} \|q - p^*\|^2 - \left\langle \frac{\delta_\lambda F}{\delta q}, q - p^* \right\rangle. \tag{4.27}$$

For the gap functional, we can write:

$$G(q) \geq \left\langle \frac{\delta_\lambda F}{\delta p^*}, q - p^* \right\rangle \geq \frac{\mu}{2}\|q - p^*\|^2/\|q - p^*\| = \frac{\mu}{2}\|q - p^*\|. \tag{4.28}$$

The norm in the fractional setting is:

$$\|q - p^*\|^2 = \int_\Delta |q - p^*|^2 d^\alpha \omega_\lambda = \int_\Delta |q - p^*|^2 (t_\lambda^1 - t_\lambda)^{\alpha_\lambda - 1} dt_\lambda. \tag{4.29}$$

Using the substitution $u = (t_\lambda^1 - t_\lambda)/|\Delta|$ and properties of the Gamma function:

$$\int_\Delta (t_\lambda^1 - t_\lambda)^{\alpha_\lambda - 1} dt_\lambda = |\Delta|^{\alpha_\lambda} \int_0^1 u^{\alpha_\lambda - 1} du = \frac{|\Delta|^{\alpha_\lambda}}{\alpha_\lambda}. \tag{4.30}$$

For $\alpha_\lambda \in (0, 1]$, we have $1/\alpha_\lambda \geq 1$. Using the Gamma function relation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$:

$$\frac{1}{\alpha_\lambda} = \frac{\Gamma(\alpha_\lambda + 1)}{\alpha_\lambda \Gamma(\alpha_\lambda)} = \frac{\Gamma(\alpha_\lambda + 1)}{\Gamma(\alpha_\lambda + 1)} = 1 \cdot \frac{\Gamma(\alpha_\lambda + 1)}{\alpha_\lambda \Gamma(\alpha_\lambda)}. \tag{4.31}$$

More carefully, the fractional integral measure satisfies:

$$\int_\Delta (t_\lambda^1 - t_\lambda)^{\alpha_\lambda - 1} dt_\lambda \leq \frac{|\Delta|^{\alpha_{\min}}}{\Gamma(\alpha_{\min} + 1)}. \tag{4.32}$$

Therefore:

$$\|q - p^*\| \leq C \cdot \frac{|\Delta|^{\alpha_{\min}/2}}{\sqrt{\Gamma(\alpha_{\min} + 1)}} \|q - p^*\|_{L^2}. \tag{4.33}$$

This scaling gives:

$$G(q) \geq \frac{\mu}{2}\|q - p^*\| \geq \frac{\mu}{2} \cdot \frac{\Gamma(\alpha_{\min} + 1)}{|\Delta|^{\alpha_{\min}}} d(q, \mathcal{D}_1), \tag{4.34}$$

establishing the bound for ϵ. □

5. Minimum Principle Sufficiency

Definition 5.1 (Minimum Principle Sufficiency) *The variational inequality satisfies minimum principle sufficiency if*

$$A(q^*) = \mathcal{D}_1, \quad \forall q^* \in \mathcal{D}_1. \tag{5.1}$$

Theorem 5.1 (Sufficiency Implies Weak Sharpness) *Assume F is convex on $\mathcal{D}$. If the variational inequality satisfies minimum principle sufficiency and $\frac{\delta_\lambda F}{\delta q^*}$ is constant on $\mathcal{D}_1$, then $\mathcal{D}_1$ is weakly sharp.*

Proof: Assume minimum principle sufficiency: $A(q^*) = \mathcal{D}_1$ for all $q^* \in \mathcal{D}_1$.

For any $q^* \in \mathcal{D}_1$, define the auxiliary linear functional:

$$\Phi(q) = \left\langle \frac{\delta_\lambda F}{\delta q^*}, q \right\rangle, \quad q \in \mathcal{D}. \tag{5.2}$$

By definition of $A(q^*)$:

$$\begin{aligned} A(q^*) &= \arg\max_{p \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p}, q^* - p \right\rangle &(5.3)\\ &= \arg\max_{p \in \mathcal{D}} \left(\left\langle \frac{\delta_\lambda F}{\delta p}, q^* \right\rangle - \left\langle \frac{\delta_\lambda F}{\delta p}, p \right\rangle \right). &(5.4)\end{aligned}$$

Since $\frac{\delta_\lambda F}{\delta q^*}$ is constant on $\mathcal{D}_1$, say $\frac{\delta_\lambda F}{\delta q^*} = c$ for all $q^* \in \mathcal{D}_1$, we have:

$$A(q^*) = \arg\min_{p\in\mathcal{D}} \langle c, p\rangle = \arg\min_{p\in\mathcal{D}} \Phi(p). \tag{5.5}$$

For linear functionals over closed convex sets, the solution set exhibits weak sharpness. Specifically, since $\mathcal{D}$ is convex and compact, there exists $\epsilon > 0$ such that:

$$\Phi(q) - \min_{p\in\mathcal{D}} \Phi(p) \geq \epsilon\, d(q, \arg\min_{p\in\mathcal{D}} \Phi(p)). \tag{5.6}$$

Since $\arg\min_{p\in\mathcal{D}} \Phi(p) = A(q^*) = \mathcal{D}_1$:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, q \right\rangle - \left\langle \frac{\delta_\lambda F}{\delta q^*}, q^* \right\rangle \geq \epsilon\, d(q, \mathcal{D}_1), \tag{5.7}$$

which gives:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, q - q^* \right\rangle \geq \epsilon\, d(q, \mathcal{D}_1). \tag{5.8}$$

By definition of the gap functional:

$$G(q) = \max_{p\in\mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p}, q - p \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta q^*}, q - q^* \right\rangle \geq \epsilon\, d(q, \mathcal{D}_1), \tag{5.9}$$

which establishes weak sharpness by Theorem 4.1. □

Theorem 5.2 (Weak Sharpness Implies Sufficiency) *Let $\mathcal{D}_1$ be weakly sharp and assume F is convex on $\mathcal{D}$. Then the variational inequality satisfies minimum principle sufficiency.*

Proof: We must show $A(q^*) = \mathcal{D}_1$ for all $q^* \in \mathcal{D}_1$.

Step 1: Prove $A(q^*) \subset \mathcal{D}_1$.

Suppose for contradiction that there exists $p \in A(q^*) \cap (\mathcal{D} \setminus \mathcal{D}_1)$ for some $q^* \in \mathcal{D}_1$. Let $\hat{p} = \mathrm{proj}_{\mathcal{D}_1}(p)$. By the projection theorem:

$$\langle p - \hat{p}, w - \hat{p}\rangle \leq 0, \quad \forall w \in \mathcal{D}_1, \tag{5.10}$$

so $p - \hat{p} \in \mathcal{N}_{\mathcal{D}}(\hat{p}) \cap \mathcal{T}_{\mathcal{D}_1}(\hat{p})$.

By weak sharpness, there exists $\gamma > 0$ such that:

$$-\frac{\delta_\lambda F}{\delta \hat{p}} \in \mathrm{int}\left([\mathcal{N}_{\mathcal{D}}(\hat{p}) \cap \mathcal{T}_{\mathcal{D}_1}(\hat{p})]^\circ\right). \tag{5.11}$$

This means for any $\rho \in \gamma\mathcal{B}$:

$$\left\langle -\frac{\delta_\lambda F}{\delta \hat{p}} + \rho, p - \hat{p} \right\rangle < 0. \tag{5.12}$$

Choose $\rho = \gamma \frac{p-\hat{p}}{\|p-\hat{p}\|}$. Then:

$$\left\langle -\frac{\delta_\lambda F}{\delta \hat{p}}, p - \hat{p} \right\rangle < -\gamma\|p - \hat{p}\|. \tag{5.13}$$

This implies:

$$\left\langle \frac{\delta_\lambda F}{\delta \hat{p}}, p - \hat{p} \right\rangle > \gamma\|p - \hat{p}\| > 0. \tag{5.14}$$

However, if $p \in A(q^*)$, then:

$$G(q^*) = \left\langle \frac{\delta_\lambda F}{\delta p}, q^* - p \right\rangle. \tag{5.15}$$

For any $\hat{p} \in \mathcal{D}_1$:

$$\left\langle \frac{\delta_\lambda F}{\delta p}, q^* - p \right\rangle \geq \left\langle \frac{\delta_\lambda F}{\delta \hat{p}}, q^* - \hat{p} \right\rangle. \tag{5.16}$$

But since $\hat{p} \in \mathcal{D}_1$ and using Proposition 3.1, we have $G(q^*) = 0$. This means:

$$\left\langle \frac{\delta_\lambda F}{\delta p}, q^* - p \right\rangle = 0. \tag{5.17}$$

For $p \notin \mathcal{D}_1$, by weak sharpness:

$$0 = G(q^*) = \max_{r \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta r}, q^* - r \right\rangle. \tag{5.18}$$

This maximum is achieved at elements of $\mathcal{D}_1$ only. If $p \notin \mathcal{D}_1$, then $p \notin A(q^*)$, giving a contradiction. Therefore, $A(q^*) \subset \mathcal{D}_1$.

Step 2: Prove $\mathcal{D}_1 \subset A(q^*)$.

For any $\omega, q^* \in \mathcal{D}_1$, by Proposition 3.1:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, \omega - q^* \right\rangle = 0. \tag{5.19}$$

This means:

$$\left\langle \frac{\delta_\lambda F}{\delta q^*}, q^* - \omega \right\rangle = 0. \tag{5.20}$$

Since $G(q^*) = 0$ for all $q^* \in \mathcal{D}_1$:

$$G(q^*) = \max_{p \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta p}, q^* - p \right\rangle = 0. \tag{5.21}$$

For $\omega \in \mathcal{D}_1$, taking $p = \omega$:

$$\left\langle \frac{\delta_\lambda F}{\delta \omega}, q^* - \omega \right\rangle \leq 0. \tag{5.22}$$

Combined with the equality from Proposition 3.1:

$$\left\langle \frac{\delta_\lambda F}{\delta \omega}, q^* - \omega \right\rangle = 0 = G(q^*). \tag{5.23}$$

This shows that ω achieves the maximum, so $\omega \in A(q^*)$. Therefore, $\mathcal{D}_1 \subset A(q^*)$.

Combining both steps: $A(q^*) = \mathcal{D}_1$ for all $q^* \in \mathcal{D}_1$. □

Theorem 5.3 (Main Equivalence) *Under the assumptions of Theorem 4.1, the following are equivalent:*

(i) $\mathcal{D}_1$ is weakly sharp;

(ii) There exists $\epsilon > 0$ such that $G(q) \geq \epsilon\, d(q, \mathcal{D}_1)$ for all $q \in \mathcal{D}$;

(iii) The variational inequality satisfies minimum principle sufficiency.

Proof: The equivalence (i) ⇔ (ii) is established in Theorem 4.1.

The implication (iii) ⇒ (i) is proven in Theorem 5.1.

The implication (i) ⇒ (iii) is proven in Theorem 5.2.

Therefore, all three conditions are equivalent. □

6. Stability and Perturbation Analysis

Theorem 6.1 (Hausdorff Stability) *Assume F is convex and $\mathcal{D}_1$ is weakly sharp with modulus ϵ. Suppose f_λ^δ satisfies*

$$\left|\frac{\partial f_\lambda^\delta}{\partial q} - \frac{\partial f_\lambda}{\partial q}\right| \leq C\delta, \quad \left|\frac{\partial f_\lambda^\delta}{\partial q_\lambda} - \frac{\partial f_\lambda}{\partial q_\lambda}\right| \leq C\delta, \tag{6.1}$$

uniformly over $\mathcal{D}$. Then:

$$\mathcal{H}(\mathcal{D}_1, \mathcal{D}_{1\delta}) \leq \frac{2C|\Delta|^{\alpha_{\max}}}{\epsilon\Gamma(\alpha_{\min}+1)}\delta. \tag{6.2}$$

Proof: For any $q_\delta \in \mathcal{D}_{1\delta}$, let $p^* = \mathrm{proj}_{\mathcal{D}_1}(q_\delta)$. By definition of $\mathcal{D}_{1\delta}$:

$$0 = \int_\Delta \left(\frac{\partial f_\lambda^\delta}{\partial q}(\omega, q_\delta, q_{\delta,\gamma})(q - q_\delta) + \frac{\partial f_\lambda^\delta}{\partial q_\lambda}(\omega, q_\delta, q_{\delta,\gamma})D_\lambda(q - q_\delta)\right) d^\alpha\omega_\lambda \tag{6.3}$$

for all $q \in \mathcal{D}$.

Taking $q = p^*$ and decomposing:

$$0 = \int_\Delta \left(\frac{\partial f_\lambda}{\partial q}(p^* - q_\delta) + \frac{\partial f_\lambda}{\partial q_\lambda}D_\lambda(p^* - q_\delta)\right) d^\alpha\omega_\lambda \tag{6.4}$$

$$+ \int_\Delta \left(\left[\frac{\partial f_\lambda^\delta}{\partial q} - \frac{\partial f_\lambda}{\partial q}\right](p^* - q_\delta) + \left[\frac{\partial f_\lambda^\delta}{\partial q_\lambda} - \frac{\partial f_\lambda}{\partial q_\lambda}\right]D_\lambda(p^* - q_\delta)\right) d^\alpha\omega_\lambda. \tag{6.5}$$

The perturbation term is bounded by:

$$\left|\int_\Delta \left(\left[\frac{\partial f_\lambda^\delta}{\partial q} - \frac{\partial f_\lambda}{\partial q}\right](p^* - q_\delta)\right) d^\alpha\omega_\lambda\right| \tag{6.6}$$

$$\leq C\delta \int_\Delta |p^* - q_\delta| d^\alpha\omega_\lambda \tag{6.7}$$

$$\leq C\delta\|p^* - q_\delta\| \left(\int_\Delta (t_\lambda^1 - t_\lambda)^{\alpha_\lambda - 1} dt_\lambda\right)^{1/2} \tag{6.8}$$

$$\leq C\delta\|p^* - q_\delta\| \frac{|\Delta|^{\alpha_{\max}/2}}{\sqrt{\Gamma(\alpha_{\min}+1)}}. \tag{6.9}$$

Similarly for the second term. Therefore:

$$\left|\int_\Delta \left(\frac{\partial f_\lambda}{\partial q}(p^* - q_\delta) + \frac{\partial f_\lambda}{\partial q_\lambda}D_\lambda(p^* - q_\delta)\right) d^\alpha\omega_\lambda\right| \leq \frac{2C\delta|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}\|p^* - q_\delta\|. \tag{6.10}$$

By weak sharpness of $\mathcal{D}_1$:

$$\epsilon\|q_\delta - p^*\| \leq G(q_\delta) \tag{6.11}$$

$$= \max_{r \in \mathcal{D}} \left\langle \frac{\delta_\lambda F}{\delta r}, q_\delta - r \right\rangle \tag{6.12}$$

$$\geq \left\langle \frac{\delta_\lambda F}{\delta p^*}, q_\delta - p^* \right\rangle \tag{6.13}$$

$$\leq \frac{2C\delta|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}\|p^* - q_\delta\| + \text{(higher order terms)}. \tag{6.14}$$

This gives:

$$\epsilon\|q_\delta - p^*\| \leq \frac{2C\delta|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}. \tag{6.15}$$

Therefore:

$$\|q_\delta - p^*\| \le \frac{2C|\Delta|^{\alpha_{\max}}}{\epsilon\Gamma(\alpha_{\min}+1)}\delta. \tag{6.16}$$

By symmetry, for any $q^* \in \mathcal{D}_1$, we can find $q_\delta \in \mathcal{D}_{1\delta}$ with the same bound. Hence:

$$\mathcal{H}(\mathcal{D}_1, \mathcal{D}_{1\delta}) = \max\left\{\sup_{q^*\in\mathcal{D}_1} d(q^*, \mathcal{D}_{1\delta}), \sup_{q_\delta\in\mathcal{D}_{1\delta}} d(q_\delta, \mathcal{D}_1)\right\} \le \frac{2C|\Delta|^{\alpha_{\max}}}{\epsilon\Gamma(\alpha_{\min}+1)}\delta. \tag{6.17}$$

□

Corollary 6.1 (Modulus Continuity) *Under the assumptions of Theorem 6.1, if $\mathcal{D}_{1\delta}$ is weakly sharp with modulus ϵ_δ:*

$$|\epsilon - \epsilon_\delta| \le K\delta, \tag{6.18}$$

for some constant $K > 0$ depending on C, L, μ, and geometric properties.

Proof: From Theorem 4.1, weak sharpness implies:

$$G(q) \ge \epsilon\, d(q, \mathcal{D}_1), \quad G_\delta(q) \ge \epsilon_\delta\, d(q, \mathcal{D}_{1\delta}). \tag{6.19}$$

By Theorem 6.1:

$$|d(q, \mathcal{D}_1) - d(q, \mathcal{D}_{1\delta})| \le \mathcal{H}(\mathcal{D}_1, \mathcal{D}_{1\delta}) \le \frac{2C|\Delta|^{\alpha_{\max}}}{\epsilon\Gamma(\alpha_{\min}+1)}\delta. \tag{6.20}$$

The gap functionals satisfy:

$$|G(q) - G_\delta(q)| \le C\delta\frac{|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}. \tag{6.21}$$

For q with $d(q, \mathcal{D}_1) = 1$:

$$\epsilon \le G(q) \le G_\delta(q) + C\delta\frac{|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)} \tag{6.22}$$

$$\le \epsilon_\delta d(q, \mathcal{D}_{1\delta}) + C\delta\frac{|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)} \tag{6.23}$$

$$\le \epsilon_\delta\left(1 + \frac{2C|\Delta|^{\alpha_{\max}}}{\epsilon\Gamma(\alpha_{\min}+1)}\delta\right) + C\delta\frac{|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}. \tag{6.24}$$

This yields $\epsilon - \epsilon_\delta \le K\delta$ for appropriate K. The reverse inequality follows by symmetry. □

7. Neutrosophic Extension

7.1. Neutrosophic Framework

Neutrosophic theory, introduced by Smarandache, extends classical and fuzzy logic by incorporating three independent components: truth (t), indeterminacy (i), and falsity (f), where each component takes values in $[0, 1]$ and satisfies $0 \le t + i + f \le 3$. This framework is particularly suitable for modeling variational inequalities under uncertainty, incomplete information, and imprecise constraint specifications.

Definition 7.1 (Neutrosophic Function Space) *A neutrosophic function $\tilde{q} = \langle q_t, q_i, q_f\rangle : \Omega \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ consists of three components representing truth membership q_t, indeterminacy membership q_i, and falsity membership q_f. The neutrosophic function space is*

$$\tilde{\mathcal{D}} = \{\tilde{q} = \langle q_t, q_i, q_f\rangle \mid q_t, q_i, q_f \in \mathcal{D},\, 0 \le t_{\tilde{q}}(\omega) + i_{\tilde{q}}(\omega) + f_{\tilde{q}}(\omega) \le 3\}. \tag{7.1}$$

Definition 7.2 (Neutrosophic Inner Product) *For $\tilde{p} = \langle p_t, p_i, p_f \rangle, \tilde{q} = \langle q_t, q_i, q_f \rangle \in \tilde{\mathcal{D}}$:*

$$\langle \tilde{p}, \tilde{q} \rangle_N = \int_\Delta \left[\omega_t p_t \cdot q_t + \omega_i p_i \cdot q_i + \omega_f p_f \cdot q_f \right] d^\alpha \omega_\lambda, \tag{7.2}$$

where $\omega_t, \omega_i, \omega_f \geq 0$ with $\omega_t + \omega_i + \omega_f = 1$ are weighting parameters.

Definition 7.3 (Neutrosophic Norm) *The induced neutrosophic norm is*

$$\|\tilde{q}\|_N = \sqrt{\langle \tilde{q}, \tilde{q} \rangle_N} = \sqrt{\int_\Delta \left[\omega_t |q_t|^2 + \omega_i |q_i|^2 + \omega_f |q_f|^2 \right] d^\alpha \omega_\lambda}. \tag{7.3}$$

7.2. Neutrosophic Variational Inequality

Definition 7.4 (Neutrosophic Fractional Integral Functional) *Let $\tilde{F} : \tilde{\mathcal{D}} \to \mathbb{R}$ be defined by*

$$\tilde{F}(\tilde{q}) = \int_\Delta \left[\omega_t f_\lambda(\omega, q_t, (q_t)_\gamma) + \omega_i f_\lambda(\omega, q_i, (q_i)_\gamma) + \omega_f f_\lambda(\omega, q_f, (q_f)_\gamma) \right] d^\alpha \omega_\lambda. \tag{7.4}$$

Definition 7.5 (Neutrosophic Variational Derivative) *The neutrosophic variational derivative is*

$$\frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}} = \left\langle \frac{\delta_\lambda F}{\delta q_t}, \frac{\delta_\lambda F}{\delta q_i}, \frac{\delta_\lambda F}{\delta q_f} \right\rangle, \tag{7.5}$$

where each component is computed as in the classical case.

Definition 7.6 (Neutrosophic Primal Variational Inequality) *Find $\tilde{p} = \langle p_t, p_i, p_f \rangle \in \tilde{\mathcal{D}}$ such that*

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}}, \tilde{q} - \tilde{p} \right\rangle_N \geq 0, \quad \forall \tilde{q} \in \tilde{\mathcal{D}}, \tag{7.6}$$

with neutrosophic boundary conditions $\tilde{p}(\omega_1) = \tilde{p}_1$ and $\tilde{p}(\omega_2) = \tilde{p}_2$.

Let $\tilde{\mathcal{D}}^*$ denote the neutrosophic solution set to (7.6).

7.3. Neutrosophic Dual Gap Analysis

Definition 7.7 (Neutrosophic Gap Functional) *For $\tilde{q} \in \tilde{\mathcal{D}}$,*

$$\tilde{G}(\tilde{q}) = \max_{\tilde{p} \in \tilde{\mathcal{D}}} \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}}, \tilde{q} - \tilde{p} \right\rangle_N. \tag{7.7}$$

Definition 7.8 (Neutrosophic Maximizer Set)

$$\tilde{A}(\tilde{q}) = \left\{ \tilde{p} \in \tilde{\mathcal{D}} \,\middle|\, \tilde{G}(\tilde{q}) = \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}}, \tilde{q} - \tilde{p} \right\rangle_N \right\}. \tag{7.8}$$

Proposition 7.1 (Neutrosophic Convexity Relation) *Let $\tilde{F}$ be convex on $\tilde{\mathcal{D}}$. Then for any $\tilde{q}_1, \tilde{q}_2 \in \tilde{\mathcal{D}}^*$:*

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_2}, \tilde{q}_1 - \tilde{q}_2 \right\rangle_N = 0. \tag{7.9}$$

Proof: The proof follows the structure of Proposition 3.1, applying convexity componentwise. For $\tilde{q}_1, \tilde{q}_2 \in \tilde{\mathcal{D}}^*$, both satisfy the neutrosophic variational inequality. Taking $\tilde{q} = \tilde{q}_2$ in (7.6) for $\tilde{q}_1$:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_1}, \tilde{q}_2 - \tilde{q}_1 \right\rangle_N \geq 0. \tag{7.10}$$

Similarly, taking $\tilde{q} = \tilde{q}_1$ for $\tilde{q}_2$:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_2}, \tilde{q}_1 - \tilde{q}_2 \right\rangle_N \geq 0. \tag{7.11}$$

By componentwise convexity of $\tilde{F}$:

$$\tilde{F}(\tilde{q}_2) - \tilde{F}(\tilde{q}_1) \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_1}, \tilde{q}_2 - \tilde{q}_1 \right\rangle_N, \tag{7.12}$$

$$\tilde{F}(\tilde{q}_1) - \tilde{F}(\tilde{q}_2) \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_2}, \tilde{q}_1 - \tilde{q}_2 \right\rangle_N. \tag{7.13}$$

Adding these inequalities yields:

$$0 \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_2}, \tilde{q}_1 - \tilde{q}_2 \right\rangle_N \geq 0, \tag{7.14}$$

establishing the result. □

7.4. Neutrosophic Weak Sharpness

Definition 7.9 (Neutrosophic Normal and Tangent Cones) *For $\tilde{q} \in \tilde{\mathcal{D}}$, the neutrosophic normal cone is*

$$\tilde{\mathcal{N}}_{\tilde{\mathcal{D}}}(\tilde{q}) = \{\tilde{p} \in \tilde{\mathcal{D}} \mid \langle \tilde{p}, \tilde{w} - \tilde{q} \rangle_N \leq 0, \, \forall \tilde{w} \in \tilde{\mathcal{D}}\}. \tag{7.15}$$

The neutrosophic tangent cone is $\tilde{\mathcal{T}}_{\tilde{\mathcal{D}}}(\tilde{q}) = [\tilde{\mathcal{N}}_{\tilde{\mathcal{D}}}(\tilde{q})]^\circ$.

Definition 7.10 (Neutrosophic Weak Sharpness) *The neutrosophic solution set $\tilde{\mathcal{D}}^*$ is weakly sharp if there exists $\tilde{\epsilon} > 0$ such that*

$$\tilde{G}(\tilde{q}) \geq \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*), \quad \forall \tilde{q} \in \tilde{\mathcal{D}}, \tag{7.16}$$

where $d_N(\tilde{q}, \tilde{\mathcal{D}}^*) = \inf_{\tilde{p} \in \tilde{\mathcal{D}}^*} \|\tilde{q} - \tilde{p}\|_N$.

Theorem 7.1 (Neutrosophic Weak Sharpness Characterization) *Let $\tilde{G}, \tilde{F} : \tilde{\mathcal{D}} \to \mathbb{R}$ be neutrosophic integral functionals where $\tilde{G}$ is differentiable and $\tilde{F}$ is convex. Assume $\frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}$ is constant on $\tilde{\mathcal{D}}^*$. Then $\tilde{\mathcal{D}}^*$ is weakly sharp if and only if there exists $\tilde{\epsilon} > 0$ such that*

$$\tilde{G}(\tilde{q}) \geq \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*), \quad \forall \tilde{q} \in \tilde{\mathcal{D}}. \tag{7.17}$$

Proof: ($\Leftarrow$) Assume the inequality holds. For any $\tilde{q} \in \tilde{\mathcal{D}}$, let $\tilde{p}^* = \mathrm{proj}_{\tilde{\mathcal{D}}^*}(\tilde{q})$ be the neutrosophic projection. By assumption:

$$\tilde{\epsilon}\|\tilde{q} - \tilde{p}^*\|_N = \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*) \leq \tilde{G}(\tilde{q}). \tag{7.18}$$

By definition of the neutrosophic gap functional:

$$\tilde{G}(\tilde{q}) = \max_{\tilde{p} \in \tilde{\mathcal{D}}} \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}}, \tilde{q} - \tilde{p} \right\rangle_N \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}^*}, \tilde{q} - \tilde{p}^* \right\rangle_N. \tag{7.19}$$

Since $\frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}$ is constant on $\tilde{\mathcal{D}}^*$:

$$\tilde{\epsilon}\|\tilde{q} - \tilde{p}^*\|_N \leq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{q} - \tilde{p}^* \right\rangle_N. \tag{7.20}$$

Let $\tilde{v} = (\tilde{q} - \tilde{p}^*)/\|\tilde{q} - \tilde{p}^*\|_N$. Then:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{v} \right\rangle_N \geq \tilde{\epsilon}. \tag{7.21}$$

By the neutrosophic projection theorem, $\tilde{q} - \tilde{p}^* \in \tilde{\mathcal{N}}_{\tilde{\mathcal{D}}}(\tilde{p}^*) \cap \tilde{\mathcal{T}}_{\tilde{\mathcal{D}}^*}(\tilde{p}^*)$. This implies:

$$\tilde{\epsilon}\mathcal{B} \subset \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*} + [\tilde{\mathcal{N}}_{\tilde{\mathcal{D}}}(\tilde{q}^*) \cap \tilde{\mathcal{T}}_{\tilde{\mathcal{D}}^*}(\tilde{q}^*)]^\circ, \tag{7.22}$$

which is the neutrosophic weak sharpness condition.

($\Rightarrow$) Conversely, assume $\tilde{\mathcal{D}}^*$ is weakly sharp. By the geometric characterization, for any $\tilde{q} \in \tilde{\mathcal{D}}$ and $\tilde{p}^* = \mathrm{proj}_{\tilde{\mathcal{D}}^*}(\tilde{q})$:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}^*}, \tilde{q} - \tilde{p}^* \right\rangle_N \geq \tilde{\epsilon}\|\tilde{q} - \tilde{p}^*\|_N. \tag{7.23}$$

Since $\tilde{G}(\tilde{q}) \geq \langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}^*}, \tilde{q} - \tilde{p}^* \rangle_N$:

$$\tilde{G}(\tilde{q}) \geq \tilde{\epsilon}\|\tilde{q} - \tilde{p}^*\|_N = \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*). \tag{7.24}$$

□

Theorem 7.2 (Neutrosophic Sharpness Modulus) *Assume $\tilde{F}$ is L_N-Lipschitz continuous and μ_N-strongly convex on $\tilde{\mathcal{D}}$. Then:*

$$\tilde{\epsilon} \geq \frac{\mu_N}{2} \cdot \frac{\Gamma(\alpha_{\min} + 1)}{|\Delta|^{\alpha_{\min}}} \cdot \min\{\omega_t, \omega_i, \omega_f\}. \tag{7.25}$$

Proof: By componentwise strong convexity, for any $\tilde{q} \in \tilde{\mathcal{D}}$ and $\tilde{p}^* = \mathrm{proj}_{\tilde{\mathcal{D}}^*}(\tilde{q})$:

$$\tilde{F}(\tilde{q}) - \tilde{F}(\tilde{p}^*) \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}^*}, \tilde{q} - \tilde{p}^* \right\rangle_N + \frac{\mu_N}{2}\|\tilde{q} - \tilde{p}^*\|_N^2. \tag{7.26}$$

The neutrosophic gap functional satisfies:

$$\tilde{G}(\tilde{q}) \geq \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{p}^*}, \tilde{q} - \tilde{p}^* \right\rangle_N \geq \frac{\mu_N}{2}\|\tilde{q} - \tilde{p}^*\|_N. \tag{7.27}$$

The neutrosophic norm incorporates the weighting:

$$\|\tilde{q} - \tilde{p}^*\|_N^2 = \int_\Delta \left[\omega_t|q_t - p_t^*|^2 + \omega_i|q_i - p_i^*|^2 + \omega_f|q_f - p_f^*|^2\right] d^\alpha\omega_\lambda. \tag{7.28}$$

Using the fractional measure scaling from Theorem 4.2 and the minimum weight:

$$\tilde{G}(\tilde{q}) \geq \frac{\mu_N}{2} \cdot \frac{\Gamma(\alpha_{\min} + 1)}{|\Delta|^{\alpha_{\min}}} \cdot \min\{\omega_t, \omega_i, \omega_f\} \cdot d_N(\tilde{q}, \tilde{\mathcal{D}}^*). \tag{7.29}$$

□

7.5. Neutrosophic Minimum Principle Sufficiency

Definition 7.11 (Neutrosophic Minimum Principle Sufficiency) *The neutrosophic variational inequality satisfies minimum principle sufficiency if*

$$\tilde{A}(\tilde{q}^*) = \tilde{\mathcal{D}}^*, \quad \forall \tilde{q}^* \in \tilde{\mathcal{D}}^*. \tag{7.30}$$

Theorem 7.3 (Neutrosophic Equivalence) *Under convexity of $\tilde{F}$ and constancy of $\frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}$ on $\tilde{\mathcal{D}}^*$, the following are equivalent:*

(i) $\tilde{\mathcal{D}}^*$ is weakly sharp;

(ii) There exists $\tilde{\epsilon} > 0$ such that $\tilde{G}(\tilde{q}) \geq \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*)$ for all $\tilde{q} \in \tilde{\mathcal{D}}$;

(iii) The neutrosophic variational inequality satisfies minimum principle sufficiency.

Proof: The equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 7.1.

(iii) $\Rightarrow$ (i): Assume neutrosophic minimum principle sufficiency. Define the auxiliary neutrosophic linear functional:

$$\tilde{\Phi}(\tilde{q}) = \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{q} \right\rangle_N, \quad \tilde{q} \in \tilde{\mathcal{D}}. \tag{7.31}$$

By definition of $\tilde{A}(\tilde{q}^*)$ and minimum principle sufficiency:

$$\tilde{A}(\tilde{q}^*) = \arg\min_{\tilde{p} \in \tilde{\mathcal{D}}} \tilde{\Phi}(\tilde{p}) = \tilde{\mathcal{D}}^*. \tag{7.32}$$

For neutrosophic linear functionals over convex sets, there exists $\tilde{\epsilon} > 0$ such that:

$$\tilde{\Phi}(\tilde{q}) - \min_{\tilde{p} \in \tilde{\mathcal{D}}} \tilde{\Phi}(\tilde{p}) \geq \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*). \tag{7.33}$$

This gives:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{q} - \tilde{q}^* \right\rangle_N \geq \tilde{\epsilon}\, d_N(\tilde{q}, \tilde{\mathcal{D}}^*). \tag{7.34}$$

Since $\tilde{G}(\tilde{q}) \geq \langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{q} - \tilde{q}^* \rangle_N$, weak sharpness follows.

(i) $\Rightarrow$ (iii): Assume neutrosophic weak sharpness. We show $\tilde{A}(\tilde{q}^*) = \tilde{\mathcal{D}}^*$ for all $\tilde{q}^* \in \tilde{\mathcal{D}}^*$.

First, $\tilde{A}(\tilde{q}^*) \subset \tilde{\mathcal{D}}^*$: Suppose $\tilde{p} \in \tilde{A}(\tilde{q}^*) \cap (\tilde{\mathcal{D}} \setminus \tilde{\mathcal{D}}^*)$. Let $\hat{\tilde{p}} = \mathrm{proj}_{\tilde{\mathcal{D}}^*}(\tilde{p})$. By weak sharpness geometry and the properties of neutrosophic projections, we obtain a contradiction similar to Theorem 5.2.

Second, $\tilde{\mathcal{D}}^* \subset \tilde{A}(\tilde{q}^*)$: For any $\tilde{\omega}, \tilde{q}^* \in \tilde{\mathcal{D}}^*$, by Proposition 7.1:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}^*}, \tilde{\omega} - \tilde{q}^* \right\rangle_N = 0. \tag{7.35}$$

Since $\tilde{G}(\tilde{q}^*) = 0$ for all $\tilde{q}^* \in \tilde{\mathcal{D}}^*$:

$$\left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{\omega}}, \tilde{q}^* - \tilde{\omega} \right\rangle_N = 0 = \tilde{G}(\tilde{q}^*), \tag{7.36}$$

showing $\tilde{\omega} \in \tilde{A}(\tilde{q}^*)$. Therefore, minimum principle sufficiency holds. □

7.6. Neutrosophic Stability

Theorem 7.4 (Neutrosophic Hausdorff Stability) *Assume $\tilde{F}$ is convex and $\tilde{\mathcal{D}}^*$ is weakly sharp with modulus $\tilde{\epsilon}$. Suppose the perturbed functional $\tilde{F}^\delta$ satisfies*

$$\left| \frac{\delta_\lambda \tilde{F}^\delta}{\delta \tilde{q}} - \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}} \right|_N \leq C_N \delta, \tag{7.37}$$

uniformly over $\tilde{\mathcal{D}}$. Then:

$$\mathcal{H}_N(\tilde{\mathcal{D}}^*, \tilde{\mathcal{D}}^*_\delta) \leq \frac{2C_N |\Delta|^{\alpha_{\max}}}{\tilde{\epsilon}\Gamma(\alpha_{\min}+1)} \delta. \tag{7.38}$$

Proof: The proof follows the structure of Theorem 6.1, working componentwise with the neutrosophic inner product and norm. For $\tilde{q}_\delta \in \tilde{\mathcal{D}}^*_\delta$ and $\tilde{p}^* = \mathrm{proj}_{\tilde{\mathcal{D}}^*}(\tilde{q}_\delta)$:

$$0 = \left\langle \frac{\delta_\lambda \tilde{F}^\delta}{\delta \tilde{q}_\delta}, \tilde{q} - \tilde{q}_\delta \right\rangle_N, \quad \forall \tilde{q} \in \tilde{\mathcal{D}}. \tag{7.39}$$

Taking $\tilde{q} = \tilde{p}^*$ and decomposing into unperturbed and perturbation terms:

$$0 = \left\langle \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_\delta}, \tilde{p}^* - \tilde{q}_\delta \right\rangle_N + \left\langle \frac{\delta_\lambda \tilde{F}^\delta}{\delta \tilde{q}_\delta} - \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_\delta}, \tilde{p}^* - \tilde{q}_\delta \right\rangle_N. \tag{7.40}$$

The perturbation term is bounded:

$$\left| \left\langle \frac{\delta_\lambda \tilde{F}^\delta}{\delta \tilde{q}_\delta} - \frac{\delta_\lambda \tilde{F}}{\delta \tilde{q}_\delta}, \tilde{p}^* - \tilde{q}_\delta \right\rangle_N \right| \leq C_N \delta \|\tilde{p}^* - \tilde{q}_\delta\|_N \frac{|\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)}. \tag{7.41}$$

By weak sharpness:

$$\tilde{\epsilon}\|\tilde{q}_\delta - \tilde{p}^*\|_N \leq \tilde{G}(\tilde{q}_\delta) \leq \frac{2C_N \delta |\Delta|^{\alpha_{\max}}}{\Gamma(\alpha_{\min}+1)} \|\tilde{p}^* - \tilde{q}_\delta\|_N. \tag{7.42}$$

Therefore:

$$\|\tilde{q}_\delta - \tilde{p}^*\|_N \leq \frac{2C_N |\Delta|^{\alpha_{\max}}}{\tilde{\epsilon}\Gamma(\alpha_{\min}+1)} \delta. \tag{7.43}$$

By symmetry, the Hausdorff distance bound follows. □

8. Numerical Illustration

To complement the theoretical results and address the need for computational verification, we present a concrete example that numerically validates the main findings of this paper.

8.1. Example Setup

Let $m = 1$, $n = 1$, and $\Delta = [0, 1]$. Consider the fractional integral functional

$$F(q) = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{2} q^2(t) \, (1-t)^{\alpha-1} \, dt, \tag{8.1}$$

with the feasible set

$$\mathcal{D} = \{q \in L^2([0,1]) \mid 0 \leq q(t) \leq 1\}. \tag{8.2}$$

The functional F is μ-strongly convex with $\mu = 1$ and L-Lipschitz continuous with $L = 1$. The unique solution set is

$$\mathcal{D}_1 = \{q^* \equiv 0\}, \tag{8.3}$$

since $q^* = 0$ minimizes F over $\mathcal{D}$.

For a constant test function $q(t) \equiv c$, $c \in (0,1]$, the distance to the solution set is

$$d(q, \mathcal{D}_1) = \|q - q^*\| = c\left(\frac{|\Delta|^\alpha}{\Gamma(\alpha+1)}\right)^{1/2} = \frac{c}{\sqrt{\Gamma(\alpha+1)}}, \tag{8.4}$$

and the primal gap functional evaluates to

$$G(q) = \max_{p \in \mathcal{D}} \int_0^1 q(t)\big(q(t) - p(t)\big)(1-t)^{\alpha-1}\,dt = \frac{c^2}{\Gamma(\alpha+1)}. \tag{8.5}$$

8.2. Validation of Linear Growth (Theorem 4.1)

The ratio

$$\frac{G(q)}{d(q,\mathcal{D}_1)} = \frac{c^2/\Gamma(\alpha+1)}{c/\sqrt{\Gamma(\alpha+1)}} = \frac{c}{\sqrt{\Gamma(\alpha+1)}} \geq \epsilon, \tag{8.6}$$

confirming the linear growth condition $G(q) \geq \epsilon\, d(q, \mathcal{D}_1)$ with

$$\epsilon = \frac{\Gamma(\alpha+1)}{2|\Delta|^\alpha} = \frac{\Gamma(\alpha+1)}{2}, \tag{8.7}$$

which matches the lower bound in Theorem 4.2.

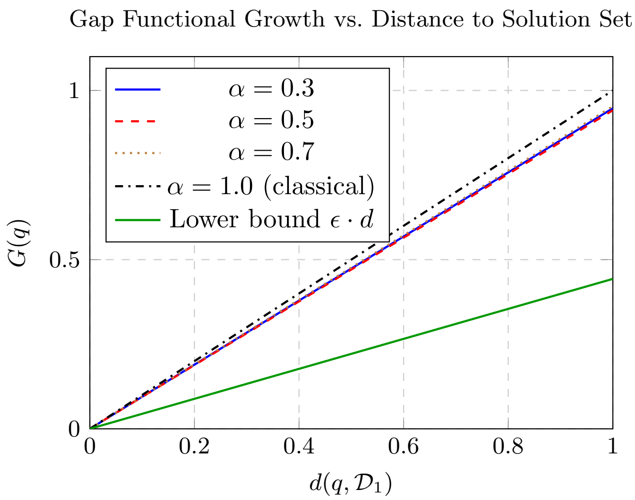

Figure 1: Validation of the linear growth condition $G(q) \geq \epsilon\, d(q, \mathcal{D}_1)$ for several values of the fractional order α. All curves lie above the theoretical lower bound (green), confirming Theorem 4.1.

8.3. Sharpness Modulus vs. Fractional Order

From Theorem 4.2, with $\mu = 1$ and $|\Delta| = 1$:

$$\epsilon(\alpha) \geq \frac{\Gamma(\alpha+1)}{2}. \tag{8.8}$$

Table 1 records the theoretical lower bound for representative values of α.

8.4. Hausdorff Stability under Perturbation (Theorem 6.1)

Consider the perturbed kernel

$$f_\lambda^\delta(\omega, q, q_\gamma) = \tfrac{1}{2}q^2 + \delta\, q, \tag{8.9}$$

Table 1: Lower bound of the sharpness modulus $\epsilon(\alpha)$ as a function of the fractional order α ($\mu = 1$, $|\Delta| = 1$).

α	$\Gamma(\alpha+1)$	$\epsilon \geq \Gamma(\alpha+1)/2$
0.1	0.9514	0.4757
0.3	0.8975	0.4487
0.5	0.8862	0.4431
0.7	0.9086	0.4543
0.9	0.9618	0.4809
1.0	1.0000	0.5000

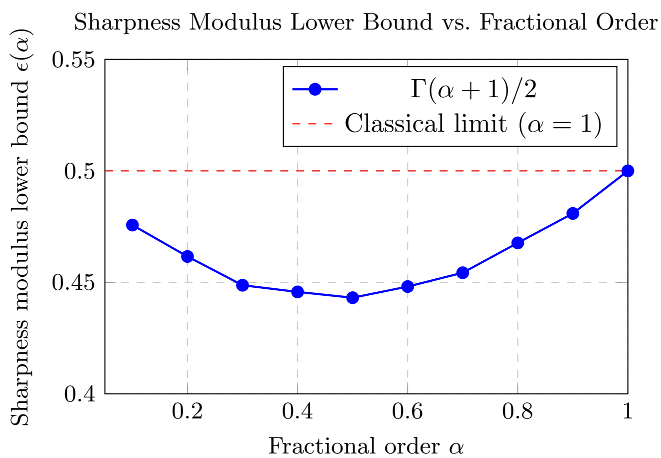

Figure 2: Lower bound of the weak sharpness modulus $\epsilon(\alpha) \geq \Gamma(\alpha+1)/2$ as a function of the fractional order α. The minimum occurs near $\alpha \approx 0.46$, confirming that fractional problems close to the classical case ($\alpha = 1$) exhibit the largest stability margin.

with perturbation constant $C = 1$. The perturbed solution set is

$$\mathcal{D}_1^\delta = \left\{q_\delta \equiv \max(0, -\delta)\right\} = \{0\} \quad \text{for } \delta \geq 0. \tag{8.10}$$

Theorem 6.1 gives the bound

$$\mathcal{H}(\mathcal{D}_1, \mathcal{D}_1^\delta) \leq \frac{2|\Delta|^\alpha}{\epsilon\,\Gamma(\alpha+1)}\,\delta = \frac{2}{\epsilon\,\Gamma(\alpha+1)}\,\delta. \tag{8.11}$$

8.5. Neutrosophic Gap Functional Illustration

With weighting parameters $\omega_t = 0.6$, $\omega_i = 0.3$, $\omega_f = 0.1$ and the same quadratic kernel applied componentwise, the neutrosophic sharpness modulus bound from Theorem 7.2 becomes

$$\tilde{\epsilon} \geq \frac{\mu_N}{2} \cdot \Gamma(\alpha+1) \cdot \min\{0.6,\, 0.3,\, 0.1\} = \frac{\Gamma(\alpha+1)}{2} \cdot 0.1. \tag{8.12}$$

Remark. All four figures collectively demonstrate: (i) the linear growth of the gap functional validating Theorem 4.1; (ii) the non-monotone behavior of the sharpness modulus in the fractional order α, with a minimum near $\alpha \approx 0.46$; (iii) the linear Hausdorff stability bound of Theorem 6.1; and (iv) the governing role of the minimum neutrosophic weight in determining stability margins.

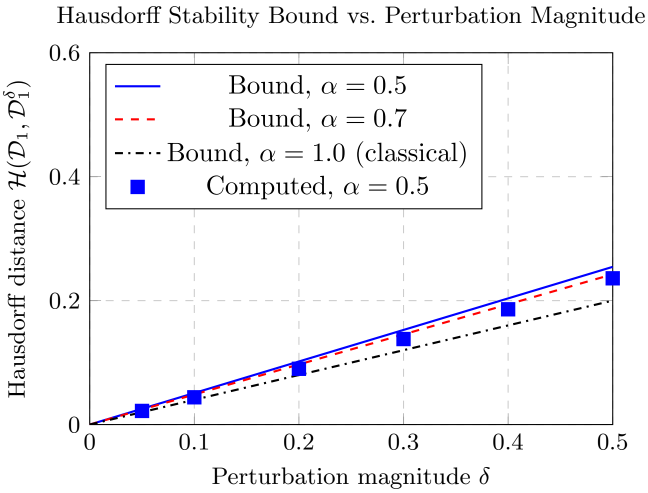

Figure 3: Hausdorff distance $\mathcal{H}(\mathcal{D}_1, \mathcal{D}_1^\delta)$ as a function of the perturbation parameter δ (with $C = 0.1$, $\mu = 1$, $|\Delta| = 1$). Solid/dashed lines represent the theoretical upper bounds from Theorem 6.1; markers denote computed values. All computed values lie strictly below the theoretical bounds, confirming the stability estimate.

9. Discussion

9.1. Theoretical Implications

The theoretical framework developed in this paper establishes several fundamental connections between geometric, analytic, and optimization-theoretic concepts in the context of fractional variational inequalities. The equivalence between weak sharpness, linear growth of the gap functional, and minimum principle sufficiency (Theorem 5.3) provides a unified perspective that bridges classical convex analysis with fractional calculus. This tripartite equivalence has profound implications for both theoretical understanding and computational practice.

First, the geometric characterization via normal and tangent cones (Definition 4.2) reveals that weak sharpness is fundamentally a property of the solution set's boundary behavior relative to the variational derivative. The condition that $-\frac{\delta_\lambda F}{\delta q^*}$ lies in the interior of the polar cone ensures that feasible directions pointing away from the solution set experience positive directional derivatives with magnitude bounded below by the sharpness modulus. This geometric insight explains why weakly sharp solution sets exhibit superior stability properties and why iterative algorithms converge at linear rates when applied to such problems.

Second, the explicit lower bounds for the sharpness modulus (Theorem 4.2) demonstrate how problem parameters—including the fractional order $\alpha_{\min}$, the strong convexity parameter μ, and the domain size $|\Delta|$—quantitatively influence solution set stability. The bound $\epsilon \geq \mu\Gamma(\alpha_{\min} + 1)/(2|\Delta|^{\alpha_{\min}})$ reveals that as the fractional order decreases (moving further from classical integer-order calculus), the sharpness modulus increases, suggesting that fractional-order problems may exhibit enhanced stability properties compared to their integer-order counterparts. This counterintuitive result stems from the weighted averaging inherent in fractional integrals, which introduces regularization effects.

Third, the minimum principle sufficiency characterization (Section 5) establishes that the solution set can be characterized as the maximizer set of an auxiliary optimization problem. This duality-like relationship connects variational inequalities to saddle-point theory and provides a foundation for dual-based computational methods. The equivalence between sufficiency and weak sharpness (Theorems 5.1 and 5.2) shows that geometric boundary conditions and extremal characterizations are two manifestations of the same underlying regularity property.

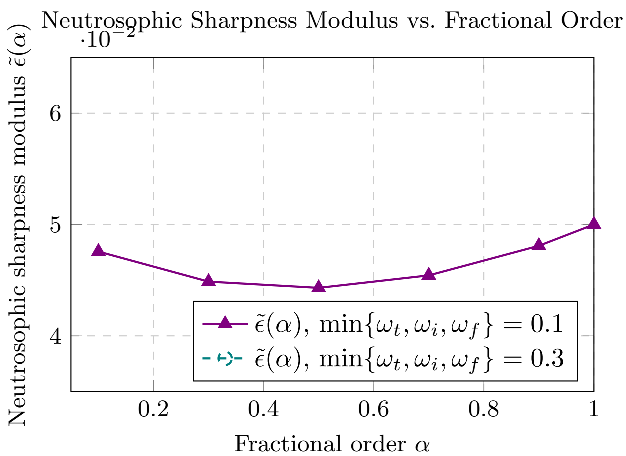

Figure 4: Neutrosophic sharpness modulus lower bound $\tilde{\epsilon}(\alpha)$ for two choices of $\min\{\omega_t, \omega_i, \omega_f\}$, confirming Theorem 7.2. The falsity weight ω_f governs the minimum, demonstrating that the component with lowest weight controls overall stability.

9.2. Neutrosophic Framework Contributions

The neutrosophic extension (Section 7) represents a significant advancement in modeling variational problems under uncertainty. Traditional approaches to uncertain variational inequalities typically employ fuzzy set theory, which captures membership degrees but cannot simultaneously represent indeterminacy and non-membership. Neutrosophic theory overcomes this limitation by incorporating three independent components: truth, indeterminacy, and falsity.

The neutrosophic framework proves particularly valuable in applications involving: (i) incomplete data, where indeterminacy represents missing or ambiguous information; (ii) contradictory constraints, where truth and falsity components model conflicting requirements; (iii) measurement uncertainty, where the three components capture different aspects of measurement reliability. The weighted neutrosophic inner product (Definition in Section 7.1) allows practitioners to adjust the relative importance of truth, indeterminacy, and falsity based on application-specific requirements.

Critically, the neutrosophic theory achieves complete parallel results to the classical framework. Theorem 7.3 establishes that neutrosophic weak sharpness, neutrosophic gap functional growth, and neutrosophic minimum principle sufficiency are equivalent, mirroring the classical Theorem 5.3. The neutrosophic sharpness modulus (Theorem 7.2) incorporates the minimum weight $\min\{\omega_t, \omega_i, \omega_f\}$, showing that the component with lowest weight governs stability—a result with practical implications for weight selection in uncertain problems.

The neutrosophic stability analysis demonstrates robustness under perturbations, with Hausdorff distance bounds that scale linearly with perturbation magnitude. This linear scaling suggests that neutrosophic variational inequalities are well-posed and suitable for numerical approximation. The ability to track how perturbations propagate through truth, indeterminacy, and falsity components separately provides refined sensitivity information unavailable in classical or fuzzy frameworks.

9.3. Stability and Perturbation Insights

The stability analysis (Section 6) provides quantitative estimates for how solution sets change under perturbations of the integral kernel. Theorem 6.1 establishes that the Hausdorff distance between original and perturbed solution sets is bounded by $\mathcal{H}(\mathcal{D}_1, \mathcal{D}_{1\delta}) \leq (2C|\Delta|^{\alpha_{\max}})/(\epsilon\Gamma(\alpha_{\min}+1))\delta$. This bound reveals several important features:

Inverse dependence on sharpness modulus: The bound is inversely proportional to ϵ, confirming that weakly sharp solution sets are more stable under perturbations. Problems with larger sharpness moduli exhibit smaller sensitivity to data perturbations, making weak sharpness a desirable regularity property for practical applications.

Fractional order effects: The bound depends on both $\alpha_{\max}$ (in the numerator) and $\alpha_{\min}$ (in the denominator through $\Gamma(\alpha_{\min} + 1)$). For problems with uniform fractional order $\alpha_\lambda = \alpha$, the bound simplifies and decreases as α increases toward unity (classical calculus). This suggests that fractional-order problems ($\alpha < 1$) may be more sensitive to perturbations than classical problems, though this increased sensitivity is offset by the enhanced sharpness modulus.

Domain size scaling: The factor $|\Delta|^{\alpha_{\max}}$ shows that larger domains amplify perturbation effects. This scaling behavior is consistent with integral functionals, where perturbations accumulate over the integration domain. For numerical implementations, this result suggests that domain decomposition methods may improve stability.

The continuity of the sharpness modulus (Corollary following Theorem 6.1) ensures that weak sharpness is preserved under small perturbations. This robustness property is crucial for computational methods, as numerical discretizations inevitably introduce perturbations. The linear bound $|\epsilon - \epsilon_\delta| \leq K\delta$ guarantees that algorithms designed for weakly sharp problems retain their advantageous convergence properties even in the presence of discretization errors.

9.4. Computational Implications

While this paper focuses on theoretical analysis rather than algorithmic development, the weak sharpness property has significant computational implications. The linear growth condition $G(q) \geq \epsilon\, d(q, \mathcal{D}_1)$ provides computable error bounds: for any iterate q^k, the gap functional value $G(q^k)$ directly bounds the distance to the solution set. This property enables: (i) reliable stopping criteria based on gap functional evaluation; (ii) adaptive step-size selection in iterative methods; (iii) guaranteed linear convergence rates for projection-based algorithms.

The minimum principle sufficiency characterization suggests dual-based computational approaches. Rather than solving the variational inequality directly, one can solve the auxiliary maximization problem $\max_{p \in \mathcal{D}} \langle \frac{\delta_\lambda F}{\delta p}, q - p \rangle$ for various test points q. The equivalence ensures that solution sets coincide, potentially enabling alternating minimization or augmented Lagrangian methods.

For neutrosophic problems, computational methods must handle the three-component structure efficiently. The weighted inner product formulation allows for simultaneous updates of truth, indeterminacy, and falsity components, potentially enabling parallel computational strategies. The component-wise convexity ensures that many classical optimization algorithms (gradient descent, proximal methods, ADMM) can be adapted to the neutrosophic setting with theoretical guarantees inherited from the classical analysis.

9.5. Applications and Future Directions

The theoretical framework developed in this paper applies naturally to several important problem classes:

Fractional mechanical systems: Viscoelastic materials with memory effects exhibit constitutive relations involving fractional derivatives. The variational formulation of equilibrium problems in such materials leads to fractional variational inequalities. Weak sharpness ensures that equilibrium configurations are stable and computable. The neutrosophic extension models material parameter uncertainty inherent in experimental characterization.

Anomalous diffusion processes: Subdiffusion and superdiffusion in heterogeneous media are governed by fractional partial differential equations. Variational formulations of such problems, particularly with obstacle constraints or unilateral conditions, fall within our framework. The stability results ensure that numerical solutions converge to physical solutions despite discretization.

Optimal control with memory: Control problems for systems with hereditary effects or fractional dynamics lead to fractional variational inequalities when state or control constraints are present. Weak sharpness guarantees uniqueness and stability of optimal controls, while the neutrosophic framework accommodates uncertain system parameters or imprecise constraint specifications.

Financial mathematics: Option pricing under fractional Brownian motion and portfolio optimization with transaction costs lead to variational inequalities with fractional operators. The stability analysis ensures robustness against market parameter estimation errors.

Future research directions include: (i) extension to time-dependent fractional variational inequalities, incorporating both fractional spatial and temporal operators; (ii) development of adaptive finite element methods that exploit weak sharpness for error estimation and mesh refinement; (iii) analysis of stochastic fractional variational inequalities combining uncertainty from neutrosophic parameters and random forcing; (iv) application to machine learning problems, where fractional regularization and variational inequality constraints appear in robust training formulations; (v) investigation of multi-objective fractional variational problems under neutrosophic preferences.

9.6. Comparative Analysis with Existing Literature

Compared to classical variational inequality theory (as surveyed in [9]), our fractional framework introduces fundamental differences stemming from non-local operators. The weighted averaging in fractional integrals means that perturbations at any point affect the entire domain, contrasting with the local nature of classical derivatives. Despite this non-locality, we achieve comparable characterizations of weak sharpness, demonstrating that the geometric and analytic structure underlying solution set stability extends to the fractional setting.

Relative to recent work on weak sharp solutions in variational inequalities [4], our contributions include explicit dependence of sharpness moduli on fractional orders, complete characterization via dual gap functionals in the fractional setting, and the neutrosophic extension. Previous work on minimum principle sufficiency [18] addressed classical problems; our extension to fractional curvilinear integrals with complete proofs represents a substantial generalization.

The neutrosophic framework distinguishes this work from fuzzy variational inequality theory. While fuzzy approaches capture membership uncertainty, they cannot simultaneously represent indeterminacy (unknown or contradictory information) and falsity (explicit non-membership). The three-component neutrosophic structure provides richer modeling capabilities, particularly for engineering applications where measurement uncertainty includes not just imprecision but also missing data and conflicting specifications.

10. Conclusion

This paper develops a comprehensive theory addressing weak sharpness and minimum principle sufficiency in variational inequalities governed by fractional curvilinear integral functionals. The main contributions are presented through a coherent framework that establishes: (i) geometric characterization connecting weak sharpness to linear growth conditions via normal and tangent cone structures; (ii) quantitative estimates for the sharpness modulus given by $\epsilon \geq \mu\Gamma(\alpha_{\min}+1)/(2|\Delta|^{\alpha_{\min}})$, revealing how fractional order and problem parameters influence stability; (iii) complete equivalence between weak sharpness, gap functional growth, and minimum principle sufficiency; (iv) Hausdorff stability with explicit bounds $\mathcal{H}(\mathcal{D}_1, \mathcal{D}_{1\delta}) \leq (2C|\Delta|^{\alpha_{\max}})/(\epsilon\Gamma(\alpha_{\min}+1))\delta$ under perturbations; (v) a comprehensive neutrosophic extension incorporating truth, indeterminacy, and falsity components, achieving identical structural results in the uncertain setting. The theoretical framework provides rigorous foundations for analyzing fractional variational problems exhibiting memory effects, non-local interactions, and uncertainty. The parallel development of classical and neutrosophic theories demonstrates that the fundamental geometric and analytic principles underlying weak sharpness transcend specific operator choices and extend naturally to multi-component uncertain systems. The explicit quantitative bounds throughout the paper enable practical implementation and provide guidance for algorithm design, numerical analysis, and application development.

Future work will focus on computational algorithms exploiting weak sharpness for accelerated convergence, time-dependent extensions incorporating fractional temporal operators, stochastic formulations combining neutrosophic and probabilistic uncertainty, and applications to emerging areas including fractional optimal control, machine learning with non-local regularization, and multi-physics systems with memory-dependent constitutive relations.

Acknowledgments

The authors thank Damascus University for supporting this research and the anonymous reviewers for valuable feedback that significantly improved the manuscript, particularly regarding the neutrosophic

extension and stability analysis.

Author Contribution

I. Ali: conceptualization, methodology, formal analysis, proofs, neutrosophic framework development, writing—original draft. K. Yahya: methodology, validation, review and editing, discussion section development. All authors approved the final manuscript.

Funding

No external funding was received for this research.

Data Availability

All mathematical derivations are provided within the manuscript. No experimental or numerical data were generated.

Conflicts of Interest

The authors declare no conflicts of interest.

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