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On Reduced Reciprocal Randić Energy of Tensor Product of Graphs

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ABSTRACT: Let $G = G_1 \otimes G_2$ be a tensor product of two graphs with $m+n$ vertices and mn edges. Let $V(G) = \{w_{ij} = (u_{ij}, v_{ij}) : \textit{ where } 1 \le i \le m \textit{ and } 1 \le j \le n\}$ be the vertex set of G. [4] defined reduced reciprocal randic matrix of a graph G on n vertices. It is denoted by $RRR(G)$ and it is defined as a $n \times n$ matrix whose (i,j) entry as $\sqrt{(d_{v_i}-1)(d_{v_j}-1)}$ if v_i and v_j are adjacent. otherwise it is 0. The Reduced reciprocal randic energy $RRRE(G)$ of a graph G is the sum of the absolute values of the eigenvalues of $RRR(G)$. In this paper, we explore the reduced reciprocal randic energy $RRRE(G)$ of tensor product $RRR(K_m \otimes K_n)$, $RRR(K_m \otimes C_n)$ and $RRR(C_m \otimes C_n)$.

Keywords: Energy of graph, tensor product, reduced reciprocal randic energy.

Contents

1. Introduction

Let G_1 and G_2 are undirected simple graphs with m and n vertices respectively. The vertex set of $G = (G_1 \otimes G_2)$ is $V(G) = \{w_{ij} = (u_i, v_j) \textit{ where } 1 \le i \le m \textit{ and } 1 \le j \le n\}$. Two vertices w_i and w_j are said to be adjacent if there is an edge connecting them. The energy of a graph G was introduced in 1978 [3], as the sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$ of a graph G. In 2014, [3] Gutman et al. introduced the reduced reciprocal randic matrix $RRR(G)$ as

$$RRR(G) = \sum_{v_i v_j \in E(G)} \sqrt{(d_{v_i}-1)(d_{v_j}-1)}$$

where $E(G)$ is the edge set of G, d_{v_i} and d_{v_j} represents degrees of vertices v_i and v_j respectively. Further we studied the work on reduced reciprocal randic matrix and reduced reciprocal randic energy in [4]. In this paper, we study the reduced reciprocal randic energy of tensor product $K_m \otimes K_n$, $K_m \otimes C_n$ and $C_m \otimes C_n$.

2. Preliminaries

Definition 2.1 [5] Let G_1 and G_2 be two graphs with m and n vertices respectively. Then Tensor product of two graphs G_1 and G_2 is such that the vertex set is the cartesian product $V(G_1) \times V(G_2)$ and joining any two vertices (u_1, u_2) and (v_1, v_2) if and only if, u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. This graph is represented by $G_1 \otimes G_2$ and is also called as Kronecker product of graphs.

Definition 2.2 [4] The reduced reciprocal randic matrix of a graph $G = G_1 \otimes G_2$, defined by

$$RRR(G) = [a_{ij}] = \begin{cases} \sqrt{(d_{v_i}-1)(d_{v_j}-1)} & if\ v_i\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

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If $\lambda_1, \lambda_2, \cdots, \lambda_{mn}$ are its eigenvalues, the reduced reciprocal randic energy of G is defined as

$$RRRE(G) = \sum_{i=1}^{mn} |\lambda_i|.$$

Definition 2.3 [1] Let $B_1, B_2, \ldots B_m$ be square matrices of order n. A block circulant matrix of order mn is of the form

$$bcirc(B_1, B_2, \ldots B_m) = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ B_m & B_1 & \cdots & B_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_3 & \cdots & B_1 \end{pmatrix}$$

If each B_i for $1 \leqslant i \leqslant n$ is circulant, then we call the above matrix a circulant block matrix with circulant blocks.

Theorem 2.1 *[1] Let C be an A-factor block circulant. Then*

$$C = V_A P(D_A) V_A^{-1},$$

where V_A is block Vandermonde matrix and $P(z)$ is the representor of C. Moreover, the set of A-factor circulants coincides with the set of matrices of the form

$$V_A diag[M_1, M_2, \ldots, M_m] V_A^{-1},$$

that is,$P(D_A) = diag[M_1, M_2, \ldots, M_m]$ for a matrix polynomial

$$P(z) = C_1 + C_2 z + \ldots + C_m z^{m-1} \textit{if and only if } [C_1 C_2 \ldots C_m] V_A = [M_1 M_2 \ldots M_m].$$

The following result is a consequence of the above Theorem 2.1

Corollary 2.1 *[1] The factor circulant C can also be expressed as*

$$C = \Re F_{mn}^{*} P(K\Omega) F_{mn} \Re^{-1}$$

where F_{mn} is a block Fourier matrix, $\Omega = diag[I, \omega I, \omega^2 I, \ldots, \omega^{m-1} I]$ $(\omega = exp(\frac{2\pi i}{m})), K$ is the principal m^{th} root of the non -singular matrix A and $\Re = diag[I K K^2 \ldots K^{m-1}]$. In particular if C is a block circulant then it can be represented as

$$C = F_{mn}^{*} P(\Omega) F_{mn}.$$

Motivated by the above, in the section 2 we discuss the reduced reciprocal randic energy of tensor product $K_m \otimes K_n$, $K_m \otimes C_n$ and $C_m \otimes C_n$.

3. Reduced reciprocal randic energy of Tensor product of some graphs

Let G_1 and G_2 be graphs with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ respectively. Then the tensor product $G = G_1 \otimes G_2$ will have mn vertices of the form $w_{ij} = (u_i, v_j)$.

In [4], the authors have discussed the reduced reciprocal randic energy of some graphs such as complete graph K_m, croen graphS_n^0, complete bipartite graph $K_{m,n}$, and cocktail party graph $K_{n\times 2}$. Motivated by this, we determine the Reduced reciprocal randic energy of the tensor product of two graphs G_1 and G_2 denoted by $G = G_1 \otimes G_2$.

Theorem 3.1 *Let K_m and K_n be complete graphs with vertices m and n respectively. Then*

$$RRRE(K_m \otimes K_n) = 4(m-1)(n-1)|mn-m-n|$$

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_m and K_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(K_m \otimes K_n)$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(K_m \otimes K_n) = bcirc(A_1, A_2, \ldots, A_n)_{mn \times mn}$$

where

$$A_1 = Z_{m \times m}$$

and

$$A_2 = \cdots = A_n = circ\,(0, mn-m-n, \cdots, mn-m-n)_{m \times m}.$$

From Theorem 1.6, the diagonal form of $RRR(K_m \otimes K_n)$ is

$$diag([\wedge_1 + (n-1)\wedge_2], [\wedge_1 - \wedge_2], \ldots, [\wedge_1 - \wedge_2])_{mn \times mn}$$

where

$$\wedge_1 = diag(0, 0, \cdots, 0)_{m \times m}$$

and

$$\wedge_2 = diag((m-1)(mn-m-n), m+n-mn, \cdots, m+n-mn)_{m \times m}$$

are the spectra of A_1 and A_2 respectively.
Consider,

$$[\wedge_1 + (n-1)\wedge_2] = diag((m-1)(n-1)(mn-m-n), (n-1)(m+n-mn), \cdots, (n-1)(m+n-mn))_{m \times m}$$

and

$$[\wedge_1 - \wedge_2] = diag((1-m)(mn-m-n), mn-m-n, \cdots, mn-m-n)_{m \times m}.$$

Hence, spectrum of $RRR(K_m \otimes K_n)$ is given by

$$\begin{cases} (m-1)(n-1)(mn-m-n) & once \\ (n-1)(m+n-mn) & (m-1)\ times \\ (1-m)(mn-m-n) & (n-1)\ times \\ mn-m-n & (m-1)(n-1)\ times \end{cases}.$$

Thus,

$$RRRE(K_m \otimes K_n) = 4(m-1)(n-1)|mn-m-n|$$

□

Theorem 3.2 *Let K_m and C_n be complete graph and cycle graph with vertices m and n respectively. Then*

$$RRRE(K_m \otimes C_n) = 4(m-1)|2m-3| \left[1 + \sum_{k=1}^{(n-1)} \left| cos\left(\frac{2\pi k}{n}\right) \right| \right].$$

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_m and C_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(K_m \otimes C_n)$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(K_m \otimes C_n) = bcirc(A_1, A_2, \ldots, A_n)_{mn \times mn}$$

where

$$A_1 = A_3 = A_4 = \cdots = A_{n-1} = Z_{m\times m},$$

and

$$A_2 = A_n = circ(0, (2m-3), \cdots, (2m-3))_{m\times m}.$$

From Theorem 1.6, the diagonal form of $RRR(K_m \otimes C_n)$ is

$$diag\left([2\wedge_2], \left[2cos\left(\frac{2\pi k}{n}\right)\wedge_2\right]\right)_{mn\times mn}$$

where $1 \leq k \leq (n-1)$,

$$\wedge_2 = diag((m-1)(2m-3), (3-2m), \cdots, (3-2m))_{m\times m}$$

is the spectrum of A_2.
Consider,

$$[2\wedge_2] = diag(2(m-1)(2m-3), 2(3-2m), \cdots, 2(3-2m))_{m\times m}$$

and

$$\left[2cos\left(\frac{2\pi k}{n}\right)\wedge_2\right]$$

$$= diag\left(2(m-1)(2m-3)cos\left(\frac{2\pi k}{n}\right), 2(3-2m)cos\left(\frac{2\pi k}{n}\right), \cdots, 2(3-2m)cos\left(\frac{2\pi k}{n}\right)\right)_{m\times m}.$$

Hence, spectrum of $RRR(K_m \otimes C_n)$ is given by

$$\begin{cases} 2(m-1)(2m-3) & once \\ 2(3-2m) & (m-1)\ times \\ 2(m-1)(2m-3)cos\left(\frac{2\pi k}{n}\right)\ where\ 1\leq k\leq (n-1) & (n-1)\ times \\ 2(3-2m)cos\left(\frac{2\pi k}{n}\right)\ where\ 1\leq k\leq (n-1) & (m-1)(n-1)\ times \end{cases}.$$

Thus,

$$RRRE(K_m \otimes C_n) = 4(m-1)|2m-3|\left[1 + \sum_{k=1}^{(n-1)}\left|cos\left(\frac{2\pi k}{n}\right)\right|\right].$$

□

Theorem 3.3 *Let C_m and C_n be cycle graphs with vertices m and n respectively. Then*

$$RRRE(C_m \otimes C_n) = 12\left[1 + \sum_{t=1}^{m-1}\left|\cos\left(\frac{2\pi t}{m}\right)\right| + \sum_{k=1}^{n-1}\left|\cos\left(\frac{2\pi k}{n}\right)\right| + \sum_{t=1}^{m-1}\sum_{k=1}^{n-1}\left|\cos\left(\frac{2\pi t}{m}\right)\cos\left(\frac{2\pi k}{n}\right)\right|\right].$$

Proof: Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_m and C_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(C_m \otimes C_n)$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(C_m \otimes C_n) = bcirc(A_1, A_2, \ldots, A_n)_{mn\times mn}$$

where

$$A_1 = A_3 = \cdots = A_{n-1} = Z_{m\times m}$$

and

$$A_2 = A_n = circ(0, 3, 0, \cdots, 0, 3)_{m\times m}.$$

From Theorem 1.6, the diagonal form of $RRR(C_m \otimes C_n)$ is

$$diag\left([2\wedge_2], \left[2cos\left(\frac{2\pi k}{n}\right)\wedge_2\right]\right)_{mn\times mn}$$

where $1 \leq k \leq (n-1)$,

$$\wedge_2 = diag\left(6, 6cos\left(\frac{2\pi t}{m}\right)\right)_{m\times m} \quad where\ 1 \leq t \leq (m-1).$$

is the spectra of A_2.
Consider,

$$[2\wedge_2] = diag\left(12, 12cos\left(\frac{2\pi t}{m}\right)\right)_{m\times m}$$

and

$$\left[2cos\left(\frac{2\pi k}{n}\right)\wedge_2\right]$$

$$= diag\left(12cos\left(\frac{2\pi k}{n}\right), 12cos\left(\frac{2\pi t}{m}\right)cos\left(\frac{2\pi k}{n}\right)\right)_{m\times m}.$$

Hence, spectrum of $RRR(C_m \otimes C_n)$ is given by

$$\begin{cases} 12 & once \\ 12cos\left(\frac{2\pi t}{m}\right)\ where\ 1 \leq t \leq (m-1) & (m-1)\ times \\ 12cos\left(\frac{2\pi k}{n}\right)\ where\ 1 \leq k \leq (n-1) & (n-1)\ times \\ 12cos\left(\frac{2\pi t}{m}\right)cos\left(\frac{2\pi k}{n}\right)\ where\ 1 \leq t \leq (m-1)\ and\ 1 \leq k \leq (n-1) & (m-1)(n-1)\ times \end{cases}.$$

Thus,

$$RRRE(C_m \otimes C_n) = 12\left[1 + \sum_{t=1}^{m-1}\left|\cos\left(\frac{2\pi t}{m}\right)\right| + \sum_{k=1}^{n-1}\left|\cos\left(\frac{2\pi k}{n}\right)\right| + \sum_{t=1}^{m-1}\sum_{k=1}^{n-1}\left|\cos\left(\frac{2\pi t}{m}\right)\cos\left(\frac{2\pi k}{n}\right)\right|\right].$$

□

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