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Bakry-Émery Ricci Curvature of Halilsoy Metric

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ABSTRACT: The objective of this paper is to investigate Ricci curvature and its generalized notion of curvature on the Halilsoy spacetime. The Bakry-Émery Ricci tensor Ric_f and $m-$Bakry-Émery Ricci tensor Ric_f^m has been calculated with the potential function f. Also, the notion of Bakry-Émery scalar curvature $\mathcal{R}_f$ and $m-$Bakry-Émery scalar curvature $\mathcal{R}_f^m$ have been introduced, which is analogous to the scalar curvature, and such quantities have determined for Halilsoy spacetime. Finally, some sufficient conditions have been obtained for the flatness with respect to the curvatures $\mathcal{R}_f$ and $\mathcal{R}_f^m$.

Keywords: Scalar curvature, Ricci tensor, Bakry-Émery Ricci tensor, $m-$Bakry-Émery Ricci tensor, Halilsoy metric, spacetime.

Contents

1. Introduction & Preliminaries

In 2015, a merger of two stellar-mass black holes about 1.3 billion light-years away was detected by LIGO (Laser Interferometer Gravitational-Wave Observatory) and it was the first direct detection of gravitational waves in the universe. Such initial observation of gravitational waves marked a transformative moment in astronomy, as it enabled exploration beyond the confines of electromagnetic radiation. From a theoretical stand-point, while the linearized form of Einstein's field equations is well established, understanding their full nonlinear behavior remains a significant challenge. For plane gravitational waves, the Einstein equations simplify to the Laplace equation, which is linear, implying that waves propagating in the same direction do not interact. To explore nontrivial dynamics, cylindrically symmetric configurations serve as the simplest and most instructive models. These models have been extensively employed in general relativity studies [6,18,15,2].

Standing waves are significant in various areas of physics. In general relativity, the nonlinear nature of Einstein's equations makes the analysis of gravitational standing waves both intriguing and challenging. However, under cylindrical symmetry, the Einstein equations become much simpler, allowing the study of standing gravitational waves in their complete nonlinear framework. A standing wave is characterized by the absence of spatial energy transport. In the framework of general relativity, the local definition of gravitational energy density is not possible, making the formulation of standing gravitational waves a challenging task. Under cylindrical symmetry, this issue was independently explored by Halilsoy [8] and Chandrasekhar [4], employing the concept of C-energy [26]. The definition suggested by Halilsoy [8] is comparatively less restrictive than that introduced by Chandrasekhar [4]. In Chandrasekhar's formulation, the C-energy remains constant over time, whereas Stephani argued that it should be temporally constant only in some average sense. Thus through the last decade the investigation on the gravitational waves became very crucial for the physicists and geometers.

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By virtue of the cylindrical symmetry, the Halilsoy spacetime [8] which is a standing wave spacetime, constitutes exact solutions to the vacuum Einstein field equations. In terms of cylindrical coordinate system (t, ρ, z, ϕ) the Halilsoy metric [8] is given by

$$ds^2 = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2.$$

Such standing gravitational wave, under the approximation, corresponds at large distances from the symmetry center to standing gravitational waves with polarization. We note that Nikiel and Szybka [16] the linear approximation for Halilsoy and Chandrasekhar standing gravitational waves. Again, very recently Szybka et al. [25] studied the backreaction effects by determining the high-frequency limit of the Halilsoy and Chandrasekhar standing gravitational wave solutions. Hence Halilsoy metric is geometrically as well as physically significant model of gravitational waves.

The article can be decomposed as follows: Section 2 is devoted with the physical significance of Halilsoy metric. In Section 3, the notion of Ricci tensor and its generalizations such as Bakry-Émery Ricci tensor, $m-$Bakry-Émery Ricci tensor have been reviewed. Section 4 consists of the main results of the article. Finally, the Section 5 is the conclusion of the article.

2. Halilsoy Metric

Geometrically, the line element of the cylindrically symmetric Halilsoy metric in (t, ρ, z, ϕ) coordinate system is given by

$$ds^2 = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2 \tag{2.1}$$

under the coordinate conditions $\rho > 0, -\infty < t, z < \infty, 0 \leq \phi < 2\pi$ and the smooth functions ψ, γ, ω depend only on t and ρ. Clearly, ∂_z, ∂_ϕ are appeared as Killing vectors as the Lie derivative of the metric g with respect to them vanishes. Again, if $\omega = 0$, the Halilsoy metric reduces to the Einstein-Rosen vacuum waves [6,20] by an appropriate selection of ψ, γ and ω. Also, Halilsoy [8] demonstrated that the basic Einstein-Rosen standing wave solution could be extended to include a secondary polarization. Halilsoy's general solution to the Einstein field equations is ([16]):

$$\begin{aligned}
\omega &= -(A \sinh\alpha)\rho \mathcal{B}_1 \sin(t/\lambda),\\
\gamma &= \frac{1}{8}A^2\left\{\left(\frac{\rho}{\lambda}\right)^2(\mathcal{B}_0^2 + \mathcal{B}_1^2) - 2\frac{\rho}{\lambda}\mathcal{B}_0\mathcal{B}_1\cos^2(t/\lambda)\right\},\\
e^{-2\psi} &= e^{A\mathcal{B}_0\cos(t/\lambda)}\sinh^2\left(\frac{\alpha}{2}\right) + e^{-A\mathcal{B}_0\cos(t/\lambda)}\cosh^2\left(\frac{\alpha}{2}\right),
\end{aligned}$$

where $\lambda > 0$ and A, α are constants. Here, $\mathcal{B}_0 = \mathcal{B}_0(\rho/\lambda)$ and $\mathcal{B}_1 = \mathcal{B}_1(\rho/\lambda)$ are Bessel's functions of the first kind of order 0 and 1. The Halilsoy metric, supplemented by the above-stated ψ, γ and ω, gives the general exact standing wave solution in vacuum general relativity. Again, if $\alpha = 0$, it becomes the original Einstein-Rosen standing wave, a solution whose properties have recently been explored in [24].

3. Ricci Curvature and its Generalizations

Let us consider a smooth metric measure space $(M^n, g, e^{-f}\,d\mathrm{vol}_g)$, where M denotes a complete and connected n-dimensional semi-Riemannian manifold equipped with a metric g of signature (n^+, n^-), f is a smooth real-valued function on M, $d\mathrm{vol}_g$ represents the Riemannian volume form associated with g and ∇ is the Levi-Civita connection on M. For $(n^+, n^-) = (n, 0)$ the manifold M is called Riamannian, and for $(n^+, n^-) = (n-1, 1)$ or $(1, n-1)$ the manifold M is called Lorentzian. We must note that the spacetime is a 4-dimensional connected Lorentzian manifold. In this article, generalized notions of Ricci tensor and has been investigated on the Halilsoy spacetime.

The Riemann curvature tensor R is a $(1,3)$-type tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all vector fields X, Y, Z on M. It measures the non-commutativity of covariant derivatives and encodes the intrinsic curvature of the manifold. In local coordinates $(\xi^1, \xi^2, \ldots, \xi^n)$ with Christoffel symbols of second kind Γ^i_{jk}, the components of R are given by

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk},$$

where $\partial_k = \frac{\partial}{\partial \xi^k}$. The Ricci curvature tensor Ric is obtained by contracting the Riemann tensor R on its first and third indices given as

$$\mathrm{Ric}_{jl} = R^i{}_{jil}.$$

Equivalently, for vector fields Y, Z,

$$\mathrm{Ric}(Y, Z) = \mathrm{trace}\{X \mapsto R(X, Y)Z\}.$$

In the Euclidean spaces, the Ricci tensor represents the measurement of the deviated volume of small geodesic balls. The scalar curvature $\mathcal{R}$ is obtained by taking the trace of the Ricci tensor Ric with respect to the metric g:

$$\mathcal{R} = g^{ij}\,\mathrm{Ric}_{ij}.$$

This scalar function on M represents the average curvature at each point.

In this context, the *Bakry-Émery Ricci tensor* Ric_f is a generalized notion of Ric, which is defined as

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f$$

for some real-valued smooth function f on M. The concept was extensively investigated and generalized by Bakry and Émery [1], who explored its geometric and analytical properties establishing its connection with diffusion processes. The Bakry-Émery tensor naturally arises in various geometric and analytic settings (see, for example, [12,17]). The tensor Ric_f is also commonly known as the ∞-Bakry-Émery Ricci tensor Ric_f^{∞} as the $m-$Bakry-Émery Ricci tensor Ric_f^m is defined by

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df.$$

The *gradient Ricci soliton* is defined by $\mathrm{Ric}_f = \lambda g$ for some constant λ, which plays a fundamental role in the theory of Ricci flow. Furthermore, the Bakry-Émery Ricci tensor admits a natural extension to metric measure spaces [13,22,23]. For a detailed analysis of the comparison of geometries associated with the Ricci tensor and Bakry–Émery Ricci tensor, we refer the readers to go through Wei and Wylie [27] for various interesting results regarding mean curvature and volume comparison.

4. Main Results

The Halilsoy metric in (t, ρ, z, ϕ) coordinate system is given by

$$g = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2, \tag{4.1}$$

where the smooth functions ψ, γ, ω depend only on t and ρ. The Christoffel symbol Γ of second kind of the Halilsoy metric are computed by

$$\begin{gathered}
\Gamma^t_{tt} = \Gamma^t_{\rho\rho} = \Gamma^t_{\rho\rho} = \gamma_t - \psi_t, \quad \Gamma^\rho_{tt} = \Gamma^t_{t\rho} = \Gamma^\rho_{\rho\rho} = \gamma_\rho - \psi_\rho, \quad \Gamma^z_{tz} = \psi_t - \tfrac{e^{4\psi}\,\omega\,\omega_t}{2\rho^2}, \quad \Gamma^\phi_{tz} = \tfrac{e^{4\psi}\,\omega_t}{2\rho^2} \\
\Gamma^z_{t\phi} = \tfrac{1}{2}\left(4\omega\psi_t + \omega_t - \tfrac{e^{4\psi}\omega^2\omega_t}{\rho^2}\right), \quad \Gamma^\phi_{t\phi} = -\psi_t + \tfrac{e^{4\psi}\omega\,\omega_t}{2\rho^2}, \quad \Gamma^z_{\rho z} = \psi_\rho - \tfrac{e^{4\psi}\,\omega\,\omega_\rho}{2\rho^2}, \quad \Gamma^\phi_{\rho z} = \tfrac{e^{4\psi}\,\omega_\rho}{2\rho^2} \\
\Gamma^z_{\rho\phi} = \tfrac{-e^{4\psi}\omega^2\,\omega_\rho + \rho(\omega(-2+4\rho\psi_\rho)+\rho\,\omega_\rho)}{2\rho^2}, \quad \Gamma^\phi_{\rho\phi} = \tfrac{1}{\rho} - \psi_\rho + \tfrac{e^{4\psi}\,\omega\,\omega_\rho}{2\rho^2}, \quad \Gamma^t_{zz} = e^{-2\gamma+4\psi}\,\psi_t \\
\Gamma^\rho_{zz} = -e^{-2\gamma+4\psi}\,\psi_\rho, \quad \Gamma^t_{z\phi} = \tfrac{1}{2}e^{-2\gamma+4\psi}\left(2\omega\psi_t + \omega_t\right), \quad \Gamma^\rho_{z\phi} = -\tfrac{1}{2}e^{-2\gamma+4\psi}\left(2\omega\psi_\rho + \omega_\rho\right) \\
\Gamma^t_{\phi\phi} = e^{-2\gamma}\big(-\rho^2\psi_t + e^{4\psi}\,\omega(\omega\psi_t + \omega_t)\big), \quad \Gamma^\rho_{\phi\phi} = e^{-2\gamma}\big(\rho(-1+\rho\psi_\rho) - e^{4\psi}\,\omega(\omega\psi_\rho + \omega_\rho)\big)
\end{gathered}$$

The components of Riemann-Christoffel curvature tensor R of type (1,3) are determined as follows:

$$\begin{gathered}
R_{t\rho t\rho} = e^{2\gamma-2\psi}(\gamma_{\rho\rho} - \psi_{\rho\rho} - \gamma_{tt} + \psi_{tt}), \quad R_{t\rho z\phi} = -\tfrac{e^{2\psi}(2\rho\,\omega_\rho\,\psi_t + (1-2\rho\,\psi_\rho)\,\omega_t)}{2\rho} \\
R_{tztz} = \tfrac{e^{6\psi}(\omega_t)^2}{4\rho^2} + e^{2\psi}((\gamma_\rho - \psi_\rho)\psi_\rho + (\gamma_t - 2\psi_t)\psi_t - \psi_{tt}) \\
R_{tzt\phi} = \tfrac{1}{4}e^{2\psi}(2(\gamma_\rho - \psi_\rho)\omega_\rho + 2(\gamma_t - 5\psi_t)\omega_t \\
+\omega(4\gamma_\rho\psi_\rho + \tfrac{e^{4\psi}(\omega_t)^2}{\rho^2} - 4((\psi_\rho)^2 - \gamma_t\psi_t + 2(\psi_t)^2 + \psi_{tt})) - 2\omega_{tt}) \\
R_{tz\rho z} = \tfrac{e^{6\psi}\omega_\rho\,\omega_t}{4\rho^2} + e^{2\psi}(\psi_\rho(\gamma_t - 3\psi_t) + \gamma_\rho\psi_t - \psi_{t\rho}) \\
R_{tz\rho\phi} = \tfrac{1}{4}e^{2\psi}(2\omega_\rho(\gamma_t - 4\psi_t) + 2(\gamma_\rho - 2\psi_\rho)\omega_t \\
+\omega(4\psi_\rho(\gamma_t - 3\psi_t) + 4\gamma_\rho\psi_t + \tfrac{e^{4\psi}\omega_\rho\omega_t}{\rho^2} - 4\psi_{t\rho}) - 2\omega_{t\rho}) \\
R_{t\phi t\phi} = \tfrac{1}{4}e^{-2\psi}(\tfrac{e^{8\psi}\omega^2(\omega_t)^2}{\rho^2} - 4\rho(\psi_\rho + \gamma_\rho(-1 + \rho\psi_\rho) - \rho((\psi_\rho)^2 - \gamma_t\psi_t + \psi_{tt})) \\
-e^{4\psi}(3(\omega_t)^2 + 4\omega^2(-(\gamma_\rho\psi_\rho) + (\psi_\rho)^2 - \gamma_t\psi_t + 2(\psi_t)^2 + \psi_{tt}) \\
+4\omega(-(\gamma_\rho) + \psi_\rho)\omega_\rho - ((\gamma_t - 5\psi_t)\omega_t) + \omega_{tt}))) \\
R_{t\phi\rho z} = \tfrac{1}{4\rho^2}(e^{6\psi}\omega\omega_\rho\omega_t + 2e^{2\psi}\rho(\rho\omega_\rho(\gamma_t - 2\psi_t) \\
+(1 + \rho\gamma_\rho - 4\rho\psi_\rho)\omega_t + 2\rho\omega(\psi_\rho(\gamma_t - 3\psi_t) + \gamma_\rho\psi_t - \psi_{t\rho}) - \rho\omega_{t\rho})) \\
R_{t\phi\rho\phi} = \tfrac{1}{4\rho^2}e^{-2\psi}(e^{8\psi}\omega^2\omega_\rho\omega_t + 4\rho^3((1 - \rho\psi_\rho)\gamma_t + \rho(-(\gamma_\rho) + \psi_\rho)\psi_t + \psi_{t\rho}) \\
+e^{4\psi}\rho(-3\rho\omega_\rho\omega_t + 4\rho\omega^2(\psi_\rho(\gamma_t - 3\psi_t) + \gamma_\rho\psi_t - \psi_{t\rho}) \\
+2\omega(2\rho\omega_\rho(\gamma_t - 3\psi_t) + (1 + 2\rho\gamma_\rho - 6\rho\psi_\rho)\omega_t - 2\rho\omega_{t\rho}))) \\
R_{\rho z\rho z} = \tfrac{e^{6\psi}(\omega_\rho)^2}{4\rho^2} + e^{2\psi}((\gamma_\rho - 2\psi_\rho)\psi_\rho - \psi_{\rho\rho} + (\gamma_t - \psi_t)\psi_t) \\
R_{\rho z\rho\phi} = \tfrac{1}{4\rho^2}(e^{6\psi}\omega(\omega_\rho)^2 + 2e^{2\psi}\rho((1 + \rho\gamma_\rho - 5\rho\psi_\rho)\omega_\rho - \rho\omega_{\rho\rho} \\
+2\rho\omega((\gamma_\rho - 2\psi_\rho)\psi_\rho - \psi_{\rho\rho} + (\gamma_t - \psi_t)\psi_t) + \rho(\gamma_t - \psi_t)\omega_t)) \\
R_{\rho\phi\rho\phi} = \tfrac{1}{4\rho^2}e^{-2\psi}(e^{8\psi}\omega^2(\omega_\rho)^2 + 4\rho^3(\psi_\rho + \gamma_\rho(1 - \rho\psi_\rho) + \rho(\psi_{\rho\rho} - \gamma_t\psi_t + (\psi_t)^2)) \\
+e^{4\psi}\rho(-3\rho(\omega_\rho)^2 + 4\rho\omega^2((\gamma_\rho - 2\psi_\rho)\psi_\rho - \psi_{\rho\rho} + (\gamma_t - \psi_t)\psi_t) \\
+4\omega((1 + \rho\gamma_\rho - 5\rho\psi_\rho)\omega_\rho - \rho\omega_{\rho\rho} + \rho(\gamma_t - \psi_t)\omega_t)))) \\
R_{z\phi z\phi} = \tfrac{1}{4}e^{-2\gamma+2\psi}(4\rho(\psi_\rho(-1 + \rho\psi_\rho) - \rho(\psi_t)^2) + e^{4\psi}((\omega_\rho)^2 - (\omega_t)^2))
\end{gathered}$$

By taking contraction of R, the components of the Ricci tensor Ric are calculated as follows:

$$\begin{gathered}
Ric_{tt} = \tfrac{e^{4\psi}(\omega_t)^2 + 2\rho(-\gamma_\rho + \psi_\rho + \rho(-\gamma_{\rho\rho} + \psi_{\rho\rho} + 2(\psi_t)^2 + \gamma_{tt} - \psi_{tt}))}{2\rho^2} \\
Ric_{t\rho} = 2\psi_\rho\,\psi_t + \tfrac{-2\rho\gamma_t + e^{4\psi}\omega_\rho\,\omega_t}{2\rho^2} \\
Ric_\rho = \tfrac{-2\rho\gamma_\rho + 2\rho\psi_\rho\,(-1 + 2\rho\psi_\rho) + e^{4\psi}(\omega_\rho)^2}{2\rho^2} + \gamma_{\rho\rho} - \psi_{\rho\rho} - \gamma_{tt} + \psi_{tt} \\
Ric_{zz} = \tfrac{e^{-2\gamma+4\psi}(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt})))}{2\rho^2} \\
Ric_{z\phi} = \tfrac{1}{2\rho^2}e^{-2\gamma+4\psi}g(\omega(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt}))) \\
+\rho((-1 + 4\rho\psi_\rho)\omega_\rho + \rho(\omega_{\rho\rho} - 4\psi_t\,\omega_t - \omega_{tt}))g) \\
Ric_{\phi\phi} = \tfrac{1}{2\rho^2}e^{-2\gamma}g(-e^{8\psi}\omega^2((\omega_\rho)^2 - (\omega_t)^2) - 2\rho^3(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt})) \\
+e^{4\psi}\rho(\rho((\omega_\rho)^2 - (\omega_t)^2) + 2\omega^2(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt})) \\
+2\omega((-1 + 4\rho\psi_\rho)\omega_\rho + \rho(\omega_{\rho\rho} - 4\psi_t\,\omega_t - \omega_{tt}))))
\end{gathered}$$

Furthermore, in a similar way the scalar curvature $\mathcal{R}$ is computed by

$$\mathcal{R} = \frac{1}{2\rho^2}\,e^{-2\gamma+2\psi}(e^{4\psi}((\omega_\rho)^2 - (\omega_t)^2) + 4\rho(-\psi_\rho + \rho(\psi_\rho)^2 + \rho(\gamma_{\rho\rho} - \psi_{\rho\rho} - (\psi_t)^2 - \gamma_{tt} + \psi_{tt}))).$$

Now, for any real-valued smooth function f, the components of the Bakry-Émery Ricci tensor Ric_f

are computed as follows:

$$\begin{gathered}
(Ric_f)_{\rho z} = f_{\rho z}, \quad (Ric_f)_{\rho\phi} = f_{\rho\phi}, \\
(Ric_f)_{tt} = -\frac{(\gamma_\rho)}{\rho} + \frac{(\psi_\rho)}{\rho} - (\gamma_{\rho\rho}) + (\psi_{\rho\rho}) + 2(\psi_t)^2 + \frac{e^{4\psi}(\omega_t)^2}{2\rho^2} + (\gamma_{tt}) - (\psi_{tt}) + (f_{tt}) \\
(Ric_f)_{t\rho} = 2(\psi_\rho)(\psi_t) + \frac{-2\rho\gamma_t + e^{4\psi}(\omega_\rho)(\omega_t)}{2\rho^2} + f_{t\rho}, \quad (Ric_f)_{tz} = f_{tz}, \quad (Ric_f)_{t\phi} = f_{t\phi} \\
(Ric_f)_{\rho\rho} = \frac{-2\rho\gamma_\rho + 2\rho(\psi_\rho)(-1+2\rho(\psi_\rho)) + e^{4\psi}(\omega_\rho)^2}{2\rho^2} + \gamma_{\rho\rho} - \psi_{\rho\rho} - \gamma_{tt} + \psi_{tt} + f_{\rho\rho} \\
(Ric_f)_{zz} = \frac{e^{-2\gamma+4\psi}\left(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt}))\right)}{2\rho^2} + f_{zz} \\
(Ric_f)_{z\phi} = \frac{1}{2\rho^2} e^{-2\gamma+4\psi}(\omega(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt}))) \\
+\rho((-1 + 4\rho\psi_\rho)(\omega_\rho) + \rho(\omega_{\rho\rho} - 4\psi_t\omega_t - \omega_{tt})) + f_{z\phi}) \\
(Ric_f)_{\phi\phi} = \frac{1}{2\rho^2} e^{-2\gamma}(-e^{8\psi}\omega^2((\omega_\rho)^2 - (\omega_t)^2) - 2\rho^3(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt})) + e^{4\psi}\rho(\rho((\omega_\rho)^2 - (\omega_t)^2) \\
+2\omega^2(\psi_\rho + \rho(\psi_{\rho\rho} - \psi_{tt})) + 2\omega((-1 + 4\rho\psi_\rho)(\omega_\rho) + \rho(\omega_{\rho\rho} - 4\psi_t\omega_t - \omega_{tt}))) + f_{\phi\phi})
\end{gathered}$$

Again, for any real number m and any real-valued smooth function f, the components of the $m-$Bakry-Émery Ricci tensor Ric_f^m are determined by

$$\begin{gathered}
(Ric_f^m)_{tt} = -\frac{(\gamma_\rho - \psi_\rho)}{\rho} - \gamma_{\rho\rho} + \psi_{\rho\rho} + 2\psi_t^2 + \frac{e^{4\psi}\omega_t^2}{2\rho^2} + \gamma_{tt} - \psi_{tt} - \frac{f_t^2}{m} + f_{tt} \\
(Ric_f^m)_{t\rho} = 2(\psi_\rho)(\psi_t) + \frac{-2\rho(\gamma_t) + e^{4\psi}(\omega_\rho)(\omega_t)}{2\rho^2} - \frac{(f_\rho)(f_t)}{m} + (f_{t\rho}) \\
(Ric_f^m)_{tz} = -\frac{(f_z)(f_t)}{m} + (f_{tz}), \quad (Ric_f^m)_{t\phi} = -\frac{(f_\phi)(f_t)}{m} + (f_{t\phi}) \\
(Ric_f^m)_{\rho\rho} = \frac{-2\rho(\gamma_\rho) + 2\rho(\psi_\rho)(-1+2\rho(\psi_\rho)) + e^{4\psi(t,\rho)}(\omega_\rho)^2}{2\rho^2} + (\gamma_{\rho\rho}) - (\psi_{\rho\rho}) - (\gamma_{tt}) + (\psi_{tt}) - \frac{(f_\rho)^2}{m} + (f_{\rho\rho}) \\
(Ric_f^m)_{\rho z} = -\frac{(f_z)(f_\rho)}{m} + (f_{\rho z}), \quad (Ric_f^m)_{\rho\phi} = -\frac{(f_\phi)(f_\rho)}{m} + (f_{\rho\phi}) \\
(Ric_f^m)_{zz} = \frac{e^{-2\gamma+4\psi}(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho((\psi_\rho) + \rho((\psi_{\rho\rho}) - (\psi_{tt}))))}{2\rho^2} - \frac{(f_z)^2}{m} + (f_{zz}) \\
(Ric_f^m)_{z\phi} = \frac{1}{2\rho^2} e^{-2\gamma+4\psi} g(\omega(e^{4\psi}(-(\omega_\rho)^2 + (\omega_t)^2) + 2\rho((\psi_\rho) + \rho((\psi_{\rho\rho}) - (\psi_{tt})))) \\
+\rho((-1 + 4\rho(\psi_\rho))(\omega_\rho) + \rho((\omega_{\rho\rho}) - 4(\psi_t)(\omega_t) - (\omega_{tt}))) - \frac{(f_\phi)(f_z)}{m} + (f_{z\phi})g) \\
(Ric_f^m)_{\phi\phi} = \frac{1}{2\rho^2} e^{-2\gamma} g(-e^{8\psi}\omega^2((\omega_\rho)^2 - (\omega_t)^2) - 2\rho^3((\psi_\rho) + \rho((\psi_{\rho\rho}) - (\psi_{tt}))) \\
+e^{4\psi}\rho(\rho((\omega_\rho)^2 - (\omega_t)^2) + 2\omega^2((\psi_\rho) + \rho((\psi_{\rho\rho}) - (\psi_{tt}))) \\
+2\omega((-1 + 4\rho(\psi_\rho))(\omega_\rho) + \rho((\omega_{\rho\rho}) - 4(\psi_t)(\omega_t) - (\omega_{tt}))))g) - \frac{(f_\phi)^2}{m} + (f_{\phi\phi})
\end{gathered}$$

The above-computed tensor components lead to the following:

Theorem 4.1 *If the functions ω, γ and ψ of the Halilsoy metric admits the following characteristics :*

1. *ω is a function of $(t + \rho)$,*
2. *γ and ψ are time (t) independent,*
3. *γ is linear and*
4. *$\frac{d^2\psi}{d\rho^2} - \left(\frac{d\psi}{d\rho}\right)^2 + \frac{1}{\rho}\frac{d\psi}{d\rho} = 0,$*

then the scalar curvature $\mathcal{R}$ vanishes.

We note that the condition (4) of Theorem 4.1 is basically an ordinary differential equation whose solution can be obtained by

$$\psi(\rho) = c2 - ln(c1 - ln(\rho)).$$

Also, the graph of the family of curves is plotted to visualize such solution for different values of $c1, c2$ in below:

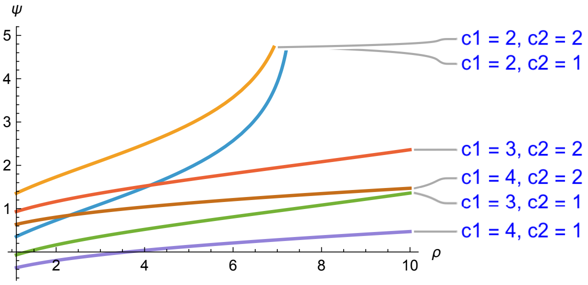

Figure 1: Graph of the solution $\psi(\rho) = c2 - ln(c1 - ln(\rho))$ of $\frac{d^2\psi}{d\rho^2} - \left(\frac{d\psi}{d\rho}\right)^2 + \frac{1}{x}\frac{d\psi}{d\rho} = 0$, for the constants $(c1, c2) \in \{2, 3, 4\} \times \{1, 2\}$.

As the scalar curvature $\mathcal{R}$ is the trace of the operator Ric by $\mathcal{R} = g^{ij}\,\mathrm{Ric}_{ij}$, we analogously define $\mathcal{R}_f$ and $\mathcal{R}_f^m$ from Ric_f and Ric_f^m respectively by

$$\mathcal{R}_f = g^{ij}\,(Ric_f)_{ij} \quad \text{and} \quad \mathcal{R}_f^m = g^{ij}\,(Ric_f^m)_{ij}.$$

In this article, we call the tensor $\mathcal{R}_f$ by *Bakry-Émery scalar curvature* and the tensor $\mathcal{R}_f^m$ by *$m-$Bakry-Émery scalar curvature*. The tensor $\mathcal{R}_f$ of Halilsoy metric are determined as follows:

$$\begin{aligned} \mathcal{R}_f &= \frac{1}{2\rho^2} e^{-2(\gamma+\psi)} \Big(e^{8\psi}\big(\omega_\rho^2 - \omega_t^2\big) + 2e^{2\gamma}\rho^2 f_{zz} \\ &+ 2e^{4\psi}\Big(e^{2\gamma}\big(f_{\phi\phi} + \omega\big(-2f_{z\phi} + \omega f_{zz}\big)\big) + \rho\big(-2\psi_\rho + 2\rho\psi_\rho^2 \\ &+ \rho\big(2\gamma_{\rho\rho} - 2\psi_{\rho\rho} - 2\psi_t^2 - 2\gamma_{tt} + 2\psi_{tt} + f_{\rho\rho} - f_{tt}\big)\big)\Big)\Big) \end{aligned}$$

This expression leads to the following:

Theorem 4.2 *If the Halilsoy metric admits the following characteristics :*

1. *ω is a function of $(t + \rho)$,*

2. *$\gamma_t = 0$ and $\gamma_{\rho\rho} = 0$,*

3. *ψ is is constant function,*

4. *$f_{zz} = f_{z\phi} = f_{\phi\phi} = 0$ and $f_{tt} = f_{\rho\rho}$,*

then the spacetime is flat with respect to Bakry-Émery scalar curvature, i.e. $\mathcal{R}_f = 0$.

We note that the solution of the second condition $f_{tt} = f_{\rho\rho}$ of (4) of Theorem 4.2 is

$$f = d_1(t - \rho) + d_2(t + \rho),$$

where d_1, d_2 is the arbitrary smooth functions. Also, the graph of such solution is given as follows:

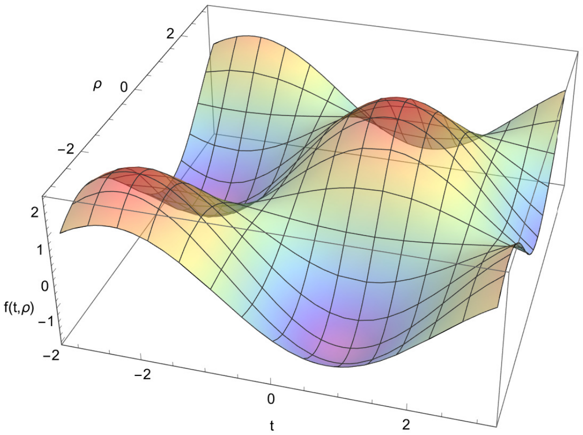

Figure 2: Graph of the solution $f(t,\rho) = \sin(t+\rho) + \cos(t-\rho)$ of the pde $f_{tt} = f_{\rho\rho}$ with the initial conditions $f(t,0) = \sin t + \cos t$, $f_\rho(t,0) = \cos t - \sin t$.

Theorem 4.3 *The $m-$Bakry-Émery scalar curvature tensor $\mathcal{R}_f^m$ of Halilsoy metric is determined by*

$$\begin{aligned}
\mathcal{R}_f^m &= \frac{1}{2m\rho^2} e^{-2(\gamma+\psi)} \Bigg(e^{8\psi}\, m(\omega_\rho)^2 - (\omega_t)^2 - 2e^{2\gamma}\rho^2 \big[(f_z)^2 - m f_{zz}\big] \\
&+ 2e^{4\psi} \Big(e^{2\gamma} \big[-(f_\phi)^2 + m f_{\phi\phi}\big] + 2\omega f_\phi f_z + \omega \Big(-2m f_{z\phi} + \omega \big[-(f_z)^2 + m f_{zz}\big] \Big) \Big) \\
&+ \rho \Big(-2m\psi_\rho + 2m\rho\psi_\rho^2 + \rho \Big(-f_\rho^2 + f_t^2 \\
&+ m \Big(2\gamma_{\rho\rho} - 2\psi_{\rho\rho} - 2\psi_t^2 - 2\gamma_{tt} + 2\psi_{tt} + f_{\rho\rho} - f_{tt} \Big) \Big) \Big) \Bigg).
\end{aligned}$$

5. Conclusion

The results reveal that the curvature characteristics of the Halilsoy spacetime depend sensitively on both the metric parameters and the choice of the potential function. Furthermore, the derived flatness conditions with respect to $\mathcal{R}_f$ and $\mathcal{R}_f^m$ establish when the Bakry-Émery geometry of this spacetime becomes curvature-free. These findings highlight the geometric richness of the Halilsoy metric and emphasize the usefulness of Bakry-Émery curvature in studying generalized Einstein metrics and standing gravitational wave spacetimes.

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