Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 3** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.79768

Numerical Approximation of the Timoshenko System with Temperature and Microtemperature Effects in the Absence of Thermal Conductivity

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ABSTRACT: This study presents a numerical investigation of a thermoelastic Timoshenko system where dissipation arises exclusively from microtemperature effects, with thermal diffusion neglected. The primary objective is to analyze the system's energy evolution and exponential decay properties. We start by formulating the problem variationally, employing transformed derivatives to derive a coupled system of four first-order variational equations. A fully discrete numerical scheme is then proposed, and its discrete stability is rigorously established. We also derive a priori error estimates for the method. To support our theoretical analysis, numerical experiments are carried out, confirming the expected decay behavior and accuracy of the solution.

Key Words: Timoshenko beam model, dissipation via microtemperature effects, numerical simulation and analysis, finite element approach, discrete exponential energy decay.

Contents

1. Introduction

Over the past few decades, a wide range of dynamic equations have been utilized as mathematical models to represent engineering phenomena. Since many physical processes are governed by partial differential equations (PDEs) that lack closed-form solutions, numerical methods have become essential tools for analysis. Techniques such as the finite element method (FEM) and the finite difference method (FDM) are commonly employed to obtain approximate solutions. These computational approaches enable the simulation and understanding of complex physical behaviors that are otherwise difficult to analyze analytically.

The numerical analysis of the Timoshenko system has garnered significant interest among researchers due to its complexity and critical applications in engineering and material sciences. This advanced system extends the classical beam theory by accounting for shear deformation and rotational inertia effects, thereby improving the accuracy of predictive models. For further reading, we refer the reader to seminal works [2,5,17,18], which provide comprehensive insights and methodologies for tackling various challenges associated with the Timoshenko system.

In recent years, there has been significant research interest in Timoshenko-type systems, with numerous researchers exploring the topic extensively. This growing attention focuses on understanding the asymptotic behavior of these systems under various damping mechanisms, including frictional, structural, and viscoelastic damping, both linear and nonlinear. Studies have been conducted with and without coupling these systems with a heat equation (see, for instance, [3,8,15,19,21]). Moreover, contemporary studies have considered linear Timoshenko systems with additional complexities like memory, delay, and second sound effects (see, e.g., [4,6,10]).

2020 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted October 27, 2025. Published January 22, 2026

Alternatively, in contrast to the Timoshenko system, the study of the asymptotic behavior of porous elastic systems with microtemperature effects has also attracted considerable attention. Noteworthy results have been established in this domain. For example, a thermoelastic system was recently examined by Dridi and Djebabla [11] under the influence of temperature and micro-temperatures. They specifically considered the following system:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, & \text{in } (0,1)\times(0,\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t = k\theta_{xx} - \gamma u_{tx} - l\varphi_t - k_1 w_x, & \text{in } (0,1)\times(0,\infty), \\ \delta w_t = k_4 w_{xx} - d\varphi_{tx} - k_2 w - k_3\theta_x, & \text{in } (0,1)\times(0,\infty). \end{cases}$$

Using the multipliers method, the authors proved exponential stability in the case of zero thermal conductivity without any conditions placed on the system's coefficients.

Understanding how the energy behavior of solutions in dynamic models depends on system parameters is crucial. Recently, there has been considerable focus on the well-posedness and stability of Timoshenko systems. Notably, Saci and Djebabla in [16] improved upon previous results, showing that dissipation solely from micro-temperatures can achieve exponential stability, contingent on specific parameters. They established that stability is attained if:

$$\beta = 0,$$

$$\chi = \chi_0 - \frac{\gamma^2}{c\rho} = 0,$$

where

$$\chi_0 = \frac{\mu}{\rho} - \frac{\delta}{J}.$$

Meradji et al., in their work [12], explored system (1.1) which is a Timoshenko system characterized by microtemperature dissipation without thermal diffusivity, described by four linear partial differential equations. These equations include two hyperbolic equations and two parabolic equations representing temperature and microtemperature differences. They used semigroup theory to establish the existence and uniqueness of solutions and proved the energy decay property using the multipliers method.

In this paper, we analyze the following model [12]:

$$\begin{cases} \rho_1 u_{tt} - k\left(u_x+\varphi\right)_x + \gamma\theta_x = 0, & \text{in } \left(0,1\right)\times\left(0,\infty\right), \\ \rho_2\varphi_{tt} - b\varphi_{xx} + k\left(u_x+\varphi\right) + \gamma w_x - \gamma\theta = 0, & \text{in } \left(0,1\right)\times\left(0,\infty\right), \\ \rho_3\theta_t + k_1 w_x + \gamma(u_x+\varphi)_t = 0, & \text{in } \left(0,1\right)\times\left(0,\infty\right), \\ w_t - k_2 w_{xx} + k_3 w + k_1\theta_x + \gamma\varphi_{tx} = 0, & \text{in } \left(0,1\right)\times\left(0,\infty\right), \end{cases} \tag{1.1}$$

with the initial and boundary conditions

$$\begin{cases} u\left(x,0\right) = u_0\left(x\right),\ u_t\left(x,0\right) = u_1\left(x\right),\ \theta\left(x,0\right) = \theta_0\left(x\right), & \text{for } x\in\left(0,1\right), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right),\ w\left(x,0\right) = w_0\left(x\right), & \text{for } x\in\left(0,1\right), \\ u_x\left(0,t\right) = u_x\left(1,t\right) = \varphi\left(0,t\right) = \varphi\left(1,t\right) = 0, & \forall t\geq 0, \\ \theta\left(0,t\right) = \theta\left(1,t\right) = w_x\left(0,t\right) = w_x\left(1,t\right) = 0, & \forall t\geq 0. \end{cases} \tag{1.2}$$

The constants ρ_1, ρ_2, ρ_3, b, k, k_1, k_2, k_3, and γ are positive and represent constitutive coefficients with well-known physical significance. The variables u, φ, θ, and w denote the displacement of the solid elastic material, the volume fraction, the temperature difference, and the microtemperature, respectively.

The initial data $u_0, u_1, \varphi_0, \varphi_1, w_0, \theta_0$ are given functions. The unknowns of the system are represented by the variables:

$$(\varphi, \psi, \theta, w) : (0,1)\times(0,\infty) \to \mathbb{R}^4.$$

2. Energy dissipation law

In this section, we summarize the results of exponential stability obtained in [12] for the problem described by equations (1.1)-(1.2). The energy associated with the solution of the system (1.1)-(1.2) is given by:

$$\mathcal{E}(t)=\frac{1}{2}\int_0^1(\rho_1 u_t^2+\rho_2\varphi_t^2+\rho_3\theta^2+w^2+b\varphi_x^2+k(u_x+\varphi)^2)dx. \tag{2.1}$$

Proposition 2.1 *Let (u,φ,θ,w) be a solution of (1.1) − (1.2). The total energy $E(t)$ of the system (1.1)-(1.2) satisfies*

$$\frac{d\mathcal{E}}{dt}=-k_2\int_0^1 w_x^2dx-k_3\int_0^1 w^2dx\le 0,\quad \forall t\ge 0.$$

The subsequent proof is an extended version of the one proposed in [12].

Proof: Multiplying equation $(1.1)_1$ by u_t and integrating by parts over $(0,1)$ and using the boundary conditions, we have

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 u_t^2dx+k\int_0^1(u_x+\varphi)u_{tx}dx-\gamma\int_0^1\theta u_{tx}dx=0. \tag{2.2}$$

Similarly, multiplying the equation $(1.1)_2$ by φ_t, the equation $(1.1)_3$ by θ, and $(1.1)_4$ by w we get

$$\frac{\rho_2}{2}\frac{d}{dt}\int_0^1\varphi_t^2dx+\frac{b}{2}\frac{d}{dt}\int_0^1\varphi_x^2dx+k\int_0^1(u_x+\varphi)\varphi_tdx+\gamma\int_0^1 w_x\varphi_tdx-\gamma\int_0^1\theta\varphi_tdx=0, \tag{2.3}$$

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1\theta^2dx+k_1\int_0^1 w_x\theta dx+\gamma\int_0^1(u_x+\varphi)_t\theta dx=0, \tag{2.4}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2dx+k_2\int_0^1 w_x^2dx+k_3\int_0^1 w^2dx-k_1\int_0^1 w_x\theta dx-\gamma\int_0^1\varphi_tw_xdx=0. \tag{2.5}$$

By summing (2.2) − (2.5), we find

$$\begin{aligned}&\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 u_t^2dx+\frac{k}{2}\frac{d}{dt}\int_0^1(u_x+\varphi)^2dx+\frac{\rho_2}{2}\frac{d}{dt}\int_0^1\varphi_t^2dx+\frac{b}{2}\frac{d}{dt}\int_0^1\varphi_x^2dx\\&+\frac{\rho_3}{2}\frac{d}{dt}\int_0^1\theta^2dx+\frac{1}{2}\frac{d}{dt}\int_0^1 w^2dx=-k_2\int_0^1 w_x^2dx-k_3\int_0^1 w^2dx.\end{aligned} \tag{2.6}$$

Finally,

$$\frac{d\mathcal{E}}{dt}=-k_2\int_0^1 w_x^2dx-k_3\int_0^1 w^2dx. \tag{2.7}$$

□

Theorem 2.1 *Let (u,φ,θ,w) be a solution of (1.1) − (1.2) and that the coefficients of the system satisfy the condition*

$$\nu=0,$$

where

$$\nu=b\rho_1-k\rho_2-\frac{\rho_2\gamma^2}{\rho_3}.$$

Then, (u,φ,θ,w) decays exponentially, i.e., there exist two positive constants λ_1,λ_2 such that

$$\mathcal{E}(t)\le\lambda_1\mathcal{E}(0)\exp(-\lambda_2 t),\quad\forall t\ge 0. \tag{2.8}$$

Proof: See Ref. [12]. □

3. Numerical approximation

In this section, we propose a finite element approximation to solve system (1.1) with initial conditions and boundary conditions (1.2).

We define the following functions: $\widetilde{u} = u_t$ and $\widetilde{\varphi} = \varphi_t$. We can then rewrite the system described in (1.1) as follows:

$$\begin{cases} \rho_1 \widetilde{u}_t - k\left(u_x + \varphi\right)_x + \gamma \theta_x = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_2 \widetilde{\varphi}_t - b\varphi_{xx} + k\left(u_x + \varphi\right) + \gamma w_x - \gamma \theta = 0, & \text{in } (0,1) \times (0,\infty), \\ \rho_3 \theta_t + k_1 w_x + \gamma\left(\widetilde{u}_x + \widetilde{\varphi}\right) = 0, & \text{in } (0,1) \times (0,\infty), \\ w_t - k_2 w_{xx} + k_3 w + k_1 \theta_x + \gamma \widetilde{\varphi}_x = 0, & \text{in } (0,1) \times (0,\infty). \end{cases} \tag{3.1}$$

We introduce an implicit Euler-type scheme utilizing finite differences for time discretization and finite elements for spatial discretization. We analyze the behavior of the discrete energy and demonstrate that this energy decays over time.

To derive the variational formulation of Problem (3.1) with boundary conditions (1.2), we define an appropriate function space and denote the scalar product in the space $L^2(0,1)$ by $(\cdot,\cdot)$, with the corresponding norm denoted by $\|\cdot\|$.

To obtain the weak form associated with system (3.1), we multiply the equations by test functions $\xi, \alpha \in H^1(0,1)$,$\chi, \eta \in H_0^1(0,1)$ and integrate by parts:

$$\begin{cases} \rho_1\left(\tilde{u}_t, \xi\right) + k\left(u_x + \varphi, \xi_x\right) - \gamma\left(\theta, \xi_x\right) = 0, \\ \rho_2\left(\tilde{\varphi}_t, \chi\right) + b\left(\varphi_x, \chi_x\right) + k\left(u_x + \varphi, \chi\right) - \gamma\left(w, \chi_x\right) - \gamma\left(\theta, \chi\right) = 0, \\ \rho_3\left(\theta_t, \eta\right) - k_1\left(w, \eta_x\right) + \gamma\left(\widetilde{u}_x + \widetilde{\varphi}, \eta\right) = 0, \\ \left(w_t, \alpha\right) + k_2\left(w_x, \alpha_x\right) + k_3\left(w, \alpha\right) + k_1\left(\theta_x, \alpha\right) + \gamma\left(\tilde{\varphi}_x, \alpha\right) = 0. \end{cases} \tag{3.2}$$

We consider a nonnegative integer J and define $h = \frac{1}{J}$ as the subdivision size for the interval $(0,1)$. This results in the points $0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = 1$, where $x_j = jh$ for $j = 0, \ldots, J$. We introduce the space

$$S^h = \left\{ u_h \in C\left[0,1\right] \; ; \; u^h_{|[x_j,x_{j+1}]} \in P_1^h\left([x_j, x_{j+1}]\right) \; j = 0, \ldots, J-1 \right\},$$

$$V^h = H^1(0,1) \cap S^h,$$

and

$$V_0^h = H_0^1(0,1) \cap S^h,$$

where the set S^h consists of functions u_h that are continuous on $[0,1]$ and piecewise linear, specifically linear on each subinterval $[x_j, x_{j+1}]$. The notation $P_1^h([x_j, x_{j+1}])$ denotes the space of polynomials of degree at most one defined on the subinterval $[x_j, x_{j+1}]$).

Given a final time T and a positive integer N, we define the time step $\Delta t = \frac{T}{N}$ and the discrete time levels $t_n = n\Delta t$ for $n = 0, \ldots, N$.

The finite element method for solving system (3.2) using the backward Euler scheme involves finding $\widetilde{u}_h^n$, $w_h^n \in V^h$ and $\widetilde{\varphi}_h^n$, $\theta_h^n \in V_0^h$ for each time step $n = 1, \ldots, N$, such that the equations hold for all test functions ξ_h, $\alpha_h \in V^h$ and χ_h, $\eta_h \in V_0^h$

$$\begin{cases} \dfrac{\rho_1}{\Delta t}\left(\tilde{u}_h^n - \tilde{u}_h^{n-1}, \xi_h\right) + k\left(\left(u_{hx}^n + \varphi_h^n\right), \xi_{hx}\right) - \gamma\left(\theta_h^n, \xi_{hx}\right) = 0, \\ \dfrac{\rho_2}{\Delta t}\left(\tilde{\varphi}_h^n - \tilde{\varphi}_h^{n-1}, \chi_h\right) + b\left(\varphi_{hx}^n, \chi_{hx}\right) + k\left(u_{hx}^n + \varphi_h^n, \chi_h\right) - \gamma\left(w_h^n, \chi_{hx}\right) \\ -\gamma\left(\theta_h^n, \chi_h\right) = 0, \\ \dfrac{\rho_3}{\Delta t}\left(\theta_h^n - \theta_h^{n-1}, \eta_h\right) - k_1\left(w_h^n, \eta_{hx}\right) + \gamma\left(\widetilde{u}_{hx}^n + \widetilde{\varphi}_h^n, \eta_h\right) = 0, \\ \dfrac{1}{\Delta t}\left(w_h^n - w_h^{n-1}, \alpha_h\right) + k_2\left(w_{hx}^n, \alpha_{hx}\right) + k_3\left(w_h^n, \alpha_h\right) + k_1\left(\theta_{hx}^n, \alpha_h\right) \\ +\gamma\left(\widetilde{\varphi}_{hx}^n, \alpha_h\right) = 0, \end{cases} \tag{3.3}$$

where, $\widetilde{u}_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}$ and $\widetilde{\varphi}_h^n = \frac{\varphi_h^n - \varphi_h^{n-1}}{\Delta t}$.
are approximations to $\tilde{u}^n = u_t(t_n)$ and $\tilde{\varphi}^n = \varphi_t(t_n)$ respectively. Here, $u_h^0, \widetilde{u}_h^0, \varphi_h^0, \widetilde{\varphi}_h^0, \theta_h^0$ and w_h^0 are given approximations to the initial conditions $u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0$ respectively.

The following result presents a discrete version of the energy decay property that is satisfied by the solution of the system (1.1).

Theorem 3.1 *The discrete energy can be defined as:*

$$\mathcal{E}_h^n = \frac{1}{2}\left[\rho_1 \|\widetilde{u}_h^n\|^2 + \rho_2 \|\widetilde{\varphi}_h^n\|^2 + \rho_3\|\theta_h^n\|^2 + \|w_h^n\|^2 \; + k\,\|u_{hx}^n + \varphi_h^n\|^2 + b\,\|\varphi_{hx}^n\|^2\right]. \tag{3.4}$$

Then, the decay property

$$\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \leq 0. \tag{3.5}$$

This holds for $n = 1, 2, \ldots, N$, where $\|\cdot\|$ denotes the norm in the space $L^2(0,1)$.

Proof: Taking

$$\xi_h = \widetilde{u}_h^n, \quad \chi_h = \widetilde{\varphi}_h^n, \quad \eta_h = \theta_h, \quad \text{and} \quad \alpha_h = w_h^n, \tag{3.6}$$

in system (3.3), we have

$$\frac{\rho_1}{\Delta t}\left(\tilde{u}_h^n - \tilde{u}_h^{n-1}, \tilde{u}_h^n\right) + k\left(u_{hx}^n + \varphi_h^n, \tilde{u}_{hx}^n\right) - \gamma\left(\theta_h^n, \tilde{u}_{hx}^n\right) = 0, \tag{3.7}$$

$$\frac{\rho_2}{\Delta t}\left(\tilde{\varphi}_h^n - \tilde{\varphi}_h^{n-1}, \tilde{\varphi}_h^n\right) + b\left(\varphi_{hx}^n, \tilde{\varphi}_{hx}^n\right) + k\left(u_{hx}^n + \varphi_h^n, \tilde{\varphi}_h^n\right) - \gamma\left(w_h^n, \tilde{\varphi}_{hx}^n\right) - \gamma\left(\theta_h^n, \tilde{\varphi}_h^n\right) = 0, \tag{3.8}$$

$$\frac{\rho_3}{\Delta t}\left(\theta_h^n - \theta_h^{n-1}, \theta_h^n\right) - k_1\left(w_h^n, \theta_{hx}^n\right) + \gamma\left(\widetilde{u}_{hx}^n + \widetilde{\varphi}_h^n, \theta_h^n\right) = 0, \tag{3.9}$$

and

$$\frac{1}{\Delta t}\left(w_h^n - w_h^{n-1}, w_h^n\right) + k_2\left(w_{hx}^n, w_{hx}^n\right) + k_3\left(w_h^n, w_h^n\right) + k_1\left(\theta_{hx}^n, w_h^n\right) + \gamma\left(\widetilde{\varphi}_{hx}^n, w_h^n\right) = 0. \tag{3.10}$$

By summing equations (3.7) through (3.10), we obtain

$$\begin{aligned}
&\frac{\rho_1}{\Delta t}\left(\tilde{u}_h^n - \tilde{u}_h^{n-1}, \tilde{u}_h^n\right) + k\left(u_{hx}^n + \varphi_h^n, \tilde{u}_{hx}^n + \tilde{\varphi}_h^n\right) + \frac{\rho_2}{\Delta t}\left(\tilde{\varphi}_h^n - \tilde{\varphi}_h^{n-1}, \tilde{\varphi}_h^n\right) \\
&+ b\left(\varphi_{hx}^n, \tilde{\varphi}_{hx}^n\right) + \frac{\rho_3}{\Delta t}\left(\theta_h^n - \theta_h^{n-1}, \theta_h^n\right) + \frac{1}{\Delta t}\left(w_h^n - w_h^{n-1}, w_h^n\right) \\
&+ k_2\left(w_{hx}^n, w_{hx}^n\right) + k_3\left(w_h^n, w_h^n\right) = 0.
\end{aligned} \tag{3.11}$$

Using the following equality:

$$(a - b, a) = \frac{1}{2}\left(\|a - b\|^2 + \|a\|^2 - \|b\|^2\right), \tag{3.12}$$

we have

$$\left(\tilde{u}_h^n - \tilde{u}_h^{n-1}, \tilde{u}_h^n\right) = \frac{1}{2}\left(\|\tilde{u}_h^n - \tilde{u}_h^{n-1}\|^2 + \|\tilde{u}_h^n\|^2 - \|\tilde{u}_h^{n-1}\|^2\right) \tag{3.13}$$

and similarly for

$$\begin{aligned}
\left(\tilde{u}_{hx}^n + \tilde{\varphi}_h^n, u_{hx}^n + \varphi_h^n\right) &= \left(\frac{u_{hx}^n - u_{hx}^{n-1}}{\Delta t} - \frac{\varphi_h^n - \varphi_h^n}{\Delta t}, u_{hx}^n + \varphi_h^n\right) \\
&= \frac{1}{\Delta t}\left(u_{hx}^n - \varphi_h^n - (u_{hx}^{n-1} - \varphi_h^{n-1}), u_{hx}^n + \varphi_h^n\right) \\
&= \frac{1}{2\Delta t}\left(\|u_{hx}^n - \varphi_h^n - (u_{hx}^{n-1} - \varphi_h^{n-1})\|^2\right. \\
&\left.+\|u_{hx}^n - \varphi_h^n\|^2 - \|u_{hx}^{n-1} - \varphi_h^{n-1}\|^2\right),
\end{aligned} \tag{3.14}$$

$$\left(\tilde{\varphi}_h^n-\tilde{\varphi}_h^{n-1},\tilde{\varphi}_h^n\right)=\frac{1}{2}\left(\|\tilde{\varphi}_h^n-\tilde{\varphi}_h^{n-1}\|^2+\|\tilde{\varphi}_h^n\|^2-\|\tilde{\varphi}_h^{n-1}\|^2\right), \tag{3.15}$$

$$\begin{aligned}(\tilde{\varphi}_{hx}^n,\varphi_{hx}^n)&=\left(\frac{\varphi_{hx}^n-\varphi_{hx}^{n-1}}{\Delta t},\varphi_{hx}^n\right)\\&=\frac{1}{\Delta t}\left(\varphi_{hx}^n-\varphi_{hx}^{n-1},\varphi_{hx}^n\right)\\&=\frac{1}{2\Delta t}\left(\|\tilde{\varphi}_{hx}^n-\tilde{\varphi}_{hx}^{n-1}\|^2+\|\tilde{\varphi}_{hx}^n\|^2-\|\tilde{\varphi}_{hx}^{n-1}\|^2\right),\end{aligned} \tag{3.16}$$

$$\left(\theta_h^n-\theta_h^{n-1},\theta_h^n\right)=\frac{1}{2}\left(\|\theta_h^n-\theta_h^{n-1}\|^2+\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2\right), \tag{3.17}$$

$$\left(w_h^n-w_h^{n-1},w_h^n\right)=\frac{1}{2}\left(\|w_h^n-w_h^{n-1}\|^2+\|w_h^n\|^2-\|w_h^{n-1}\|^2\right). \tag{3.18}$$

From the above equations, we deduce that

$$\begin{aligned}&\frac{\rho_1}{2\Delta t}\left(\|\tilde{u}_h^n-\tilde{u}_h^{n-1}\|^2+\|\tilde{u}_h^n\|^2-\|\tilde{u}_h^{n-1}\|^2\right)\\&+\frac{k}{2\Delta t}\left(\|u_{hx}^n-\varphi_h^n-(u_{hx}^{n-1}-\varphi_h^{n-1})\|^2+\|u_{hx}^n-\varphi_h^n\|^2-\|u_{hx}^{n-1}-\varphi_h^{n-1}\|^2\right)\\&+\frac{\rho_2}{2\Delta t}\left(\|\tilde{\varphi}_h^n-\tilde{\varphi}_h^{n-1}\|^2+\|\tilde{\varphi}_h^n\|^2-\|\tilde{\varphi}_h^{n-1}\|^2\right)\\&+\frac{b}{2\Delta t}\left(\|\tilde{\varphi}_{hx}^n-\tilde{\varphi}_{hx}^{n-1}\|^2+\|\tilde{\varphi}_{hx}^n\|^2-\|\tilde{\varphi}_{hx}^{n-1}\|^2\right)\\&+\frac{\rho_3}{2\Delta t}\left(\|\theta_h^n-\theta_h^{n-1}\|^2+\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2\right)\\&+\frac{1}{2\Delta t}\left(\|w_h^n-w_h^{n-1}\|^2+\|w_h^n\|^2-\|w_h^{n-1}\|^2\right)+k_2\|w_{hx}^n\|^2+k_3\|w_h^n\|^2=0.\end{aligned} \tag{3.19}$$

So,

$$\begin{aligned}&\frac{\rho_1}{2\Delta t}\left(\|\tilde{u}_h^n\|^2-\|\tilde{u}_h^{n-1}\|^2\right)+\frac{k}{2\Delta t}\left(\|u_{hx}^n-\varphi_h^n\|^2-\|u_{hx}^{n-1}-\varphi_h^{n-1}\|^2\right)\\&+\frac{\rho_2}{2\Delta t}\left(\|\tilde{\varphi}_h^n\|^2-\|\tilde{\varphi}_h^{n-1}\|^2\right)+\frac{b}{2\Delta t}\left(\|\tilde{\varphi}_{hx}^n\|^2-\|\tilde{\varphi}_{hx}^{n-1}\|^2\right)\\&+\frac{\rho_3}{2\Delta t}\left(\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2\right)+\frac{1}{2\Delta t}\left(\|w_h^n\|^2-\|w_h^{n-1}\|^2\right)+k_2\|w_{hx}^n\|^2+k_3\|w_h^n\|^2\\&\leq 0.\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned}&\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\Delta t}\\&\leq\frac{\rho_1}{2\Delta t}\left(\|\tilde{u}_h^n\|^2-\|\tilde{u}_h^{n-1}\|^2\right)+\frac{k}{2\Delta t}\left(\|u_{hx}^n-\varphi_h^n\|^2-\|u_{hx}^{n-1}-\varphi_h^{n-1}\|^2\right)\\&+\frac{\rho_2}{2\Delta t}\left(\|\tilde{\varphi}_h^n\|^2-\|\tilde{\varphi}_h^{n-1}\|^2\right)+\frac{b}{2\Delta t}\left(\|\tilde{\varphi}_{hx}^n\|^2-\|\tilde{\varphi}_{hx}^{n-1}\|^2\right)\\&+\frac{\rho_3}{2\Delta t}\left(\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2\right)+\frac{1}{2\Delta t}\left(\|w_h^n\|^2-\|w_h^{n-1}\|^2\right)+k_2\|w_{hx}^n\|^2+k_3\|w_h^n\|^2\\&\leq 0.\end{aligned} \tag{3.21}$$

This inequality implies that $\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\Delta t}\leq 0$, indicating that the discrete energy $\mathcal{E}_h$ is decreasing over time. Thus, the theorem is proven by utilizing the definition of the discrete energy. □

As a result, we derive the following stability estimates.

Corollary 3.1 *The discrete solution* $\{\widetilde{u}_h^n, \widetilde{\varphi}_h^n, u_h^n, \varphi_h^n, \theta_h^n, w_h^n\}$, *generated by discrete problem* (3.3), *satisfies*

$$\|\widetilde{u}_h^n\|^2 + \|\widetilde{\varphi}_h^n\|^2 + \|u_{hx}^n + \varphi_h^n\|^2 + \|\varphi_{hx}^n\|^2 + \|\theta_h^n\|^2 + \|w_h^n\|^2 + \Delta t \sum_{i=1}^{n} \left\|w_{hx}^i\right\|^2 + \Delta t \sum_{i=1}^{n} \left\|w_h^i\right\|^2 \leq C.$$

Proof: By summing (3.20) over n, the result follows. □

4. Error analysis: a priori error estimates

In this section, we will establish a priori error estimates for the numerical approximation. These estimates will show that the algorithm converges linearly, given that certain additional regularity conditions are satisfied.

Theorem 4.1 *There exists a positive constant* C, *independent of the discretization parameters* h *and* Δt *such that for all* $\{\xi_h^i,\ \alpha_h^i\}_{i=0}^N \subset V^h$ *and* $\{\chi_h^i,\ \eta_h^i\}_{i=0}^N \subset V_0^h$,

$$\begin{aligned}
&\max_{0\leq n\leq N}\{\|\widetilde{u}^n - \widetilde{u}_h^n\|^2 + \|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\|^2 + \|(u_x^n + \varphi^n) - (u_{hx}^n + \varphi_h^n)\|^2 \\
&+ \|\varphi_x^n - \varphi_{hx}^n\|^2 + \|\theta^n - \theta_h^n\|^2 + \|w^n - w_h^n\|^2\} \\
&\leq C\Delta t \sum_{i=1}^N \Bigg(\|\widetilde{u}_t^n - \delta\widetilde{u}^n\|^2 + \|\widetilde{u}^n - \xi_h\|^2 + \|\widetilde{u}_x^n - \xi_{hx}\|^2 + \|\widetilde{\varphi}_t^n - \delta\widetilde{\varphi}^n\|^2 \\
&+ \|\widetilde{\varphi}^n - \chi_h\|^2 + \|\varphi_{xt}^n - \delta\varphi_x^n\|^2 + \|(u_x^n + \varphi^n)_t - \delta(u_x^n + \varphi^n)\|^2 + \|\theta_t^n - \delta\theta^n\|^2 \\
&+ \|\widetilde{\varphi}_x^n - \chi_{hx}\|^2 + \|\theta_t^n - \delta\theta^n\|^2 + \|\theta^n - \eta_h\|^2 + \|\widetilde{u}_x^n + \widetilde{\varphi}^n - \widetilde{u}_{hx}^n - \widetilde{\varphi}_h^n\|^2 \\
&+ \|\theta^n - \eta_h\|^2 + \|w_t^n - \delta w^n\|^2 + \|w^n - \alpha_h\|^2 + \|w_x^n - w_{hx}^n\|^2 + \|w_x^n - \alpha_{hx}\|^2 \\
&+ \|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\|^2 \Bigg) + \frac{C}{\Delta t} \sum_{i=1}^{N-1} \left\|\widetilde{\varphi}^i - \chi_h^i - \left(\widetilde{\varphi}^{i+1} - \chi_h^{i+1}\right)\right\|^2 \\
&+ \frac{C}{\Delta t} \sum_{i=1}^{N-1} \left\|\theta^i - \eta_h^i - \left(\theta^{i+1} - \eta_h^{i+1}\right)\right\|^2 + \frac{C}{\Delta t} \sum_{i=1}^{N-1} \left\|\widetilde{u}^i - \xi_h^i - \left(\widetilde{u}^{i+1} - \xi_h^{i+1}\right)\right\|^2 \\
&+ \frac{C}{\Delta t} \sum_{i=1}^{N-1} \left\|w^i - \alpha^i - \left(w^{i+1} - \alpha_h^{i+1}\right)\right\|^2 + C \max_{0\leq n\leq N} \left(\|\widetilde{\varphi}^n - \chi_h^n\|^2 + \|\widetilde{u}^n - \xi_h^n\|^2\right. \\
&\left. + \|\theta^n - \eta_h^n\|^2 + \|w^n - \alpha_h^n\|^2\right) + C\Bigg(\left\|\widetilde{u}^0 - \widetilde{u}_h^0\right\|^2 + \left\|\widetilde{\varphi}^0 - \widetilde{\varphi}_h^0\right\|^2 \\
&+ \left\|(u_x^0 + \varphi^0) - (u_{hx}^0 + \varphi_h^0)\right\|^2 + \left\|\varphi_x^0 - \varphi_{hx}^0\right\|^2 + \left\|\theta^0 - \theta_h^0\right\|^2 + \left\|w^0 - w_h^0\right\|^2 \Bigg).
\end{aligned} \tag{4.1}$$

Proof: Recall that $\widetilde{u} = u_t$ and $\widetilde{\varphi} = \varphi_t$. For a continuous function $f(t)$, we denote $f^n = f(t_n)$, and for a sequence $\{f^n\}_{n=1}^N$, we define $\delta f^n = \frac{f^n - f^{n-1}}{\Delta t}$. By subtracting equation $(3.3)_1$ at time $t = t_n$ for $\xi = \xi_h \in V^h$ from $(3.2)_1$, we obtain:

$$\rho_1\left(\widetilde{u}_t^n - \delta\widetilde{u}_h^n, \xi_h\right) + k\left((u_x^n + \varphi^n) - (u_{hx}^n + \varphi_h^n), \xi_{hx}\right) + \gamma\left(\theta_x^n - \theta_{hx}^n, \xi_h\right) = 0, \tag{4.2}$$

Thus, for all $\xi_h \in V^h$, we obtain

$$\begin{aligned}
&\rho_1\left(\widetilde{u}_t^n - \delta\widetilde{u}_h^n, \widetilde{u}^n - \widetilde{u}_h^n\right) + k\left((u_x^n + \varphi^n) - (u_{hx}^n + \varphi_h^n), \widetilde{u}_x^n - \widetilde{u}_{hx}^n\right) \\
&-\gamma\left(\theta^n - \theta_h^n, \widetilde{u}_x^n - \widetilde{u}_{hx}^n\right) \\
= \ &\rho_1\left(\widetilde{u}_t^n - \delta\widetilde{u}_h^n, \widetilde{u}^n - \xi_h\right) + k\left((u_x^n + \varphi^n) - (u_{hx}^n + \varphi_h^n), \widetilde{u}_x^n - \xi_{hx}\right) \\
&-\gamma\left(\theta^n - \theta_h^n, \widetilde{u}_x^n - \xi_{hx}\right).
\end{aligned} \tag{4.3}$$

Additionally, from equations $(3.3)_2$-$(3.3)_4$ and $(3.2)_2$-$(3.2)_4$, we deduce that for all $\chi_h, \eta_h \in V_0^h$ and $\alpha_h \in V^h$, the following holds:

$$\begin{aligned}
&\rho_2\left(\widetilde{\varphi}_t^n - \delta\widetilde{\varphi}_h^n, \widetilde{\varphi}^n - \widetilde{\varphi}_h^n\right) + b\left(\varphi_x^n - \varphi_{hx}^n, \widetilde{\varphi}_x^n - \widetilde{\varphi}_{hx}^n\right) + k\left(u_x^n + \varphi^n - u_{hx}^n - \varphi_h^n, \widetilde{\varphi}^n - \widetilde{\varphi}_h^n\right) \\
&-\gamma\left(w^n - w_h^n, \widetilde{\varphi}_x^n - \widetilde{\varphi}_{hx}^n\right) - \gamma\left(\theta^n - \theta_h^n, \widetilde{\varphi}^n - \widetilde{\varphi}_h^n\right) \\
= \ &\rho_2\left(\widetilde{\varphi}_t^n - \delta\widetilde{\varphi}_h^n, \widetilde{\varphi}^n - \chi_h\right) + b\left(\varphi_x^n - \varphi_{hx}^n, \widetilde{\varphi}_x^n - \chi_{hx}\right) + k\left(u_x^n + \varphi^n - u_{hx}^n - \varphi_h^n, \widetilde{\varphi}^n - \chi_h\right) \\
&-\gamma\left(w^n - w_h^n, \widetilde{\varphi}_x^n - \chi_{hx}\right) - \gamma\left(\theta^n - \theta_h^n, \widetilde{\varphi}^n - \chi_h\right),
\end{aligned} \tag{4.4}$$

$$\begin{array}{rl} & \rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\theta_h^n\right)-k_1\left(w^n-w_h^n,\theta_x^n-\theta_{hx}^n\right)+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_h^n-\widetilde{\varphi}_h^n,\theta^n-\theta_h^n\right)\\ = & \rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)-k_1\left(w^n-w_h^n,\theta_x^n-\eta_{hx}\right)+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n,\theta^n-\eta_h\right),\end{array} \tag{4.5}$$

$$\begin{array}{rl} & \left(w_t^n-\delta tw_h^n,w^n-w_h^n\right)+k_2\left(w_x^n-w_{hx}^n,w_x^n-w_{hx}^n\right)+k_3\left(w^n-w_h^n,w^n-w_h^n\right)\\ & +k_1\left(\theta_x^n-\theta_{hx}^n,w^n-w_h^n\right)+\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-w_h^n\right)\\ = & \left(w_t^n-\delta tw_h^n,w^n-\alpha_h\right)+k_2\left(w_x^n-w_{hx}^n,w_x^n-\alpha_{hx}\right)+k_3\left(w^n-w_h^n,w^n-\alpha_h\right)\\ & +k_1\left(\theta_x^n-\theta_{hx}^n,w^n-\alpha_h\right)+\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-\alpha_h\right).\end{array} \tag{4.6}$$

By adding these last equations and applying some simplifications, we obtain

$$\begin{array}{l} \rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\widetilde{u}_h^n\right)+k\left((u_x^n+\varphi^n)_t-\delta(u_{hx}^n+\varphi_h^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)\\ +\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+b\left(\varphi_{xt}^n-\delta\varphi_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)+\rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\theta_h^n\right)\\ +\left(w_t^n-\delta w_h^n,w^n-w_h^n\right)+k_2\left\|w_x^n-w_{hx}^n\right\|^2+k_3\left\|w^n-w_h^n\right\|^2\\ =\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)+k\left((u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n),\widetilde{u}_x^n-\xi_{hx}\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{u}_x^n-\xi_{hx}\right)\\ +\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)+b\left(\varphi_x^n-\varphi_{hx}^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+k\left(u_x^n+\varphi^n-u_{hx}^n-\varphi_h^n,\widetilde{\varphi}^n-\chi_h\right)\\ -\gamma\left(w^n-w_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+\rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)\\ -k_1\left(w^n-w_h^n,\theta_x^n-\eta_{hx}\right)+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n,\theta^n-\eta_h\right)+\left(w_t^n-\delta tw_h^n,w^n-\alpha_h\right)\\ +k_2\left(w_x^n-w_{hx}^n,w_x^n-\alpha_{hx}\right)+k_3\left(w^n-w_h^n,w^n-\alpha_h\right)+k_1\left(\theta_x^n-\theta_{hx}^n,w^n-\alpha_h\right)\\ +\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-\alpha_h\right).\end{array} \tag{4.7}$$

Applying the equality $(a-b)a=\frac{1}{2}\left((a-b)^2+a^2-b^2\right)$, we derive

$$\begin{array}{l} \left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\widetilde{u}_h^n\right)\\ =\left(\widetilde{u}_t^n-\delta\widetilde{u}^n,\widetilde{u}^n-\widetilde{u}_h^n\right)+\left(\delta\widetilde{u}^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\widetilde{u}_h^n\right)\\ =\left(\widetilde{u}_t^n-\delta\widetilde{u}^n,\widetilde{u}^n-\widetilde{u}_h^n\right)+\frac{1}{2\Delta t}\left(\left\|\widetilde{u}^n-\widetilde{u}_h^n-\left(\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right)\right\|^2+\left\|\widetilde{u}^n-\widetilde{u}_h^n\right\|^2\right.\\ \left.-\left\|\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right\|^2\right).\end{array} \tag{4.8}$$

In the same way, we find

$$\begin{array}{l} \left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)\\ =\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+\left(\delta\widetilde{\varphi}^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)\\ =\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+\frac{1}{2\Delta t}\left(\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n-\left(\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right)\right\|^2+\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right\|^2\right.\\ \left.-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right),\end{array} \tag{4.9}$$

$$\begin{array}{l} \left(\theta_t^n-\delta\theta_h^n,\theta^n-\theta_h^n\right)\\ =\left(\theta_t^n-\delta\theta^n,\theta^n-\theta_h^n\right)+\left(\delta\theta^n-\delta\theta_h^n,\theta^n-\theta_h^n\right)\\ =\left(\theta_t^n-\delta\theta^n,\theta^n-\theta_h^n\right)+\frac{1}{2\Delta t}\left(\left\|\theta^n-\theta_h^n-\left(\theta^{n-1}-\theta_h^{n-1}\right)\right\|^2+\left\|\theta^n-\theta_h^n\right\|^2\right.\\ \left.-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right),\end{array} \tag{4.10}$$

$$\begin{array}{l} \left(w_t^n-\delta w_h^n,w^n-w_h^n\right)\\ =\left(w_t^n-\delta w^n,w^n-w_h^n\right)+\left(\delta w^n-\delta w_h^n,w^n-w_h^n\right)\\ =\left(w_t^n-\delta w^n,w^n-w_h^n\right)+\frac{1}{2\Delta t}\left(\left\|w^n-w_h^n-\left(w^{n-1}-w_h^{n-1}\right)\right\|^2+\left\|w^n-w_h^n\right\|^2\right.\\ \left.-\left\|w^{n-1}-w_h^{n-1}\right\|^2\right),\end{array} \tag{4.11}$$

$$\begin{array}{l} \left((u_x^n+\varphi^n)_t-\delta(u_{hx}^n+\varphi_h^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)\\ =\left((u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)\\ +\left(\delta(u_x^n+\varphi^n)-\delta(u_{hx}^n+\varphi_h^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)\\ =\left((u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)\\ +\frac{1}{2\Delta t}\left(\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)-\left((u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right)\right\|^2\right.\\ \left.+\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right\|^2-\left\|(u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right\|^2\right).\end{array} \tag{4.12}$$

$$\begin{aligned}
&(\varphi_{xt}^n-\delta\varphi_{hx}^n,\varphi_x^n-\varphi_{hx}^n)\\
&=(\varphi_{xt}^n-\delta\varphi_x^n,\varphi_x^n-\varphi_{hx}^n)+(\delta\varphi_x^n-\delta\varphi_{hx}^n,\varphi_x^n-\varphi_{hx}^n)\\
&=(\varphi_{xt}^n-\delta\varphi_x^n,\varphi_x^n-\varphi_{hx}^n)+\tfrac{1}{2\Delta t}\left(\left\|\varphi_x^n-\varphi_{hx}^n-\left(\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right)\right\|^2+\|\varphi_x^n-\varphi_{hx}^n\|^2\right.\\
&\left.-\left\|\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right\|^2\right).
\end{aligned}\tag{4.13}$$

From the latest equations, we achieve

$$\begin{aligned}
&\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}^n,\widetilde{u}^n-\widetilde{u}_h^n\right)+\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)+b\left(\varphi_{xt}^n-\delta\varphi_x^n,\varphi_x^n-\varphi_{hx}^n\right)\\
&+k\left((u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)+\rho_3\left(\theta_t^n-\delta\theta^n,\theta^n-\theta_h^n\right)\\
&+\left(w_t^n-\delta w^n,w^n-w_h^n\right)+k_2\left\|w_x^n-w_{hx}^n\right\|^2+k_3\left\|w^n-w_h^n\right\|^2\\
&+\tfrac{\rho_1}{2\Delta t}\left(\left\|\widetilde{u}^n-\widetilde{u}_h^n-\left(\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right)\right\|^2+\left\|\widetilde{u}^n-\widetilde{u}_h^n\right\|^2-\left\|\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right\|^2\right)\\
&+\tfrac{k}{2\Delta t}\left(\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)-\left((u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right)\right\|^2\right.\\
&\left.+\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right\|^2-\left\|(u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right\|^2\right)\\
&+\tfrac{\rho_2}{2\Delta t}\left(\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n-\left(\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right)\right\|^2+\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right)\\
&+\tfrac{b}{2\Delta t}\left(\left\|\varphi_x^n-\varphi_{hx}^n-\left(\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right)\right\|^2+\left\|\varphi_x^n-\varphi_{hx}^n\right\|^2-\left\|\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right\|^2\right)\\
&+\tfrac{\rho_3}{2\Delta t}\left(\left\|\theta^n-\theta_h^n-\left(\theta^{n-1}-\theta_h^{n-1}\right)\right\|^2+\left\|\theta^n-\theta_h^n\right\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)\\
&+\tfrac{1}{2\Delta t}\left(\left\|w^n-w_h^n-\left(w^{n-1}-w_h^{n-1}\right)\right\|^2+\left\|w^n-w_h^n\right\|^2-\left\|w^{n-1}-w_h^{n-1}\right\|^2\right)\\
&=\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)+k\left((u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n),\widetilde{u}_x^n-\xi_{hx}\right)\\
&-\gamma\left(\theta^n-\theta_h^n,\widetilde{u}_x^n-\xi_{hx}\right)+\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)+b\left(\varphi_x^n-\varphi_{hx}^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)\\
&+k\left(u_x^n+\varphi^n-u_{hx}^n-\varphi_h^n,\widetilde{\varphi}^n-\chi_h\right)-\gamma\left(w^n-w_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)\\
&-\gamma\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+\rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)-k_1\left(w^n-w_h^n,\theta_x^n-\eta_{hx}\right)\\
&+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n,\theta^n-\eta_h\right)+\left(w_t^n-\delta w_h^n,w^n-\alpha_h\right)\\
&+k_2\left(w_x^n-w_{hx}^n,w_x^n-\alpha_{hx}\right)+k_3\left(w^n-w_h^n,w^n-\alpha_h\right)+k_1\left(\theta_x^n-\theta_x^n,w^n-\alpha_h\right)\\
&+\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-\alpha_h\right).
\end{aligned}\tag{4.14}$$

So, which implies

$$\begin{aligned}
&\tfrac{\rho_1}{2\Delta t}\left(\left\|\widetilde{u}^n-\widetilde{u}_h^n-\left(\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right)\right\|^2+\left\|\widetilde{u}^n-\widetilde{u}_h^n\right\|^2-\left\|\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right\|^2\right)\\
&+\tfrac{k}{2\Delta t}\left(\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)-\left((u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right)\right\|^2\right.\\
&\left.+\left\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right\|^2-\left\|(u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right\|^2\right)\\
&+\tfrac{\rho_2}{2\Delta t}\left(\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n-\left(\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right)\right\|^2+\left\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right)\\
&+\tfrac{b}{2\Delta t}\left(\left\|\varphi_x^n-\varphi_{hx}^n-\left(\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right)\right\|^2+\left\|\varphi_x^n-\varphi_{hx}^n\right\|^2-\left\|\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right\|^2\right)\\
&+\tfrac{\rho_3}{2\Delta t}\left(\left\|\theta^n-\theta_h^n-\left(\theta^{n-1}-\theta_h^{n-1}\right)\right\|^2+\left\|\theta^n-\theta_h^n\right\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)\\
&+\tfrac{1}{2\Delta t}\left(\left\|w^n-w_h^n-\left(w^{n-1}-w_h^{n-1}\right)\right\|^2+\left\|w^n-w_h^n\right\|^2-\left\|w^{n-1}-w_h^{n-1}\right\|^2\right)\\
&+k_2\left\|w_x^n-w_{hx}^n\right\|^2+k_3\left\|w^n-w_h^n\right\|^2\\
&=-\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}^n,\widetilde{u}^n-\widetilde{u}_h^n\right)-\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)-b\left(\varphi_{xt}^n-\delta\varphi_x^n,\varphi_x^n-\varphi_{hx}^n\right)\\
&-k\left((u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)-\rho_3\left(\theta_t^n-\delta\theta^n,\theta^n-\theta_h^n\right)\\
&-\left(w_t^n-\delta w^n,w^n-w_h^n\right)+\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{u}_x^n-\xi_{hx}\right)\\
&+k\left((u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n),\widetilde{u}_x^n-\xi_{hx}\right)+\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)\\
&+b\left(\varphi_x^n-\varphi_{hx}^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+k\left(u_x^n+\varphi^n-u_{hx}^n-\varphi_h^n,\widetilde{\varphi}^n-\chi_h\right)\\
&-\gamma\left(w^n-w_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+\rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)\\
&-k_1\left(w^n-w_h^n,\theta_x^n-\eta_{hx}\right)+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n,\theta^n-\eta_h\right)\\
&+\left(w_t^n-\delta w_h^n,w^n-\alpha_h\right)+k_2\left(w_x^n-w_{hx}^n,w_x^n-\alpha_{hx}\right)+k_3\left(w^n-w_h^n,w^n-\alpha_h\right)\\
&+k_1\left(\theta_x^n-\theta_{hx}^n,w^n-\alpha_h\right)+\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-\alpha_h\right).
\end{aligned}\tag{4.15}$$

It follows that

$$
\begin{aligned}
&\tfrac{\rho_1}{2\Delta t}\left(\|\widetilde{u}^n-\widetilde{u}_h^n\|^2-\left\|\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right\|^2\right)+\tfrac{\rho_2}{2\Delta t}\left(\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right)\\
&+\tfrac{k}{2\Delta t}\left(\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\|^2-\left\|(u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right\|^2\right)\\
&+\tfrac{b}{2\Delta t}\left(\|\varphi_x^n-\varphi_{hx}^n\|^2-\left\|\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right\|^2\right)+\tfrac{\rho_3}{2\Delta t}\left(\|\theta^n-\theta_h^n\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)\\
&+\tfrac{1}{2\Delta t}\left(\|w^n-w_h^n\|^2-\left\|w^{n-1}-w_h^{n-1}\right\|^2\right)\\
&\le-\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}^n,\widetilde{u}^n-\widetilde{u}_h^n\right)-\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n,\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\right)-b\left(\varphi_{xt}^n-\delta\varphi_x^n,\varphi_x^n-\varphi_{hx}^n\right)\\
&-k\left((u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n),(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\right)-\rho_3\left(\theta_t^n-\delta\theta^n,\theta^n-\theta_h^n\right)\\
&-\left(w_t^n-\delta w^n,w^n-w_h^n\right)+\rho_1\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{u}_x^n-\xi_{hx}\right)\\
&+k\left((u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n),\widetilde{u}_x^n-\xi_{hx}\right)+\rho_2\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)\\
&+b\left(\varphi_x^n-\varphi_{hx}^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+k\left(u_x^n+\varphi^n-u_{hx}^n-\varphi_h^n,\widetilde{\varphi}^n-\chi_h\right)\\
&-\gamma\left(w^n-w_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)-\gamma\left(\theta^n-\theta_h^n,\widetilde{\varphi}_x^n-\chi_{hx}\right)+\rho_3\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)\\
&-k_1\left(w^n-w_h^n,\theta_x^n-\eta_{hx}\right)+\gamma\left(\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n,\theta^n-\eta_h\right)\\
&+\left(w_t^n-\delta w_h^n,w^n-\alpha_h\right)+k_2\left(w_x^n-w_{hx}^n,w_x^n-\alpha_{hx}\right)+k_3\left(w^n-w_h^n,w^n-\alpha_h\right)\\
&+k_1\left(\theta_x^n-\theta_{hx}^n,w^n-\alpha_h\right)+\gamma\left(\widetilde{\varphi}_x^n-\widetilde{\varphi}_{hx}^n,w^n-\alpha_h\right).
\end{aligned}
\tag{4.16}
$$

By applying the Cauchy–Schwarz inequality and Young's inequality, we derive:

$$
\begin{aligned}
&\tfrac{\rho_1}{2\Delta t}\left(\|\widetilde{u}^n-\widetilde{u}_h^n\|^2-\left\|\widetilde{u}^{n-1}-\widetilde{u}_h^{n-1}\right\|^2\right)+\tfrac{\rho_2}{2\Delta t}\left(\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2-\left\|\widetilde{\varphi}^{n-1}-\widetilde{\varphi}_h^{n-1}\right\|^2\right)\\
&+\tfrac{k}{2\Delta t}\left(\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\|^2-\left\|(u_x^{n-1}+\varphi^{n-1})-(u_{hx}^{n-1}+\varphi_h^{n-1})\right\|^2\right)\\
&+\tfrac{b}{2\Delta t}\left(\|\varphi_x^n-\varphi_{hx}^n\|^2-\left\|\varphi_x^{n-1}-\varphi_{hx}^{n-1}\right\|^2\right)+\tfrac{\rho_3}{2\Delta t}\left(\|\theta^n-\theta_h^n\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)\\
&+\tfrac{1}{2\Delta t}\left(\|w^n-w_h^n\|^2-\left\|w^{n-1}-w_h^{n-1}\right\|^2\right)\\
&\le\frac{C}{2}\Bigg(\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)+\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)+\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)\\
&+\left(w_t^n-\delta w_h^n,w^n-\alpha_h\right)+\|\widetilde{u}_t^n-\delta\widetilde{u}^n\|^2+\|\widetilde{u}^n-\widetilde{u}_h^n\|^2+\|\widetilde{u}^n-\xi_h\|^2\\
&+\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\|^2+\|\widetilde{u}_x^n-\xi_{hx}\|^2+\|\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n\|^2+\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2\\
&+\|\widetilde{\varphi}^n-\chi_h\|^2+\|\varphi_{xt}^n-\delta\varphi_x^n\|^2+\|(u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n)\|^2+\|\theta_t^n-\delta\theta^n\|^2\\
&+\|\varphi_x^n-\varphi_{hx}^n\|^2+\|\widetilde{\varphi}_x^n-\chi_{hx}\|^2+\|\theta_t^n-\delta\theta^n\|^2+\|\theta^n-\eta_h\|^2+\|\theta^n-\theta_h^n\|^2\\
&+\|\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n\|^2+\|\theta^n-\eta_h\|^2+\|w_t^n-\delta w^n\|^2+\|w^n-\alpha_h\|^2\\
&+\|w_x^n-w_{hx}^n\|^2+\|w_x^n-\alpha_{hx}\|^2+\|w^n-w_h^n\|^2+\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2\Bigg).
\end{aligned}
\tag{4.17}
$$

Thus, summing (4.17) over n yields, for all $\left\{\xi_h^i,\ \alpha_h^i\right\}_{i=0}^N\subset V^h$ and $\left\{\chi_h^i,\ \eta_h^i\right\}_{i=0}^N\subset V_0^h$, we derive

$$
\begin{aligned}
&\|\widetilde{u}^n-\widetilde{u}_h^n\|^2+\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2+\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\|^2+\|\varphi_x^n-\varphi_{hx}^n\|^2\\
&+\|\theta^n-\theta_h^n\|^2+\|w^n-w_h^n\|^2\\
&\le C\Delta t\sum_{i=1}^N\Bigg(\left(\widetilde{u}_t^n-\delta\widetilde{u}_h^n,\widetilde{u}^n-\xi_h\right)+\left(\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}_h^n,\widetilde{\varphi}^n-\chi_h\right)+\left(\theta_t^n-\delta\theta_h^n,\theta^n-\eta_h\right)\\
&+\left(w_t^n-\delta w_h^n,w^n-\alpha_h\right)+\|\widetilde{u}_t^n-\delta\widetilde{u}^n\|^2+\|\widetilde{u}^n-\widetilde{u}_h^n\|^2+\|\widetilde{u}^n-\xi_h\|^2\\
&+\|(u_x^n+\varphi^n)-(u_{hx}^n+\varphi_h^n)\|^2+\|\widetilde{u}_x^n-\xi_{hx}\|^2+\|\widetilde{\varphi}_t^n-\delta\widetilde{\varphi}^n\|^2+\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2\\
&+\|\widetilde{\varphi}^n-\chi_h\|^2+\|\varphi_{xt}^n-\delta\varphi_x^n\|^2+\|(u_x^n+\varphi^n)_t-\delta(u_x^n+\varphi^n)\|^2+\|\theta_t^n-\delta\theta^n\|^2\\
&+\|\varphi_x^n-\varphi_{hx}^n\|^2+\|\widetilde{\varphi}_x^n-\chi_{hx}\|^2+\|\theta_t^n-\delta\theta^n\|^2+\|\theta^n-\eta_h\|^2+\|\theta^n-\theta_h^n\|^2\\
&+\|\widetilde{u}_x^n+\widetilde{\varphi}^n-\widetilde{u}_{hx}^n-\widetilde{\varphi}_h^n\|^2+\|\theta^n-\eta_h\|^2+\|w_t^n-\delta w^n\|^2+\|w^n-\alpha_h\|^2\\
&+\|w_x^n-w_{hx}^n\|^2+\|w_x^n-\alpha_{hx}\|^2+\|w^n-w_h^n\|^2+\|\widetilde{\varphi}^n-\widetilde{\varphi}_h^n\|^2\Bigg)\\
&+C\Bigg(\left\|\widetilde{u}^0-\widetilde{u}_h^0\right\|^2+\left\|\widetilde{\varphi}^0-\widetilde{\varphi}_h^0\right\|^2+\left\|(u_x^0+\varphi^0)-(u_{hx}^0+\varphi_h^0)\right\|^2+\left\|\varphi_x^0-\varphi_{hx}^0\right\|^2\\
&+\left\|\theta^0-\theta_h^0\right\|^2+\left\|w^0-w_h^0\right\|^2\Bigg)
\end{aligned}
\tag{4.18}
$$

Finally, considering that (as referenced in [1]):

$$
\begin{aligned}
\Delta t \sum_{i=1}^{N} \left(\delta\widetilde{u}^i - \delta\widetilde{u}_h^i, \widetilde{u}^i - \xi_h\right) &= \left(\widetilde{u}^N - \widetilde{u}_h^N, \widetilde{u}^N - \xi_h^N\right) + \left(\widetilde{u}_h^0 - u^1, \widetilde{u}^1 - \xi_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\widetilde{u}^j - \widetilde{u}_h^j, \widetilde{u}^j - \xi_h^j - \left(\widetilde{u}^{j+1} - \xi_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\delta\widetilde{\varphi}^i - \delta\widetilde{\varphi}_h^i, \widetilde{\varphi}^i - \chi_h\right) &= \left(\widetilde{\varphi}^N - \widetilde{\varphi}_h^N, \widetilde{\varphi}^N - \chi_h^N\right) + \left(\widetilde{\varphi}_h^0 - \varphi^1, \widetilde{\varphi}^1 - \chi_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\widetilde{\varphi}^j - \widetilde{\varphi}_h^j, \widetilde{\varphi}^j - \chi_h^j - \left(\widetilde{\varphi}^{j+1} - \chi_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\delta\theta^i - \delta\theta_h^i, \theta^i - \eta_h\right) &= \left(\theta^N - \theta_h^N, \theta^N - \eta_h^N\right) + \left(\theta_h^0 - \theta^0, \theta^1 - \eta_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\theta^j - \theta_h^j, \theta^j - \eta_h^j - \left(\theta^{j+1} - \eta_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\delta w^i - \delta w_h^i, w^i - \alpha_h\right) &= \left(w^N - w_h^N, w^N - \alpha_h^N\right) + \left(w_h^0 - w^0, w^1 - \alpha_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(w^j - w_h^j, w^j - \alpha_h^j - \left(w^{j+1} - \alpha_h^{j+1}\right)\right).
\end{aligned}
$$

The proof is concluded by utilizing the previous estimates and applying a discrete version of Gronwall's inequality (as referenced in [7]). □

Based on the previous estimates, linear convergence of the approximations can be achieved if the solution to the continuous problem possesses sufficient additional regularity.

Corollary 4.1 *Suppose that the solution to the continuous problem is sufficiently regular, that is*

$$
\begin{aligned}
&u,\ \varphi \in H^3\left(0,T;L^2(0,L)\right) \cap H^2\left(0,T;H^1(0,L)\right) \cap C^1\left([0,T];H^2(0,L)\right), \\
&w \in H^2\left(0,T;L^2(0,L)\right) \cap H^1\left(0,T;H^1(0,L)\right) \cap C\left([0,T];H^2(0,L)\right), \\
&\theta \in H^1\left(0,T;L^2(0,L)\right) \cap H^1\left(0,T;H^1(0,L)\right) \cap C\left([0,T];H^2(0,L)\right).
\end{aligned}
$$

Then, there exists a positive constant C, independent of the discretization parameters h and Δt, such that

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ &\|\widetilde{u}^n - \widetilde{u}_h^n\|^2 + \|\widetilde{\varphi}^n - \widetilde{\varphi}_h^n\|^2 + \|(u_x^n + \varphi^n) - (u_{hx}^n + \varphi_h^n)\|^2 \\
&+ \|\varphi_x^n - \varphi_{hx}^n\|^2 + \|\theta^n - \theta_h^n\|^2 + \|w^n - w_h^n\|^2 \Big\} \le C\left(h + (\Delta t)\right).
\end{aligned}
$$

5. Numerical Simulation

In this section, we provide numerical examples to evaluate energy decay and provide a priori error estimates. Our initial focus is on validating the accuracy of our numerical approach, emphasizing the error analysis related to problem (5.1). By selecting specific external forces g_l for $l = 1,2,3,4$, we ensure the system has a known exact solution. We consider the problem defined by system (3.3) with artificial forces. The finite element method P_1 is applied to (5.1), with boundary and initial conditions (1.2), using the backward Euler scheme.

$$\begin{cases} \frac{\rho_1}{\Delta t}\left(\widetilde{u}_h^n-\widetilde{u}_h^{n-1},\xi_h\right)+k\left(\left(u_{hx}^n+\varphi_h^n\right),\xi_{hx}\right)-\gamma\left(\theta_h^n,\xi_{hx}\right)=\left(g_{1,h}^n,\xi_h\right)\\ \frac{\rho_2}{\Delta t}\left(\widetilde{\varphi}_h^n-\widetilde{\varphi}_h^{n-1},\chi_h\right)+b\left(\varphi_{hx}^n,\chi_{hx}\right)+k\left(u_{hx}^n+\varphi_h^n,\chi_h\right)+\gamma\left(w_{hx}^n,\chi_h\right)\\ -\gamma\left(\theta_h^n,\chi_h\right)=\left(g_{3,h}^n,\chi_h\right)\\ \frac{\rho_3}{\Delta t}\left(\theta_h^n-\theta_h^{h-1},\eta_h\right)+k_1\left(w_{hx}^n,\eta_h\right)+\gamma\left(\widetilde{u}_{hx}^n+\widetilde{\varphi}_h^n,\eta_h\right)=\left(g_{3,h}^n,\eta_h\right)\\ \frac{1}{\Delta t}\left(w_h^n-w_h^{n-1},\alpha_h\right)+k_2\left(w_{hx}^n,\alpha_{hx}\right)+k_3\left(w_h^n,\alpha_h\right)-k_1\left(\theta_h^n,\alpha_{hx}\right)\\ -\gamma\left(\widetilde{\varphi}_h^n,\alpha_{hx}\right)=\left(g_{4,h}^n,\alpha_h\right) \end{cases} \tag{5.1}$$

where,

$$\widetilde{u}_h^n=\frac{u_h^n-u_h^{n-1}}{\Delta t}\ \text{and}\ \widetilde{\varphi}_h^n=\frac{\varphi_h^n-\varphi_h^{n-1}}{\Delta t}. \tag{5.2}$$

The solution of (5.1) takes the form:

$$\begin{gathered} u_h^n=\sum_{i=0}^J u_{h,i}^n e_i(x),\ \widetilde{u}_h^n=\sum_{i=0}^J \widetilde{u}_{h,i}^n e_i(x),\ \ \varphi_h^n=\sum_{i=0}^J \varphi_{h,i}^n e_i(x),\ \ \widetilde{\varphi}_h^n=\sum_{i=0}^J \widetilde{\varphi}_{h,i}^n e_i(x),\\ \theta_h^n=\sum_{i=0}^J \theta_{h,i}^n e_i(x),\ \ w_h^n=\sum_{i=0}^J w_{h,i}^n e_i(x), \end{gathered} \tag{5.3}$$

for the equation's right-hand side

$$g_{l,h}^n=(g_{l,j}^n)_{0\le j\le J}=(g_l(x_j,t_n)_{0\le j\le J}\ \text{for}\ l\in\{1,2,3,4\}, \tag{5.4}$$

where e_i are the linear basis functions of space S^h. Taking $\xi_h=\chi_h=\eta_h=\alpha_h=e_j$ for all $j=0,\dots,J$ and lets denote

$$\begin{gathered} U^n=(u_{h,i}^n)_{0\le i\le J},\ \widetilde{U}^n=(u_{h,i}^n)_{0\le i\le J},\ \Phi^n=(\varphi_{h,i}^n)_{0\le i\le J},\\ \widetilde{\Phi}^n=(\widetilde{\varphi}_{h,i}^n)_{0\le i\le J},\ \Theta^n=(\theta_{h,i}^n)_{0\le i\le J},\ W^n=(w_{h,i}^n)_{0\le i\le J}, \end{gathered} \tag{5.5}$$

and let

$$\begin{gathered} M_1=((e_i,e_j))_{0\le i,j\le J},\ M_2=((e_{ix},e_j))_{0\le i,j\le J},\ M_3=((e_i,e_{jx}))_{0\le i,j\le J},\\ M_4=((e_i,e_j))_{0\le i,j\le J},\ G_l^n=\left(g_{l,j}^n,e_j\right)_{1\le l\le 4,0\le j\le J}. \end{gathered} \tag{5.6}$$

Finally we get from (5.1) – (5.5), the following linear system:

$$A_1Y^n=A_2Y^{n-1}+G^n,$$

where the block matrices defined as follows:

$$A_1=\begin{pmatrix} I_{J-1} & -\Delta t I_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1}\\ k\Delta t M_4 & \rho_1 M_1 & k\Delta t M_3 & O_{J-1} & -\gamma\Delta t M_3 & O_{J-1}\\ O_{J-1} & O_{J-1} & I_{J-1} & -\Delta t I_{J-1} & O_{J-1} & O_{J-1}\\ k\Delta t M_2 & O_{J-1} & b\Delta t M_4+k\Delta t M_1 & \rho_2 M_1 & -\gamma\Delta t M_1 & \gamma\Delta t M_2\\ O_{J-1} & \gamma\Delta t M_2 & O_{J-1} & \gamma\Delta t M_1 & \rho_3 M_1 & k_1\Delta t M_2\\ O_{J-1} & O_{J-1} & O_{J-1} & -\gamma\Delta t M_3 & -k_1\Delta t M_3 & M_1+k_3\Delta t M_1+k_2\Delta t M_4 \end{pmatrix},$$

$$A_2=\begin{pmatrix} I_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1}\\ O_{J-1} & \rho_1 M_1 & O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1}\\ O_{J-1} & O_{J-1} & I_{J-1} & O_{J-1} & O_{J-1} & O_{J-1}\\ O_{J-1} & O_{J-1} & O_{J-1} & \rho_2 M_1 & O_{J-1} & O_{J-1}\\ O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & \rho_3 M_1 & O_{J-1}\\ O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & O_{J-1} & M_1 \end{pmatrix}$$

and vectors

$$Y^n=\begin{pmatrix} U^n\\ \widetilde{U}^n\\ \Phi^n\\ \widetilde{\Phi}^n\\ \Theta^n\\ W^n \end{pmatrix},\qquad F^n=\begin{pmatrix} O_{J-1,1}\\ G_1^n\\ O_{J-1,1}\\ G_2^n\\ G_3^n\\ G_4^n \end{pmatrix}$$

Example 1: Scheme Convergence

First, we conducted a simulation to assess the numerical error. To do this, we addressed problem (5.1), where g_1, g_2, g_3 and g_4, in addition to the initial data, were determined using the exact solution for all $x \in (0,1)$ and $t \geq 0$ described below:

$$\begin{aligned} u(x,t) &= x^4(1-x)^2 \exp(\frac{t^2}{2}+t), \quad \varphi(x,t) = x\cos(\pi x + \frac{\pi}{2})\exp(t), \\ \theta(x,t) &= \sin(\pi x)\exp(t), \quad w(x,t) = x^3(1-x)^3\exp(\frac{t}{2}). \end{aligned} \tag{5.7}$$

Initially, we take the following parameters of the model:

$$\begin{gathered} \rho_1 = 0.1,\ \rho_2 = 0.3,\ \rho_3 = 0.4,\ b = 0.1,\ k = 0.1,\ k_1 = 0.3,\ k_2 = 1, \\ k_3 = 0.7,\ \gamma = 2. \end{gathered} \tag{5.8}$$

The calculated errors at time $T = 1$ are displayed in Table 1, with $Error$ being defined as follows:

$$\begin{aligned} Error = {} & \|\widetilde{u}_h^n - \widetilde{u}_h\|^2 + \|\widetilde{\varphi}_h^n - \widetilde{\varphi}_h\|^2 + \|\theta_h^n - \theta_h\|^2 + \|w_h^n - w_h\|^2 \\ & + \|u_{hx}^n + \varphi_h^n - (u_{hx} + \varphi_h)\|^2 + \|\varphi_{hx}^n - \varphi_{hx}\|^2 . \end{aligned} \tag{5.9}$$

We observe that the error decreases by a factor of 1, indicating a linear convergence rate, as illustrated in Fig. 3, which corresponds to the result stated in Corollary 4.1.

Table 1: Computed numerical errors for a final time T=1 and for some values of J and Δt.

$J \downarrow \Delta t \rightarrow$	0.1	0.05	0.0025	0.0125	0.00625	0.003125
25	8.044485	8.022602	8.014106	8.010503	8.008887	8.008167
50	3.874869	3.851257	3.842182	3.838326	3.836565	3.835724
100	1.903136	1.883671	1.876862	1.874181	1.873019	1.872483
200	0.948229	0.932160	0.927122	0.925336	0.924622	0.924309
400	0.479031	0.464991	0.460993	0.459728	0.459273	0.459090
800	0.246568	0.233626	0.230188	0.229202	0.228886	0.228772

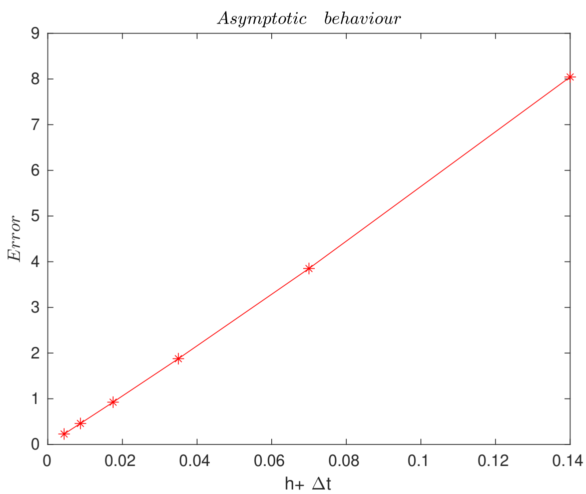

Figure 1: Asymptotic behavior of $Error$.

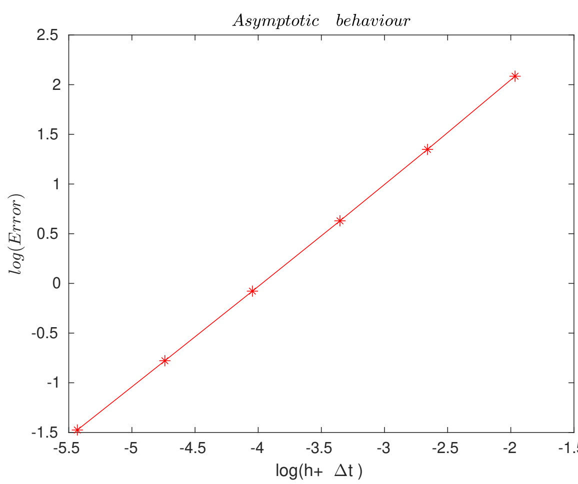

Figure 2: Behavior of log(Error).

Figure 3: Error estimate.

Example 2: Energy Behavior

We recall that the energy is exponentially stable if (5.10) holds.

$$\nu = b\rho_1 - k\rho_2 - \frac{\rho_2\gamma^2}{\rho_3} = 0. \tag{5.10}$$

In this instance, we consider the initial system without external forces g_l for $l = 1, ..., 4$ and we observed how energy decay evolves over time. The discrete energy is given by

$$\begin{aligned} \mathcal{E}_h^n = \frac{1}{2}\Big[&\rho_1 \left\|\widetilde{u}_h^n\right\|^2 + \rho_2 \left\|\widetilde{\varphi}_h^n\right\|^2 + \rho_3\|\theta_h^n\|^2 + \|w_h^n\|^2 \\ &+k \left\|u_{hx}^n + \varphi_h^n\right\|^2 + b \left\|\varphi_{hx}^n\right\|^2\Big]. \end{aligned} \tag{5.11}$$

The discretization parameters are:

$$h = \frac{1}{J},\ \Delta t = \frac{T}{N},\ J = 200,\ T = 20,\ N = 2000. \tag{5.12}$$

Along with the initial conditions specified for all $\forall x \in (0, 1)$:

$$\begin{aligned} u_0(x) = u_1(x) = x^4(1-x)^2,\ \varphi_0(x) = \varphi_1(x) = x\cos(\pi x + \frac{\pi}{2}), \\ \theta_0(x) = x(1-x)\cos(\pi x),\ w_0(x) = x^3(1-x)^3. \end{aligned} \tag{5.13}$$

In the present example, we have chosen the following values:

$$\begin{aligned} \rho_1 = 1,\ \rho_2 = 3,\ \rho_3 = 4, b = 15.0075,\ k = 5,\ k_1 = 0.5,\ k_2 = 0.6, \\ k_3 = 1,\ \gamma = 0.1. \end{aligned} \tag{5.14}$$

we display the energy evolution in both natural scale (Fig. 4) and semi-log scale (Fig. 5). It is evident that the discrete energy approaches zero, demonstrating that exponential energy decay is observed when the condition in (5.10) is met.

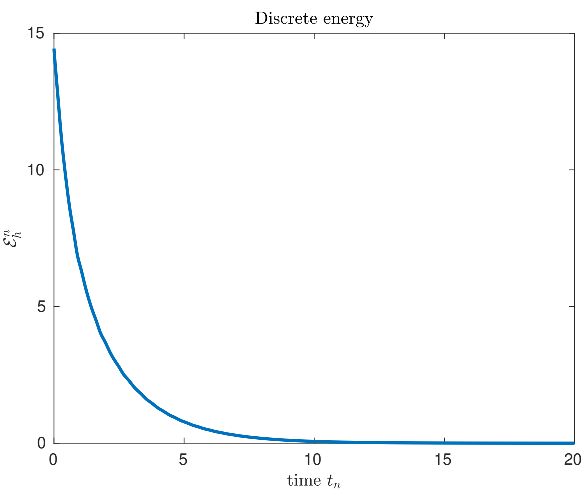

Figure 4: Natural scale behavior of $\mathcal{E}_h^n$.

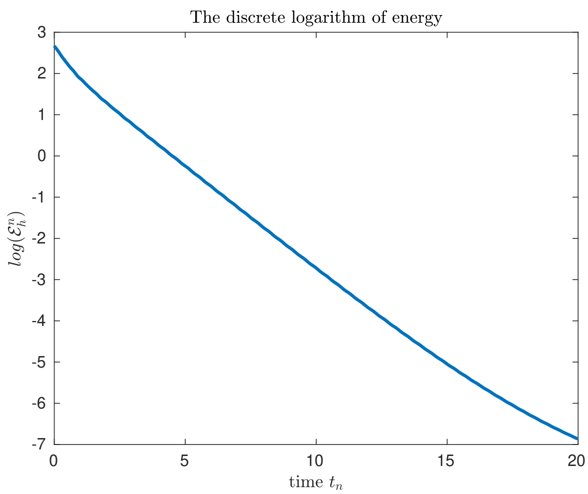

Figure 5: Semi-log scale behavior of $\mathcal{E}_h^n$.

In Fig. 8, the evolution over time of both the displacement of the solid elastic material (Fig. 6) and the volume fraction (Fig. 7) at several points is presented. Additionally, in Fig. 11, both the temperature difference (Fig. 7) and the microtemperature (Fig. 9) are shown at different time instants. As expected, all physical quantities converge to zero over time and space.

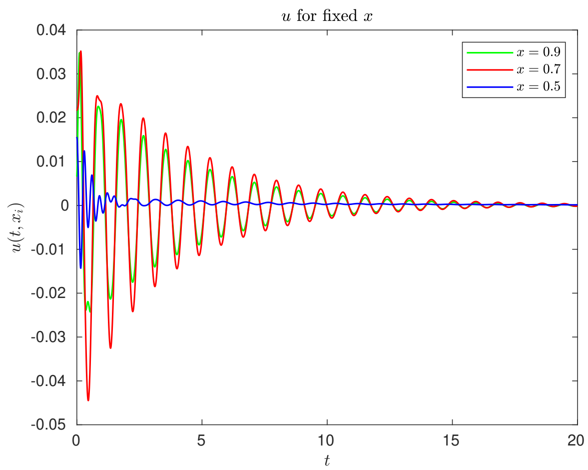

Figure 6: The displacement of the solid elastic material u.

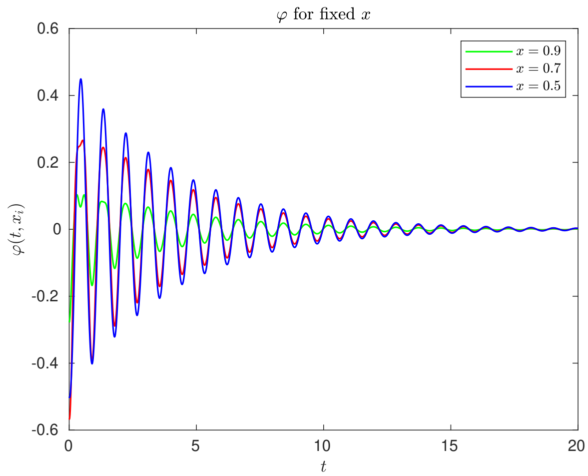

Figure 7: The volume fraction φ.

Figure 8: The displacement of the solid elastic material and the volume fraction over time for various fixed values of x.

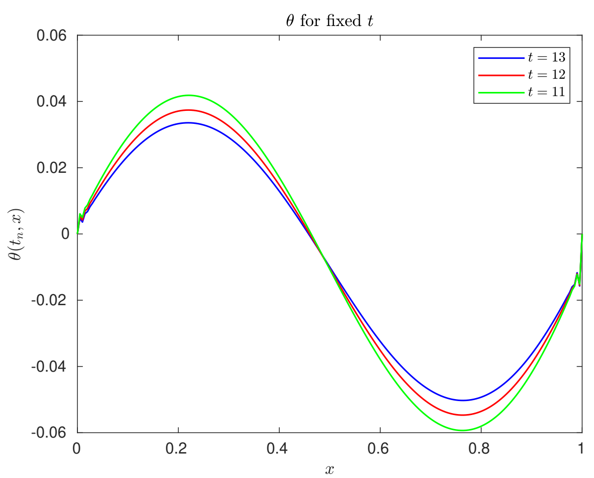

Figure 9: The temperature difference θ.

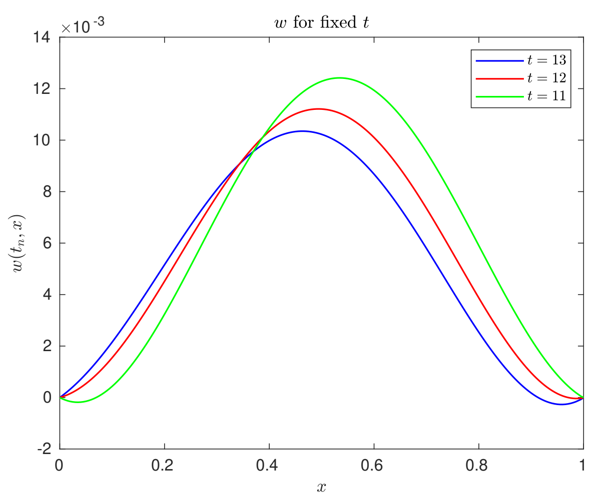

Figure 10: The microtemperature w.

Figure 11: The temperature difference and the microtemperature over space for various fixed values of t.

The numerical schemes were implemented using MATLAB on an Intel Core i5-6006U CPU @ 2.00 GHz.

Conclusion

In this study, we performed a numerical analysis of a one-dimensional problem involving the Timoshenko system with microtemperature effects and no thermal conductivity. We began by deriving the variational form of the linear system using integration by parts and introduced a fully discrete approximation through the classical finite element method with linear elements and the backward Euler scheme. We established the discrete stability of the approximations and provided an a priori error analysis. After implementing the algorithm in MATLAB, we conducted numerical simulations. In the first example, we demonstrated the linear convergence of the approximations. In the second example, we illustrated the theoretical exponential energy decay by analyzing the discrete energy curve, showing that the solution converges to zero over time and space.

Author contributions

All authors contributed equally to this manuscript.

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

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