



Some Properties of Frobenius–Sigmoid–Fibonacci Polynomials with Their Applications

Haitham Qawaqneh and Waseem Ahmad Khan and Musharraf Ali and Ulfat Ansari

ABSTRACT: In this paper, we introduce two-variable Frobenius–Sigmoid–Fibonacci polynomials and their associated numbers within the approach of Golden F –Calculus. Utilizing generating functions, we derive several fundamental properties, including summation theorems, recurrence relations, symmetry properties, and F –derivative identities. We further establish connections with Stirling–Fibonacci numbers of the second kind and present multiple summation formulas and convolution–type identities. Additionally, we introduce novel parametric forms of these polynomials, incorporating trigonometric generating structures, and analyze their behavior using the F –differential operator and functional equation techniques. The proposed approach enriches the theory of Fibonacci–based special polynomials and opens new avenues for applications in combinatorics, number theory, approximation theory, and matrix analysis.

Keywords: Golden calculus, Sigmoid polynomials, Frobenius–Sigmoid–Fibonacci polynomials, symmetric identities, Stirling numbers of the second kind.

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1. Introduction

Recently, many mathematicians [11,16,17,18] have focused on defining and constructing generating functions for new families of special polynomials, such as F –Bernoulli, F –Euler, and F –Genocchi polynomials, by employing Frobenius–Euler–Fibonacci and Frobenius–Euler–Genocchi–Fibonacci polynomials. These investigations have yielded fundamental properties and revealed diverse applications of such polynomials. In particular, several explicit and implicit summation formulas, recurrence relations, symmetric identities, and deep connections with classical polynomials have been extensively established. Moreover, novel and elegant correlations among various special polynomials have also been discovered. The aforementioned frameworks further enable the derivation of useful properties through comparatively simple procedures and contribute to the construction of new families of special polynomials. Motivated by these developments, in this work we introduce Frobenius–Sigmoid–Fibonacci polynomials and explore their properties by establishing various relations and applications. Specifically, we derive addition theorems, recurrence formulas, implicit summation identities, and relations with previously studied polynomials in the literature (see [1], [2], [21],[22], [23],[24], [25], [26], [27],[28],[29],[30],[31]).

The Apostol–type Frobenius–Euler polynomials $\mathcal{H}_\omega^{(\alpha)}(\xi; \mathfrak{U}; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by [19,32,36]:

$$\left(\frac{1 - \mathfrak{U}}{\lambda e^d - \mathfrak{U}} \right)^\alpha e^{\xi d} = \sum_{\omega=0}^{\infty} \mathcal{H}_\omega^{(\alpha)}(\xi; \mathfrak{U}; \lambda) \frac{d^\omega}{\omega!}, \quad (1.1)$$

where $\mathfrak{U} \in \mathcal{C} \setminus \{1\}$, $\xi \in \mathcal{R}$ and $|d| < \left| \log \left(\frac{\lambda}{\mathfrak{U}} \right) \right|$.

At the point $\xi = 0$, $\mathcal{H}_\omega^{(\alpha)}(\mathfrak{u}; \lambda) = \mathcal{H}_\omega^{(\alpha)}(0; \mathfrak{u}; \lambda)$ are called the Apostol–type Frobenius–Euler numbers of order α . From (1.1), we find

$$\mathcal{H}_\omega^{(\alpha)}(\xi; \mathfrak{u}; \lambda) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathcal{H}_\nu^{(\alpha)}(\mathfrak{u}; \lambda) \xi^{\omega-\nu}. \quad (1.2)$$

For $w \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Stirling numbers of the first kind are defined by [33,34,35]

$$(\xi)_w = \sum_{p=0}^w S_1(w, p) \xi^p, \quad (1.3)$$

where $(\xi)_0 = 1$, and $(\xi)_w = \xi(\xi - 1) \cdots (\xi - w + 1)$, ($w \geq 1$). From (1.3), we get

$$\frac{1}{r!} (\log(1 + d))^r = \sum_{w=r}^{\infty} S_1(w, r) \frac{d^w}{w!}, \quad (r \geq 0). \quad (1.4)$$

For $w \in \mathbb{N}_0$, the Stirling numbers of the second kind are defined by [37,38]

$$\xi^w = \sum_{q=0}^w S_2(w, q) (\xi)_q. \quad (1.5)$$

From (1.5), we see that

$$\frac{1}{w!} (e^d - 1)^w = \sum_{j=w}^{\infty} S_2(j, w) \frac{d^j}{j!}. \quad (1.6)$$

For any non-negative integer r , the r -Stirling numbers $S_{2,r}(j, w)$ of the second kind are defined by [3]

$$\frac{1}{w!} e^{rd} (e^d - 1)^w = \sum_{j=w}^{\infty} S_{2,r}(j + r, w + r) \frac{d^j}{j!}. \quad (1.7)$$

The Sigmoid function is expressed as

$$s(d) = \frac{1}{e^{-d} + 1},$$

where $d \in \mathbb{C}$. Extensive research has explored the diverse applications of the Sigmoid function [4]. Notably, a three-parameter extension of the Sigmoid function has been utilized to model and analyze hybrid Sigmoid networks [5,6]. Additionally, significant attention has been directed towards investigating the zeros of Sigmoid polynomials, their generalized forms within various analytical frameworks, and their relationships with other established families of polynomials [7,8,9]. The generating function of the sigmoid polynomials are defined by [10]

$$\frac{1}{e^{-d} + 1} e^{\xi d} = \sum_{w=0}^{\infty} S_w(\xi) \frac{d^w}{w!}. \quad (1.8)$$

For $\xi = 0$, $S_w(0)$ become the Sigmoid numbers. In [11], Khan and Kızılateş introduced Frobenius Sigmoid polynomials as an alternative generalization of Sigmoid polynomials. Additionally, they investigated several properties of these polynomials through the application of their generating function.

We hereby introduce specific definitions and properties pertinent to Golden Calculus, also referred to as F -calculus. The Fibonacci sequence is defined by the following recurrence relation:

$$F_w = F_{w-1} + F_{w-2}, \quad w \geq 2$$

with $F_0 = 0$, $F_1 = 1$. Fibonacci numbers can be expressed explicitly as

$$F_w = \frac{\alpha^w - \beta^w}{\alpha - \beta}, \quad w \in \mathbb{N}_0$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ (Golden ratio) and $\beta = \frac{1-\sqrt{5}}{2}$. The use of the golden ratio in various branches of science and mathematics is well-documented. Additionally, this mystical number also makes appearances in the fields of architecture and art. Recently, the properties of F -calculus have been exhaustively defined and studied by Pashaev and Nalci [12]. Please refer to the following papers for further reading: Krot [14], Özvatan [15], Pashaev [13], and Kuş et al. [16]. These sources provide additional information and insights on the topic at hand.

For $w \in \mathbb{N}$, the F -factorial, derived from the Fibonacci sequence, was formally characterized as follows:

$$F_1 F_2 F_3 \dots F_w = F_w!, \quad (1.9)$$

where $F_0! = 1$.

Presented here is the formal representation of the binomial theorem as it applies to F -analogues, often referred to as the Golden binomial theorem:

$$(a + b)_F^w = \sum_{s=0}^w (-1)^{\binom{2}{s}} \binom{w}{s}_F a^{w-s} b^s, \quad (1.10)$$

or

$$(a +_F b)^w = \sum_{s=0}^w \binom{w}{s}_F a^{w-s} b^s, \quad (1.11)$$

in terms of the Golden binomial coefficients, called as Fibonomials

$$\binom{u}{s}_F = \frac{F_u!}{F_{u-s}! F_s!},$$

with u and s being non-negative integers, $u \geq s$. The Golden derivative (F -Derivative Formula) defined as follows:

$$\frac{d_F}{d_F \xi} (g(\xi)) = \frac{g(\alpha\xi) - g\left(-\frac{\xi}{\alpha}\right)}{\left(\alpha - \left(-\frac{1}{\alpha}\right)\right) \xi} = \frac{g(\alpha\xi) - g(\beta\xi)}{(\alpha - \beta) \xi}. \quad (1.12)$$

The first and second types of Golden exponential functions are represented as

$$e_F^\xi = \sum_{s=0}^{\infty} \frac{(\xi)_F^s}{F_s!}, \quad (1.13)$$

and

$$E_F^\xi = \sum_{s=0}^{\infty} (-1)^{\binom{2}{s}} \frac{(\xi)_F^s}{F_s!}, \quad (1.14)$$

respectively. Briefly, we use this notation throughout the article

$$e_F^\xi = \sum_{s=0}^{\infty} \frac{\xi^s}{F_s!}, \quad (1.15)$$

and

$$E_F^\xi = \sum_{s=0}^{\infty} (-1)^{\binom{2}{s}} \frac{\xi^s}{F_s!}. \quad (1.16)$$

Using these functions (see [14,12]), the following equations can be given:

$$e_F^\xi e_F^\eta = e_F^{(\xi+F\eta)}, \quad (1.17)$$

$$e_F^\xi E_F^\eta = e_F^{(\xi+\eta)_F}. \quad (1.18)$$

The Fibonacci cosine and sine functions, commonly known as the Golden trigonometric functions, are expressed through the power series [18]

$$\cos_F(\xi) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s}}{F_{2s}!}, \quad (1.19)$$

and

$$\sin_F(\xi) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s+1}}{F_{2s+1}!}. \quad (1.20)$$

For arbitrary number ϕ , Golden derivatives of $e_F^{\phi\xi}$, $E_F^{\phi\xi}$, $\cos_F(\phi\xi)$, and $\sin_F(\phi\xi)$ functions are given as follows:

$$\frac{d_F}{d_F\xi} \left(e_F^{\phi\xi} \right) = \phi e_F^{\phi\xi}, \quad (1.21)$$

$$\frac{d_F}{d_F\xi} \left(E_F^{\phi\xi} \right) = \phi E_F^{-\phi\xi}, \quad (1.22)$$

$$\frac{d_F}{d_F\xi} \left(\cos_F(\phi\xi) \right) = -\phi \sin_F(\phi\xi), \quad (1.23)$$

and

$$\frac{d_F}{d_F\xi} \left(\sin_F(\phi\xi) \right) = \phi \cos_F(\phi\xi). \quad (1.24)$$

In 2024, Alatawi et al. [17] introduced the generalized Apostol-type Frobenius-Euler-Fibonacci polynomials as:

$$\left(\frac{1 - \mathfrak{U}}{e_F^d - \mathfrak{U}} \right)^\alpha e_F^{\zeta d} E_F^{\eta d} = \sum_{w=0}^{\infty} \mathcal{H}_{w,F}^{(\alpha)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!}, \quad |d| \neq \pi, \quad (1.25)$$

where $\mathfrak{U} \in \mathcal{C}$ with $\mathfrak{U} \neq 1$, and $w \in \mathcal{N}_0$.

Note that, for $\xi = 0$, the generalized Apostol-type Frobenius-Euler-Fibonacci polynomials become the generalized Apostol-type Frobenius-Euler-Fibonacci numbers. The author then gave several identities for these types of polynomials and numbers. For more details related to Fibonacci-based special polynomials, please refer to [17, 18, 22, 23].

The object of the paper is as follows. In section 2, we introduce two variable Frobenius-Sigmoid polynomials and find some zero values of these polynomials. We derive some properties of these polynomials. In section 3, we introduce parametric type of Frobenius-Sigmoid-Fibonacci polynomials and find some properties of these polynomials.

2. Properties of Frobenius-Sigmoid-Fibonacci Polynomials

In this section, we define Frobenius-Sigmoid-Fibonacci polynomials and numbers. Then, we derive the basic properties such as additivity and symmetrically regarding related to it. Lastly, we introduce certain properties of these numbers and polynomials.

Let $\mathfrak{U} \in \mathcal{C}$ with $\mathfrak{U} \neq 1$, and $w \in \mathcal{N}_0$. Then, we can define the Frobenius-Sigmoid-Fibonacci polynomials $\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U})$:

$$\frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{\zeta d} E_F^{\eta d} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!}, \quad |d| \neq \pi. \quad (2.1)$$

A few of them are:

$$\begin{aligned}\mathcal{S}_{0,F}(\zeta, \eta; \mathfrak{U}) &= \frac{1}{2}, \\ \mathcal{S}_{1,F}(\zeta, \eta; \mathfrak{U}) &= \frac{1}{2}(\zeta + \eta)_F + \frac{1}{2(1 - \mathfrak{U})}, \\ \mathcal{S}_{2,F}(\zeta, \eta; \mathfrak{U}) &= \frac{1}{2}(\zeta + \eta)_F^2 - \frac{(\zeta + \eta)_F}{2(1 - \mathfrak{U})} - \frac{1}{2(1 - \mathfrak{U})} - \frac{1}{2(1 - \mathfrak{U})}, \\ \mathcal{S}_{3,F}(\zeta, \eta; \mathfrak{U}) &= \frac{1}{2}(\zeta + \eta)_F^3 + \frac{1}{1 - \mathfrak{U}}(\zeta + \eta)_F^2 - \left(\frac{1}{2(1 - \mathfrak{U})} + \frac{1}{2(1 - \mathfrak{U})^2} \right) (\zeta + \eta)_F - \frac{2}{(1 - \mathfrak{U})^2} - \frac{1}{(1 - \mathfrak{U})^3}.\end{aligned}$$

Letting $\mathfrak{U} = -1$ in (2.1), we get

$$\frac{1}{e_F^{-d} + 1} e_F^{\zeta d} E_F^{\eta d} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta, \eta) \frac{d^w}{F_w!},$$

where, $\mathcal{S}_{w,F}(\zeta, \eta)$ are called the w^{th} -Sigmoid-Fibonacci polynomials.

Again, put $\zeta = \eta = 0$ in (2.1), we note that

$$\sum_{w=0}^{\infty} \mathcal{S}_{w,F}(0; \mathfrak{U}) \frac{d^w}{F_w!} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\mathfrak{U}) \frac{d^w}{F_w!} = \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})},$$

where, $\mathcal{S}_{w,F}(\mathfrak{U})$ are called the w^{th} -Frobenius-Sigmoid-Fibonacci numbers.

To derive structural properties of these polynomials, we now prove a series of summation, recurrence, and convolution-type identities.

Theorem 2.1 *The following summation formulas for the polynomials $\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U})$ hold true:*

$$\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) = \sum_{j=0}^w \binom{w}{j}_F \mathcal{S}_{j,F}(\mathfrak{U}) (\zeta + \eta)_F^{w-j}, \quad (2.2)$$

$$\mathcal{S}_{j,F}(\zeta, \eta; \mathfrak{U}) = \sum_{j=0}^w \binom{w}{j}_F \mathcal{S}_{j,F}(0, \eta; \mathfrak{U}) \zeta^{w-j}, \quad (2.3)$$

$$\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) = \sum_{j=0}^w \binom{w}{j}_F \mathcal{S}_{w-j,F}(\zeta; \mathfrak{U}) (-1)^{\frac{j(j-1)}{2}} \eta^j. \quad (2.4)$$

Proof: By appropriately applying equations (1.10) and (1.18) within the generating function (2.1), three distinct forms are derived. Subsequently, the F -Cauchy product rule is utilized in the resulting expressions, and by comparing the corresponding powers of d on both sides of the resultant equation, we obtain formulas (2.2)-(2.4). \square

Theorem 2.2 *The following recursive formulas for the two variable Frobenius-Sigmoid-Fibonacci polynomials $\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U})$ hold true:*

$$\frac{\partial_F}{\partial_F \zeta} \{ \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \} = F_w \mathcal{S}_{w-1,F}(\zeta, \eta; \mathfrak{U}), \quad (2.5)$$

and

$$\frac{\partial_F}{\partial_F \eta} \{ \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \} = F_w \mathcal{S}_{w-1,F}(\zeta, -\eta; \mathfrak{U}), \quad (2.6)$$

where $\frac{\partial_F}{\partial_F \zeta}$ is F -partial derivative operator.

Proof: Differentiating generating function (2.1) with respect to ζ and η , utilizing equations (1.12) and (2.3), and subsequently simplifying through the application of the F -Cauchy product rule formulas yields the results given in (2.5) and (2.6). \square

The subsequent definition delineates the r -Stirling-Fibonacci numbers of the second kind.

$$\sum_{k=m}^{\infty} S_{2,r}^F(k+r, m+r) \frac{d^k}{F_k!} = e_F^{rd} \frac{(e_F^d - 1)^m}{F_m!}. \quad (2.7)$$

For $r = 0$, we have Stirling-Fibonacci numbers of the second kind [20,17]

$$\sum_{k=m}^{\infty} S_2^F(k, m) \frac{d^k}{F_k!} = \frac{(e_F^d - 1)^m}{F_m!}. \quad (2.8)$$

Theorem 2.3 *Let $w \geq 0$ be an integer. Then we have*

$$\mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) = \sum_{k=0}^w \sum_{m=0}^k \binom{w}{k} \mathcal{S}_{w-k,F}(\eta; \mathfrak{U})(\zeta)_m S_{2,F}(k, m). \quad (2.9)$$

Proof: By virtue (2.1) and (2.8), we find

$$\begin{aligned} \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} (e_F^d - 1 + 1)^\zeta E_F^{\eta d} \\ &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} E_F^{\eta d} \sum_{m=0}^{\infty} (\zeta)_m \frac{(e_F^d - 1)^m}{F_m!} \\ &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} E_F^{\eta d} \sum_{m=0}^{\infty} (\zeta)_m \sum_{k=m}^{\infty} S_{2,F}(k, m) \frac{d^k}{F_k!} \\ &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\eta; \mathfrak{U}) \frac{d^w}{F_w!} \sum_{k=0}^{\infty} \sum_{m=0}^k (\zeta)_m S_{2,F}(k, m) \frac{d^k}{F_k!} \\ &= \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \sum_{m=0}^k \binom{w}{k} \mathcal{S}_{w-k,F}(\eta; \mathfrak{U})(\zeta)_m S_{2,F}(k, m) \right) \frac{d^w}{F_w!}. \quad (2.10) \end{aligned}$$

To derive the result, we equate the coefficients of $\frac{d^w}{F_w!}$ on both sides of the last equation. \square

Theorem 2.4 *Let $w \geq 0$ be an integer. Then we have*

$$\mathcal{S}_{w,F}(\zeta + r, \eta; \mathfrak{U}) = \sum_{k=0}^w \sum_{m=0}^k \binom{w}{k}_F \mathcal{S}_{w-k,F}(\eta; \mathfrak{U})(\zeta)_m S_{2,F}(k+r, m+r). \quad (2.11)$$

Proof: By virtue of (2.7), (2.8) and (2.1), we find

$$\begin{aligned}
 \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta + r, \eta; \mathfrak{U}) \frac{d^w}{F_w!} &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{rd} E_F^{\eta d} (e^d - 1 + 1)^\zeta \\
 &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{rd} E_F^{\eta d} \sum_{m=0}^{\infty} (\zeta)_m \frac{(e^d - 1)^m}{F_m!} \\
 &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} E_F^{\eta d} \sum_{m=0}^{\infty} (\zeta)_m \sum_{k=m}^{\infty} S_{2,F}(k + r, m + r) \frac{d^k}{F_k!} \\
 &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\eta; \mathfrak{U}) \frac{d^w}{F_w!} \sum_{k=0}^{\infty} \sum_{m=0}^k (\zeta)_m S_{2,F}(k + r, m + r) \frac{d^k}{F_k!} \\
 &= \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \sum_{m=0}^k \binom{w}{k}_F \mathcal{S}_{w-k,F}(\eta; \mathfrak{U}) (\zeta)_m S_{2,F}(k + r, m + r) \right) \frac{d^w}{F_w!}. \quad (2.12)
 \end{aligned}$$

To derive the result, we equate the coefficients of $\frac{d^w}{F_w!}$ on both sides of the last equation. \square

Theorem 2.5 *Let $w \geq 0$ be an integer. then we have*

$$\sum_{k=0}^w \binom{n}{k}_F \mathcal{S}_{k,F}(\mathfrak{U}) (-1)^{w-k} - \mathfrak{U} \mathcal{S}_{w,F}(\mathfrak{U}) = \begin{cases} \frac{1-\mathfrak{U}}{2}, & \text{if } w = 0, \\ 0, & \text{if } w \geq 1. \end{cases} \quad (2.13)$$

Proof: If $e_F^{-d} \neq \mathfrak{U}$ in the generating function of the Frobenius–Sigmoid–Fibonacci number, then we find

$$\begin{aligned}
 \sum_{w=0}^{\infty} 2\mathcal{S}_{w,F}(\mathfrak{U}) \frac{d^w}{F_w!} \left(\sum_{w=0}^{\infty} (-1)^w \frac{d^w}{F_w!} - \mathfrak{U} \right) &= 1 - \mathfrak{U} \\
 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \mathcal{S}_{k,F}(\mathfrak{U}) (-1)^{w-k} - \mathfrak{U} \mathcal{S}_{w,F}(\mathfrak{U}) \right) \frac{d^w}{F_w!} &= 1 - \mathfrak{U}.
 \end{aligned}$$

Hence, complete proof of the theorem. \square

Corollary 2.1 *Using the similar method of proof from Theorem 2.5, we have*

$$\sum_{k=0}^w \binom{w}{k}_F \mathcal{S}_{k,F}(\zeta, \eta; \mathfrak{U}) (-1)^{w-k} - \mathfrak{U} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) = (\zeta + \eta) \frac{d^w}{F_w!}. \quad (2.14)$$

Theorem 2.6 *Let $w \geq 0$ be an integer. Then we have*

$$\begin{aligned}
 \sum_{k=0}^w \binom{w}{k}_F \alpha^{w-k} \beta^k \mathcal{S}_{w-k,F}(\alpha^{-1}\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(\beta^{-1}\phi, \xi; \mathfrak{U}) \\
 = \sum_{k=0}^w \binom{w}{k}_F \beta^{w-k} \alpha^k \mathcal{S}_{w-k,F}(\beta^{-1}\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(\alpha^{-1}\phi, \xi; \mathfrak{U}). \quad (2.15)
 \end{aligned}$$

Proof: Let

$$A = \frac{(1 - \mathfrak{U})^2}{4(e^{-\alpha d} - \mathfrak{U})(e^{-\beta d} - \mathfrak{U})} e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}. \quad (2.16)$$

Then, we have

$$\begin{aligned} A &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\alpha^{-1}\zeta, \eta; \mathfrak{U}) \frac{(\alpha d)^w}{F_w!} \sum_{k=0}^{\infty} \mathcal{S}_{k,F}(\beta^{-1}\phi, \xi; \mathfrak{U}) \frac{(\beta d)^k}{F_k!} \\ &= \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \alpha^{w-k} \beta^k \mathcal{S}_{w-k,F}(\alpha^{-1}\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(\beta^{-1}\phi, \xi; \mathfrak{U}) \right) \frac{d^w}{F_w!}. \end{aligned} \quad (2.17)$$

Similarly, we have

$$A = \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \beta^{w-k} \alpha^k \mathcal{S}_{w-k,F}(\beta^{-1}\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(\alpha^{-1}\phi, \xi; \mathfrak{U}) \right) \frac{d^w}{F_w!}. \quad (2.18)$$

By virtue of (2.17) and (2.18), we attain the result. \square

Theorem 2.7 *Let $w \geq 0$ be an integer. Then we have*

$$\mathfrak{U} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) = \mathcal{S}_{w,F}(\zeta - 1, \eta; \mathfrak{U}) - \frac{(1 - \mathfrak{U})}{2} (\zeta + \eta)_F^w. \quad (2.19)$$

Proof: Suppose

$$\frac{\mathfrak{U}}{(e_F^{-d} - \mathfrak{U})e_F^{-d}} = \frac{1}{e_F^{-d} - \mathfrak{U}} - \frac{1}{e_F^{-d}}.$$

Evaluating the following fraction using the above identity, we find

$$\begin{aligned} \frac{\mathfrak{U}(1 - \mathfrak{U})e_F^{\zeta d} E_F^{\eta d}}{2(e_F^{-d} - \mathfrak{U})e_F^{-d}} &= \frac{(1 - \mathfrak{U})e_F^{\zeta d} E_F^{\eta d}}{2(e_F^{-d} - \mathfrak{U})} - \frac{(1 - \mathfrak{U})e_F^{\zeta d} E_F^{\eta d}}{2e_F^{-d}} \\ \mathfrak{U} \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{w!} &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta - 1, \eta; \mathfrak{U}) \frac{d^w}{F_w!} - \frac{(1 - \mathfrak{U})}{2} \sum_{w=0}^{\infty} (\zeta + \eta)_F^w \frac{d^w}{F_w!}. \end{aligned}$$

Hence, the proof of the theorem is complete. \square

Corollary 2.2 *On setting $\mathfrak{U} = -1$ in Theorem 2.7, we have*

$$(\zeta + \eta)_F^w = \mathcal{S}_{w,F}(\zeta, \eta; -1) + \mathcal{S}_{w,F}(\zeta - 1, \eta; -1).$$

Theorem 2.8 *Let $w \geq 0$ be an integer. Then we have*

$$\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(a\zeta, \eta; \mathfrak{U}) \mathcal{H}_{k,F}(b\phi, \xi; \mathfrak{U})}{a^{w-k} b^k} = \sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(b\zeta, \eta; \mathfrak{U}) \mathcal{H}_{k,F}(a\phi, \xi; \mathfrak{U})}{b^{w-k} a^k}, \quad (2.20)$$

where $\mathcal{H}_{w,F}(\zeta, \eta; \mathfrak{U})$ is the Frobenius–Euler–Fibonacci polynomials.

Proof: Let

$$B(d) = \frac{(1 - \mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e_F^{-\frac{d}{a}} - \mathfrak{U})(e_F^{-\frac{d}{b}} - \mathfrak{U})}.$$

Now, we can write

$$B(d) = 2 \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(a\zeta, \eta; \mathfrak{U}) \frac{d^w}{a^w F_w!} \sum_{k=0}^{\infty} \mathcal{H}_{k,F}(b\phi, \xi; \mathfrak{U}) \frac{d^k}{b^k F_k!}$$

$$= 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(a\zeta, \eta; \mathfrak{U}) \mathcal{H}_{k,F}(b\phi, \xi; \mathfrak{U})}{a^{w-k} b^k} \right) \frac{d^w}{F_w!}. \quad (2.21)$$

Also, we can transform the form $B(d)$ as follows:

$$B(d) = 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(b\zeta, \eta; \mathfrak{U}) \mathcal{H}_{k,F}(a\phi, \xi; \mathfrak{U})}{b^{w-k} a^k} \right) \frac{d^w}{F_w!}. \quad (2.22)$$

By virtue of (2.21) and (2.22), we obtain the result. \square

Corollary 2.3 *Letting $a = 1$ in Theorem 2.8, we get*

$$\sum_{k=0}^w \binom{w}{k} \frac{\mathcal{S}_{w-k}(\xi; \mathfrak{U}) \mathcal{H}_k(b\eta; \mathfrak{U})}{b^k} = \sum_{k=0}^w \binom{w}{k} \frac{\mathcal{S}_{w-k}(b\xi; \mathfrak{U}) \mathcal{H}_k(\eta; \mathfrak{U})}{b^{w-k}}.$$

Theorem 2.9 *Let $a, b > 0$ and $a \neq b$. Then*

$$\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(a\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(b\phi, \xi; \mathfrak{U})}{a^{w-k} b^k} = \sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(b\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(a\phi, \xi; \mathfrak{U})}{b^{w-k} a^k}. \quad (2.23)$$

Proof: Let

$$C(d) = \frac{(1 - \mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e^{-\frac{d}{a}} - \mathfrak{U})(e^{-\frac{d}{b}} - \mathfrak{U})}.$$

Now, we can write

$$\begin{aligned} C(d) &= 4 \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(a\zeta, \eta; \mathfrak{U}) \frac{d^w}{a^w F_w!} \sum_{k=0}^{\infty} \mathcal{S}_{k,F}(b\phi, \xi; \mathfrak{U}) \frac{d^k}{b^k F_k!} \\ &= 4 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(a\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(b\phi, \xi; \mathfrak{U})}{a^{w-k} b^k} \right) \frac{d^w}{F_w!}. \end{aligned} \quad (2.24)$$

Also, we can transform the form $C(d)$ as follows:

$$C(d) = 4 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(b\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(a\phi, \xi; \mathfrak{U})}{b^{w-k} a^k} \right) \frac{d^w}{F_w!}. \quad (2.25)$$

In virtue of (2.24) and (2.25), we obtain the result. \square

Theorem 2.10 *Let $a, b > 0$ and $a \neq b$. Then*

$$\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(-a\zeta, \eta; \mathfrak{U}) \mathcal{H}_k(b\phi, \xi; \mathfrak{U})}{(-a)^{w-k} b^k} = \sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(b\zeta, \eta; \mathfrak{U}) \mathcal{S}_{k,F}(\phi, \xi; \mathfrak{U})}{b^{w-k} a^k}. \quad (2.26)$$

Proof: Let

$$E(d) = \frac{(1 - \mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e^{\frac{d}{a}} + \mathfrak{U})(e^{\frac{d}{b}} - \mathfrak{U})}.$$

Now, we can write

$$E(d) = 2 \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(-a\zeta, \eta; \mathfrak{U}) \frac{d^w}{(-a)^w F_w!} \sum_{k=0}^{\infty} \mathcal{H}_{k,F}(b\phi, \xi; \mathfrak{U}) \frac{d^k}{b^k F_k!}$$

$$= 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(-a\zeta, \eta; \mathfrak{U}) \mathcal{H}_k(b\phi, \xi; \mathfrak{U})}{(-a)^{w-k} b^k} \right) \frac{d^w}{F_w!}. \quad (2.27)$$

Also, we can transform the form B as follows:

$$E(d) = 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F \frac{\mathcal{S}_{w-k,F}(-b\zeta, \eta; \mathfrak{U}) \mathcal{H}_{k,F}(a\phi, \xi; \mathfrak{U})}{b^{w-k} a^k} \right) \frac{d^w}{F_w!}. \quad (2.28)$$

By (2.27) and (2.28), we obtain the result. \square

3. Parametric type of Frobenius–Sigmoid–Fibonacci Numbers and Polynomials

This section extends the two-variable Sigmoid–Fibonacci polynomials by integrating parametric trigonometric elements through the application of the golden cosine and sine functions. Let $p, q \in \mathbb{R}$. The Taylor series of the functions $e_F^{pd} \cos_F(qd)$ and $e_F^{pd} \sin_F(qd)$ are given as follows [17]:

$$e_F^{pd} \cos_F(qd) = \sum_{w=0}^{\infty} \mathcal{C}_{w,F}(p, q) \frac{d^w}{F_w!}, \quad (3.1)$$

and

$$e_F^{pd} \sin_F(qd) = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(p, q) \frac{d^w}{F_w!}, \quad (3.2)$$

where

$$\mathcal{C}_{w,F}(p, q) = \sum_{k=0}^{\lfloor \frac{w}{2} \rfloor} (-1)^k \binom{w}{2k}_F p^{w-2k} q^{2k}, \quad (3.3)$$

$$\mathcal{S}_{w,F}(p, q) = \sum_{k=0}^{\lfloor \frac{w-1}{2} \rfloor} (-1)^k \binom{w}{2k+1}_F p^{w-2k-1} q^{2k+1}. \quad (3.4)$$

With the help of Equation (2.1), we define Frobenius–Sigmoid–Fibonacci polynomials represented by the following generating function:

$$\frac{1 - \mathfrak{U}}{2(e^{-d} - \mathfrak{U})} e_F^{\zeta d} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta; \mathfrak{U}) \frac{d^w}{F_w!}. \quad (3.5)$$

If we substitute $\zeta = 0$ into Equation (3.5), we obtain Frobenius–Sigmoid–Fibonacci numbers as follows:

$$\frac{1 - \mathfrak{U}}{2(e^{-d} - \mathfrak{U})} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\mathfrak{U}) \frac{d^w}{F_w!}. \quad (3.6)$$

Based on the aforementioned definitions of $\mathcal{C}_{w,F}(p, q)$ and $\mathcal{S}_{w,F}(p, q)$, as well as the number of a $\mathcal{S}_{w,F}$ we can specify two parameters for Frobenius–Sigmoid–Fibonacci polynomials as follows:

Definition 3.1 Two parametric families of the Frobenius–Sigmoid–Fibonacci polynomials are hereby defined via the following generating functions:

$$\frac{1 - \mathfrak{U}}{2(e^{-d} - \mathfrak{U})} e_F^{\zeta d} \cos_F \eta d = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!}, \quad (3.7)$$

and

$$\frac{1 - \mathfrak{U}}{2(e^{-d} - \mathfrak{U})} e_F^{\zeta d} \sin_F \eta d = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(s)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!}. \quad (3.8)$$

Based on the aforementioned definitions, we have arrived at the following principal results.

Theorem 3.1 *Let $w \geq 0$ be an integer. Then*

$$\mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) = \sum_{l=0}^w \binom{w}{l}_F \mathcal{S}_{l,F}(\mathfrak{U}) \mathcal{C}_{w-l,F}(\zeta, \eta), \quad (3.9)$$

and

$$\mathcal{S}_{w,F}^{(s)}(\zeta, \eta; \mathfrak{U}) = \sum_{l=0}^w \binom{w}{l}_F \mathcal{S}_{l,F}(\mathfrak{U}) \mathcal{S}_{w-l,F}(\zeta, \eta). \quad (3.10)$$

Proof: Using (3.1) and (3.5), we have

$$\begin{aligned} \frac{1 - \mathfrak{U}}{2(e^{-d} - \mathfrak{U})} e_F^{\zeta d} \cos_F(\eta d) &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\mathfrak{U}) \frac{d^w}{F_w!} \sum_{w=0}^{\infty} \mathcal{C}_w(\zeta, \eta) \frac{d^w}{F_w!} \\ &= \sum_{w=0}^{\infty} \left(\sum_{l=0}^w \binom{w}{l}_F \mathcal{S}_{l,F}(\mathfrak{U}) \mathcal{C}_{w-l}(\zeta, \eta) \right) \frac{d^w}{F_w!} \\ \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} &= \sum_{w=0}^{\infty} \left(\sum_{l=0}^w \binom{w}{l}_F \mathcal{S}_{l,F}(\mathfrak{U}) \mathcal{C}_{w-l,F}(\zeta, \eta) \right) \frac{d^w}{F_w!}. \end{aligned} \quad (3.11)$$

By equating the coefficients of $\frac{d^w}{F_w!}$ on both sides of the above equation, we derive the intended result (3.9). Equation (3.10) can be obtain in a similar manner. \square

Theorem 3.2 *The following identities hold for:*

$$\mathcal{S}_{w,F}^{(c)}(\zeta + \theta, \eta; \mathfrak{U}) = \sum_{l=0}^w \binom{w}{l}_F (-1)^{\binom{l}{2}} \mathcal{S}_{w-l,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \theta^l, \quad (3.12)$$

and

$$\mathcal{S}_{w,F}^{(s)}(\zeta + \theta, \eta; \mathfrak{U}) = \sum_{l=0}^w \binom{w}{l}_F (-1)^{\binom{l}{2}} \mathcal{S}_{w-l,F}^{(s)}(\zeta, \eta; \mathfrak{U}) \theta^l. \quad (3.13)$$

Proof: Utilizing (3.5), we obtain the following functional equation through derivation:

$$\begin{aligned} \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta + \theta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} &= \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{\zeta d} E_F^{\theta d} \cos_F \eta d \\ &= \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} \sum_{w=0}^{\infty} (-1)^{\binom{w}{2}} \theta^w \frac{d^w}{F_w!} \\ &= \sum_{w=0}^{\infty} \left(\sum_{l=0}^w \binom{w}{l}_F (-1)^{\binom{l}{2}} \mathcal{S}_{w-l,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \theta^l \right) \frac{d^w}{F_w!}. \end{aligned}$$

By equating the coefficients of $\frac{d^w}{F_w!}$ on both sides of the aforementioned equation, it can be observed that

$$\mathcal{S}_{w,F}^{(c)}(\zeta + \theta, \eta; \mathfrak{U}) = \sum_{l=0}^w \binom{w}{l}_F (-1)^{\binom{l}{2}} \mathcal{S}_{w-l,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \theta^l.$$

The proof of the result (3.12) is provided. The assertion (3.13) can also be established in a similar manner. \square

Theorem 3.3 (*F-Derivative Formulas*) *It is known that the following identities hold for:*

$$\frac{\partial_F}{\partial_F \zeta} \left(\mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) = F_w \mathcal{S}_{w-1,F}^{(c)}(\zeta, \eta; \mathfrak{U}), \quad (3.14)$$

$$\frac{\partial_F}{\partial_F \zeta} \left(\mathcal{S}_{w,F}^{(s)}(\zeta, \eta; \mathfrak{U}) \right) = F_w \mathcal{S}_{w-1,F}^{(s)}(\zeta, \eta; \mathfrak{U}), \quad (3.15)$$

$$\frac{\partial_F}{\partial_F \eta} \left(\mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) = -F_w \mathcal{S}_{w-1,F}^{(s)}(\zeta, \eta; \mathfrak{U}), \quad (3.16)$$

$$\frac{\partial_F}{\partial_F \eta} \left(\mathcal{S}_{w,F}^{(s)}(\zeta, \eta; \mathfrak{U}) \right) = F_w \mathcal{S}_{w-1,F}^{(c)}(\zeta, \eta; \mathfrak{U}). \quad (3.17)$$

Proof: Utilizing the Golden derivative operator $\frac{\partial_F}{\partial_F \zeta}$, and using (1.19) and (3.5), we have

$$\begin{aligned} \sum_{w=0}^{\infty} \frac{\partial_F}{\partial_F \zeta} \left(\mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) \frac{d^w}{F_w!} &= \frac{\partial_F}{\partial_F \zeta} \left\{ \frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{\zeta d} \cos_F \eta d \right\} = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^{w+1}}{F_w!} \\ &= \sum_{w=1}^{\infty} \mathcal{S}_{w-1,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_{w-1}!}. \end{aligned}$$

Since $\frac{\partial_F}{\partial_F \zeta} \left(\mathcal{S}_{0,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) = 0$; for $w \geq 1$, by analyzing the coefficients of $\frac{d^w}{F_w!}$ on both sides of the final equation, the desired result is achieved in (3.14). The remaining results can then be easily deduced. \square

Theorem 3.4 *The following identities holds true:*

$$C_{w,F}(\zeta, \eta) = \frac{2}{1 - \mathfrak{U}} \left[\sum_{k=0}^w \binom{w}{k}_F (-1)^{w-k} \mathcal{S}_{k,F}^{(c)}(\zeta, \eta; \mathfrak{U}) - \mathfrak{U} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right], \quad (3.18)$$

and

$$S_{w,F}(\zeta, \eta) = \frac{2}{1 - \mathfrak{U}} \left[\sum_{k=0}^w \binom{w}{k}_F (-1)^{w-k} \mathcal{S}_{k,F}^{(s)}(\zeta, \eta; \mathfrak{U}) - \mathfrak{U} \mathcal{S}_{w,F}^{(s)}(\zeta, \eta; \mathfrak{U}) \right]. \quad (3.19)$$

Proof: By using (3.5), we have

$$\frac{1 - \mathfrak{U}}{2(e_F^{-d} - \mathfrak{U})} e_F^{\zeta d} \cos_F(\eta d) = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!},$$

and

$$\begin{aligned} (1 - \mathfrak{U}) e_F^{\zeta d} \cos_F(\eta d) &= 2(e_F^{-d} - \mathfrak{U}) \sum_{k=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} \\ &= 2 \sum_{w=0}^{\infty} (-1)^w \frac{d^w}{F_w!} \sum_{k=0}^{\infty} \mathcal{S}_{k,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^k}{F_k!} - 2\mathfrak{U} \sum_{w=0}^{\infty} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \frac{d^w}{F_w!} \\ &= 2 \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F (-1)^{w-k} \mathcal{S}_{k,F}^{(c)}(\zeta, \eta; \mathfrak{U}) - \mathfrak{U} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) \frac{d^w}{F_w!}. \end{aligned}$$

From (3.1), we get

$$\sum_{w=0}^{\infty} C_{w,F}(\zeta, \eta) \frac{d^w}{F_w!} = \frac{2}{1 - \mathfrak{U}} \sum_{w=0}^{\infty} \left(\sum_{k=0}^w \binom{w}{k}_F (-1)^{w-k} \mathcal{S}_{k,F}^{(c)}(\zeta, \eta; \mathfrak{U}) - \mathfrak{U} \mathcal{S}_{w,F}^{(c)}(\zeta, \eta; \mathfrak{U}) \right) \frac{d^w}{F_w!}.$$

By equating the coefficients of $\frac{d^w}{F_w!}$, we obtain the desired result. Equation (3.19) can be derived in a similar manner. \square

4. Conclusion

In this article, we have introduced the Frobenius–Sigmoid–Fibonacci polynomials as the F -analogs of classical Frobenius–Sigmoid polynomials, formulated in two-variable within the approach of Golden Calculus. Building upon and extending the results presented in the previously published study (see [17]), we have established the generating functions of these polynomials and derived several fundamental properties, including recurrence relations, summation identities, and derivative formulas. Furthermore, we have defined the Frobenius–Sigmoid–Fibonacci numbers associated with these polynomials and explored their essential properties and interrelations. Our findings provide a unified and generalized perspective on the interplay between Frobenius–Sigmoid–type structures and Fibonacci-based frameworks, opening a new direction for the study of special polynomials. This framework not only deepens the theoretical understanding of polynomial families under the Golden Calculus but also creates potential bridges to diverse areas such as combinatorics, number theory, approximation theory, and applied mathematical modeling.

References

1. H. Alsamir, M.S.M. Noorani, W. Shatanawi, H. Aydi and H. Qawaqneh, *Fixed point results in metric-like spaces via σ -simulation functions*, European Journal of Pure and Applied Mathematics, 12(1), 88-100, (2019).
2. H. Alsamir, H. Qawaqneh, G. Al-Musannef and R. Khalil, *Common Fixed Point of Generalized Berinde Type Contraction and an Application*, European Journal of Pure and Applied Mathematics, 17(4), 2492-2504, (2024).
3. A. Z. Border, *The r -Stirling numbers*, Disc. Math., 49, 241-259, (1984).
4. H. K. Kwan, *Simple sigmoid-like activation function suitable for digital hardware implementation*, Stat. Methods Med. Res., 28, 1379-1380, (1992). doi:10.1049/el:19920877.
5. J. Han, C. Moraga, *The influence of the sigmoid function parameters on the speed of backpropagation learning*, Int. Workshop Artif. Neural Netw., 930, 195-201, (2005).
6. J. Han, R. S. Wilson, S. E. Leurgans, *Sigmoidal mixed models for longitudinal data*, Stat. Methods Med. Res., 27, 863-875, (2018). <https://doi.org/10.1177/0962280216645632>.
7. J. Y. Kang, *Some Properties and Distribution of the Zeros of the q -Sigmoid Polynomials*, Discret. Dyn. Nat. Soc., 2020, 1-10, (2020).
8. J. Y. Kang, *Determination of the properties of (p, q) -sigmoid polynomials and the structure of their roots*, In *Number Theory and Its Applications*, IntechOpen: London, UK), 1-19, (2020). <http://dx.doi.org/10.5772/intechopen.91862>.
9. J. Y. Kang, *Explicit properties of q -sigmoid polynomials combining q -cosine function*, J. Appl. Math. Inform., 39, 541-552, (2021).
10. J. Y. Kang, *Some relationships between sigmoid polynomials and other polynomials*, J. Pure Appl. Math., 1, 57-67, (2019).
11. W. A. Khan, C. Kızılateş, *Some properties of Frobenius–Sigmoid polynomials*, In: Duman, O., Erkus-Duman, E. (eds) *Approximation Theory and Special Functions. ATSF 2024*. Springer Proceedings in Mathematics & Statistics, vol 503. Springer, Cham. 2026.
12. O. K. Pashaev, S. Nalci, *Golden quantum oscillator and Binet–Fibonacci calculus*, J. Phys A: Math. Theor., 45, 1-19, (2011).
13. O. K. Pashaev, *Quantum calculus of Fibonacci divisors and infinite hierarchy of bosonic–fermionic golden quantum oscillators*, Internat. J. Geom. Methods Modern Phys., 18, 2150075, (2021).
14. E. Krot, *An introduction to finite fibonomial calculus*, Centr. Eur. J. Math., 2, 754-766, (2004).
15. M. Özvatan, *Generalized golden-Fibonacci calculus and applications*, Ph.D. thesis, Izmir Institute of Technology, (2018).
16. S. Kuş, N. Tuğlu, T. Kim, *Bernoulli F -polynomials and Fibo–Bernoulli matrices*, Adv. Differ. Equ., 2019, 145, (2019).
17. M. S. Alatawi, W. A. Khan, C. Kızılateş, C. S. Ryoo, *Some Properties of Generalized Apostol–Type Frobenius–Euler–Fibonacci Polynomials*, Mathematics, 12, 800, (2024).
18. H. Guan, W. A. Khan, C. Kızılateş, C. S. Ryoo, *On Certain Properties of Parametric Kinds of Apostol–Type Frobenius–Euler–Fibonacci Polynomials*, Axioms, 13, 348, (2024).
19. N. Tuğlu, E. Ercan, *Some properties of Apostol Bernoulli Fibonacci and Apostol Euler Fibonacci Polynomials*, Icmee-2021, 32-34.
20. O. Dişkaya, *Fibonacci–based generalizations of degenerate Stirling numbers*, Ramanujan J., 68, 51, (2025).
21. H. Qawaqneh, *Fractional analytic solutions and fixed point results with some applications*, Adv. Fixed Point Theory, 14(1), 2024.

22. H. Qawaqneh, M.S.Md. Noorani, H. Aydi, A. Zraiqat, A.H. Ansari, *On fixed point results in partial b-metric spaces*, Journal of Function Spaces, 2021, 8769190, 9 pages, (2021).
23. H. Qawaqneh, M.S.Md. Noorani, H. Aydi, *Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications*, AIMS Mathematics, 8(3), 6682-6696, (2023).
24. H. Qawaqneh, H.A. Hammad and H. Hassen Aydi, *Exploring new geometric contraction mappings and their applications in fractional metric spaces*, AIMS Mathematics, 9(1), 521-541, (2024).
25. H. Qawaqneh, *New Functions For Fixed Point Results In Metric Spaces With Some Applications*, Indian Journal of Mathematics, 66(1), 55-84, (2024).
26. H. Qawaqneh, M.S. Noorani, H. Aydi, and W. Shatanawi, *On Common Fixed Point Results for New Contractions with Applications to Graph and Integral Equations*, Mathematics, 7(11), 1082, (2019).
27. M. Elbes, T. Kanan, M. Alia and M. Ziad, *COVID-19 detection platform from X-ray images using deep learning*, International Journal of Advances in Soft Computing and its Applications, 14(1), 197-211, (2022).
28. T. Kanan, M. Elbes, K. Abu Maria, M. Alia, *Exploring the potential of IoT-based learning environments in education*, International Journal of Advances in Soft Computing and its Applications, 15 (2), (2023).
29. I. M. Batiha, S. A. Njadat, R. M. Batyha, A. Zraiqat, A. Dababneh and Sh. Momani, *Design fractional-order PID controllers for single-joint robot arm model*, International Journal of Advances in Soft Computing and its Applications, 14(2), 96-114, (2022).
30. A. Aledamat, W. A. Khan, U. Duran, C. S. Ryoo, *Construction for q-hypergeometric Bernoulli polynomials of a complex variable with applications computer modeling*, Advanced Mathematical Models and Applications, 9(1), 68-92, (2024).
31. Z. Chen, N. Omur, S. Koparal, W. A. Khan, *Some identities with multi generalized q-hyper harmonic numbers of order r*, Symmetry, 15:917, 1-10, (2023).
32. H. Guan, N. Omur, S. Koparal, W. A. Khan, *q-analogue of $H_{n,m}$ and their applications*, Mathematics, 15:943, 1-10, (2023).
33. W. A. Khan, M. S Alatawi, C. S. Ryoo, U. Duran, *Novel properties of q-Sine based and q-Cosine-based q-Fubini polynomials*, Symmetry, 15:356, 1-18, (2023).
34. W. A. Khan, *A note on q-analogue of degenerate Catalan numbers associated p-adic integral on Z_p* , Symmetry, 14:119, 1-10, (2022). doi.org/10.3390/sym14061119.
35. W. A. Khan, *A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials*, Journal of Mathematics, Article ID 9486880, 9 pages, (2022). https://doi.org/10.1155/2022/9486880.
36. Y. Rao, W. A. Khan, S. Araci, C. S. Ryoo, *Explicit properties of Apostol-type Frobenius-Euler polynomials involving q-trigonometric function with applications in computer modeling*. Mathematics, 11:2386, 1-21, (2023).
37. W. A. Khan, *A note on q-analogue of degenerate Catalan numbers associated p-adic integral on Z_p* , Symmetry, 14(119), 1-10, (2022). doi.org/10.3390/sym14061119.
38. W. A. Khan, M. S Alatawi, C. S. Ryoo, U. Duran, *Novel properties of q-Sine based and q-Cosine-based q-Fubini polynomials*, Symmetry, 15:356, 1-18, (2023).

Haitham Qawaqneh,
Department of Basic Sciences,
Al-Zaytoonah University of Jordan, Amman 11733, Jordan
E-mail address: h.alqawaqneh@zuj.edu.jo

and

Waseem Ahmad Khan,
Department of Electrical Engineering,
Prince Mohammad Bin Fahd University,
P.O Box 1664, Al Khobar 31952
Saudi Arabia.
E-mail address: wkhan1@pmu.edu.sa

and

Musharraf Ali
Department of Mathematics,

Gandhi Faiz-E-Aam College

Shahjahanpur-242001, Affiliated to Mahatma Jyotiba Phule Rohilkhand University,

Bareilly 243006

E-mail address: drmusharrafali@gmail.com

and

Ulfat Ansari

Department of Mathematics,

Gandhi Faiz-E-Aam College

Shahjahanpur-242001, Affiliated to Mahatma Jyotiba Phule Rohilkhand University,

Bareilly 243006

E-mail address: ansari.ulfat96@gmail.com