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μ-ξ-$\mathcal{I}$-Open Sets in Ideal Generalized Topological Spaces*

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ABSTRACT: In this paper, we introduce a new class of generalized-$\mathcal{I}$-open sets namely, μ-ξ-$\mathcal{I}$-open sets and specifically, we focus the μ-β-$\mathcal{I}$-open sets. Further, we define the generalized-$\mathcal{I}$-operators such as μ-ξ-$\mathcal{I}$-interior and μ-ξ-$\mathcal{I}$-closure and analyze the ideal generalized topological concepts through these operators. Moreover, we obtain the various types of ideal generalized continuous such as (μ-ξ-$\mathcal{I}$, ν)-continuous and (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous and discuss their relationships between them.

Keywords: μ-ξ-$\mathcal{I}$-open, μ-ξ-$\mathcal{I}$-closed, (μ-ξ-$\mathcal{I}$, ν)-continuous and (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous.

Contents

1. Introduction

A. Csaszar [3,4,5,6,7,8,9] introduced the concept of generalized topology and he obtained the concepts of μ-open sets, μ-ξ-open sets (where "ξ" stands for α, semi, pre and β). Further he defined the operators μ-interior (resp. μ-ξ-interior) and μ-closure (resp. resp. μ-ξ-closure) in generalized topological spaces. Also, he initiated generalized continuity such as (μ, ν)-continuous, (μ-ξ, ν)-continuous and (μ-ξ, ν-ξ)-continuous and studied their essential relationships between the functions through the notions of μ-open sets and μ-ξ-open sets in generalized topological spaces. Saravanakumar et.al [20,21,26] initiated a $\tilde{\gamma}$-open (resp. γ^*-pre-open) set in operation topology and $\tilde{\mu}$-open set in generalized topology and studied $\tilde{\gamma}$-T_i (resp. γ^*-pre-T_i, $\tilde{\mu}$-T_i) ($i = 0, \frac{1}{2}, 1, 2$) spaces using through the $\tilde{\gamma}$-open (resp. γ^*-pre-open, $\tilde{\mu}$-open) and $\tilde{\gamma}$-closed (resp. γ^*-pre-closed, $\tilde{\mu}$-closed) sets. Saravanakumar et.al[20 - 29] created various types of operation continuous mappings in operation topology as well as generalized continuous in generalized topology and discussed some important properties. Abd El-Monsef et.al [1,2] defined the β-open (resp. $\mathcal{I}$-open) sets and β-continuous (resp. $\mathcal{I}$-continuous) functions in topological space. In [9,10,14,15,16,18,19,31,34], introduced semiopen, preopen, α-open, regular-open, g-open, δ-open, θ-open sets and semicontinuous, precontinuous, α-continuous, regular-continuous, g-continuous, δ-continuous, θ-continuous functions in topological spaces and analyzed their important properties. In [11,12,13], created new topologies by some local operators and generated innovative open sets. Kuratowski [13] and Vaidyanathaswamy [33] defined new concept of topologies namely ideal topological spaces and investigated some of their essential properties by some local operators. Jankovic et.al [12], generated the concept of ideal topological spaces and introduced $\mathcal{I}$-open sets by $()^*$ operator. Modak [12], applied the concept of ideal in generalized topological space and he created new types of open sets namely, μ^*-open [17] and μ^*-closed [17].

In this paper, we introduced the concept of generalized-$\mathcal{I}$-open sets such as μ-$\mathcal{I}$-open, almost-μ-$\mathcal{I}$-open, μ-semi-$\mathcal{I}$-open, μ-α-$\mathcal{I}$-open, μ-pre-$\mathcal{I}$-open, μ-regular-$\mathcal{I}$-open, μ-β-$\mathcal{I}$-open in ideal generalized topological spaces. Mainly, we focused the concept of μ-β-$\mathcal{I}$-open sets and discussed their relationship

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with some of other generalized-$\mathcal{I}$-open sets. Generally, we represented the generalized-$\mathcal{I}$-open sets as μ-ξ-open sets (where "ξ" stands for α, semi, pre and β) and characterized ideal generalized topological spaces by some operators namely, μ-ξ-$\mathcal{I}$-interior and μ-ξ-$\mathcal{I}$-closure. Also, we defined ideal generalized continuous such as (μ-ξ-$\mathcal{I}$, ν)-continuous and (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous and analyzed some of their essential characterizations. Moreover, we proved that (μ-β-$\mathcal{I}$, ν)-continuous and (μ^*_d-$\mathcal{I}$, ν)-continuous implies that the concepts of (almost-μ-$\mathcal{I}$, ν)-continuous and proved that X is a μ-ξ-$\mathcal{I}$-T_2 space whenever $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is a (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous injective function and Y is ν-ξ-$\mathcal{J}$-T_2.

2. Preliminaries

Let $X \neq \emptyset$ and let $\mathcal{P}(X)$ be the power set of X. Then $\mu \subseteq \mathcal{P}(X)$ is said to be a generalized topology [3] (shortly, GT) on X if and only if $\emptyset \in \mu$, and U_i $(\neq \emptyset) \in \mu$ for $i \in J$ implies that $\bigcup_{i\in J} U_i \in \mu$. The pair (X, μ) is called a generalized topological space [3] (shortly, GTS) and the elements of μ are called μ-open sets [3]. For a subset $A \subseteq X$, $X \setminus A$ is called a μ-closed set if A is μ-open; μ-interior [4] of A is defined by $i_\mu(A) = \cup\{U : U \in \mu \text{ and } U \subseteq A\}$ (clearly, it is the largest μ-open set contained in A; μ-closure [4] of A is defined by $c_\mu(A) = \cap\{V : X \setminus V \in \mu \text{ and } A \subseteq V\}$ (clearly, the smallest μ-closed set containing A). The operators i_μ and c_μ satisfies the properties [4]: $i_\mu(A) \subseteq A$; $c_\mu(A) \supseteq A$; $i_\mu(i_\mu(A)) = i_\mu(A)$; $c_\mu(c_\mu(A)) = c_\mu(A)$; $X \setminus i_\mu(A) = c_\mu(X \setminus A)$; $X \setminus c_\mu(A) = i_\mu(X \setminus A)$. For $A \subseteq X$ and $U \in \mu$, $i_\mu(A \cup U) \subseteq i_\mu(A) \cup U$ and $c_\mu(A) \cap U \subseteq c_\mu(A \cap U)$. A subset A of X is said to be a μ-semi-open [5] (resp. μ-α-open [5], μ-preopen [5], μ-regular open [31], μ-β-open [5]) set if $A \subseteq c_\mu(i_\mu(A))$ (resp. $A \subseteq i_\mu(c_\mu(i_\mu(A)))$, $A \subseteq i_\mu(c_\mu(A))$, $A = i_\mu(c_\mu(A))$, $A \subseteq c_\mu(i_\mu(c_\mu(A))))$. The complement of a μ-semi-open (resp. μ-α-open, μ-preopen, μ-regularopen, μ-β-open) set is said to be μ-semi-closed (resp. μ-α-closed, μ-preclosed, μ-regularclosed, μ-β-closed). The family of μ-semi-open (resp. μ-α-open, μ-preopen, μ-regular open, μ-β-open) sets is denoted by $\mu SO(X)$ (resp. $\mu\alpha O(X)$, $\mu PO(X)$, $\mu RO(X)$, $\mu\beta O(X)$). For a subset $A \subseteq X$, μ-ξ-interior [5] of A is defined by $i_{\xi_\mu}(A) = \cup\{U : U \in \mu\xi O(X) \text{ and } U \subseteq A\}$ (clearly, it is the largest μ-ξ-open set contained in A; μ-ξ-closure [5] of A is defined by $c_{\xi_\mu}(A) = \cap\{V : X \setminus V \in \mu\xi O(X) \text{ and } A \subseteq V\}$ (clearly, the smallest μ-ξ-closed set containing A), (where "ξ" stands for α, semi, pre and β). According to [8], let μ be a generalized topology on X, $A \subseteq X$ and $x \in X$, then (i) $x \in c_{\xi_\mu}(A)$ if and only if $M \cap A \neq \emptyset$ for each μ-ξ-open set M containing x; (ii) $c_{\xi_\mu}(X \setminus A) = X \setminus i_{\xi_\mu}(A)$ and (iii) $c_{\xi_\mu}(c_{\xi_\mu}) = c_{\xi_\mu}$. Let (X, μ) and (Y, ν) be any two generalized topological spaces with the generalized topologies μ, ν respectively. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (i) (μ, ν)-continuous [5] if for any $V \in \nu$, $f^{-1}(V) \in \mu$; (ii) (μ-ξ, ν)-continuous [5] if for any $V \in \nu$, $f^{-1}(V) \in \mu\xi O(X)$; (iii) ($\mu$-$\xi$, ν-ξ)-continuous [5] if for any $V \in \nu\xi O(Y)$, $f^{-1}(V) \in \mu\xi O(X)$, (where "$\xi$" stands for α, semi, pre and β).

An ideal $\mathcal{I}$ [13,33] on X is a nonempty collection $\mathcal{I} \subseteq \mathcal{P}(X)$ satisfying the following conditions: (i) $A \subseteq B$, $B \in \mathcal{I}$ implies $A \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Let $(X, \mu, \mathcal{I})$ [17] be an ideal generalized topological space (briefly, IGTS) and a generalized local function $()^*_\mu$ [17] is a mapping form $\mathcal{P}(X)$ into $\mathcal{P}(X)$ defined by $A^*_\mu = A^*_\mu(\mathcal{I}) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for each } U \in \mu(x)\}$ where $\mu(x) = \{U \in \mu : x \in U\}$. For $A \subseteq X$, $*$-generalized closure operator $c^*_\mu(A)$ [17] is defined by $c^*_\mu(A) = A \cup A^*_\mu$ and $*$-generalized topology [17] defined by $\mu^* = \mu^*(\mathcal{I}) = \{U \subset X : c^*_\mu(X \setminus U) = X \setminus U\}$. Clearly μ^* is finer than μ. The elements of μ^* are called μ^*-open [17] and their complements μ^*-closed [17].

Definition 2.1 *[17] A subset A of an IGTS $(X, \mu, \mathcal{I})$ is said to be*
(i) μ^*-dense if $A \subseteq A^*_\mu$;
(ii) μ^*-perfect if $A = A^*_\mu$.

Proposition 2.1 *[17] Let $(X, \mu, \mathcal{I})$ be an IGTS and A, $B \subseteq X$. Then*
(i) $(A^*_\mu)^* \subseteq A^*_\mu \subseteq c^*_\mu(A) \subseteq c_\mu(A)$;
(ii) $A^*_\mu \subseteq B^*_\mu$ if $A \subseteq B$;
(iii) $A^*_\mu \in \mu^c$;
(iv) $A^*_\mu(\mathcal{J}) \subseteq A^*_\mu(\mathcal{I})$ if $\mathcal{I} \subseteq \mathcal{J}$;
(v) $U \cap (U \cap A)^*_\mu \subseteq U \cap A^*_\mu$, where $U \in \mu$;
(vi) $(A \setminus I)^*_\mu = A^*_\mu = (A \cup I)^*_\mu$, where $I \in \mathcal{I}$;
(vii) $A^*_\mu = c_\mu(A)$ if $\mathcal{I} = \emptyset$;
(viii) $A^*_\mu = \emptyset$ if $\mathcal{I} = \mathcal{P}(X)$;

(ix) $c^*_\mu(\emptyset) = \emptyset$ *and* $c^*_\mu(X) = X$;
(x) $c^*_\mu(A) \subseteq c^*_\mu(B)$ *if* $A \subseteq B$;
(xi) $c^*_\mu(A) \cup c^*_\mu(B) \subseteq c^*_\mu(A \cup B)$;
(xii) $X \setminus c^*_\mu(A) = i^*_\mu(X \setminus A)$;
(xiii) $X \setminus i^*_\mu(A) = c^*_\mu(X \setminus A)$;
(xiv) $c^*_\mu(c^*_\mu(A)) = c^*_\mu(A)$;
(xv) $i^*_\mu(i^*_\mu(A)) = i^*_\mu(A)$.

3. Generalized-$\mathcal{I}$-open sets

Definition 3.1 *A subset* A *of an IGTS* $(X, \mu, \mathcal{I})$ *is said to be*
(i) μ-$\mathcal{I}$-*open if* $A \subseteq i_\mu(A^*_\mu)$,
(ii) almost-μ-$\mathcal{I}$-*open if* $A \subseteq c_\mu(i_\mu(A^*_\mu))$,
(iii) μ-*semi*-$\mathcal{I}$-*open if* $A \subseteq c^*_\mu(i_\mu(A))$,
(iv) μ-α-$\mathcal{I}$-*open if* $A \subseteq i_\mu(c^*_\mu(i_\mu(A)))$,
(v) μ-*pre*-$\mathcal{I}$-*open if* $A \subseteq i_\mu(c^*_\mu(A))$,
(vi) μ-*regular*-$\mathcal{I}$-*open if* $A = i_\mu(c^*_\mu(A))$,
(vii) μ-β-$\mathcal{I}$-*open if* $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$,

The complement of a μ-$\mathcal{I}$-*open (resp. almost-*μ-$\mathcal{I}$-*open,* μ-*semi*-$\mathcal{I}$-*open,* μ-α-$\mathcal{I}$-*open,* μ-*pre*-$\mathcal{I}$-*open,* μ-*regular*-$\mathcal{I}$-*open,* μ-β-$\mathcal{I}$-*open) set is called a* μ-$\mathcal{I}$-*closed (resp. almost-*μ-$\mathcal{I}$-*closed,* μ-*semi*-$\mathcal{I}$-*closed,* μ-α-$\mathcal{I}$-*closed,* μ-*pre*-$\mathcal{I}$-*closed,* μ-*regular*-$\mathcal{I}$-*closed,* μ-β-$\mathcal{I}$-*closed).*

The family of μ-$\mathcal{I}$-*open (resp. almost-*μ-$\mathcal{I}$-*open,* μ-*semi*-$\mathcal{I}$-*open,* μ-α-$\mathcal{I}$-*open,* μ-*pre*-$\mathcal{I}$-*open,* μ-*regular*-$\mathcal{I}$-*open,* μ-β-$\mathcal{I}$-*open) sets is denoted by* $\mu\mathcal{I}O(X)$ *(resp.* $A\mu\mathcal{I}O(X)$, $\mu S\mathcal{I}O(X)$, $\mu\alpha\mathcal{I}O(X)$, $\mu P\mathcal{I}O(X)$, $\mu R\mathcal{I}O(X)$, $\mu\beta\mathcal{I}O(X)$*) and the family of* μ-$\mathcal{I}$-*closed (resp. almost-*μ-$\mathcal{I}$-*closed,* μ-*semi*-$\mathcal{I}$-*closed,* μ-α-$\mathcal{I}$-*closed,* μ-*pre*-$\mathcal{I}$-*closed,* μ-*regular*-$\mathcal{I}$-*closed,* μ-β-$\mathcal{I}$-*closed) sets is denoted by* $\mu\mathcal{I}C(X)$ *(resp.* $A\mu\mathcal{I}C(X)$, $\mu S\mathcal{I}C(X)$, $\mu\alpha\mathcal{I}C(X)$, $\mu P\mathcal{I}C(X)$, $\mu R\mathcal{I}C(X)$, $\mu\beta\mathcal{I}C(X)$*).*

Theorem 3.1 *Let* $(X, \mu, \mathcal{I})$ *be an IGTS. Then*
(i) $\mu\mathcal{I}O(X) \subseteq \mu P\mathcal{I}O(X)$,
(ii) $\mu\mathcal{I}O(X) \subseteq A\mu\mathcal{I}O(X) \subseteq \mu\beta\mathcal{I}O(X)$,
(iii) $\mu \subseteq \mu\alpha\mathcal{I}O(X) = \mu S\mathcal{I}O(X) \cap \mu P\mathcal{I}O(X)$,
(iv) $\mu S\mathcal{I}O(X) \subseteq \mu\beta\mathcal{I}O(X)$,
(v) $\mu R\mathcal{I}O(X) \subseteq \mu P\mathcal{I}O(X) \subseteq \mu\beta\mathcal{I}O(X)$,
(vi) $\mu S\mathcal{I}O(X) \cup \mu P\mathcal{I}O(X) \subseteq \mu\beta\mathcal{I}O(X)$.

Proof: (i) Let $A \in \mu\mathcal{I}O(X)$. Then $A \subseteq i_\mu(A^*_\mu) \subseteq i_\mu(A \cup A^*_\mu) = i_\mu(c^*_\mu(A))$. Hence $A \in \mu P\mathcal{I}O(X)$.

(ii) Let $A \in \mu\mathcal{I}O(X)$. Then $A \subseteq i_\mu(A^*_\mu) \subseteq c_\mu(i_\mu(A^*_\mu))$. Hence $A \in A\mu\mathcal{I}O(X)$, this implies that $A \subseteq c_\mu(i_\mu(A^*_\mu)) \subseteq c_\mu(i_\mu(A \cup A^*_\mu)) = c_\mu(i_\mu(c^*_\mu(A)))$. Thus $A \in \mu\beta\mathcal{I}O(X)$.

(iii) Let $A \in \mu$. Then $A \subseteq i_\mu(A) = i_\mu(i_\mu(A)) \subseteq i_\mu(i_\mu(A) \cup (i_\mu(A))^*_\mu) = i_\mu(c^*_\mu(i_\mu(A)))$. Hence $A \in \mu\alpha\mathcal{I}O(X) \Leftrightarrow A \subseteq i_\mu(c^*_\mu(i_\mu(A))) \Leftrightarrow A \subseteq [c^*_\mu(i_\mu(A)) \cap i_\mu(c^*_\mu(A))] \Leftrightarrow A \subseteq c^*_\mu(i_\mu(A))$ and $A \subseteq i_\mu(c^*_\mu(A)) \Leftrightarrow A \in \mu S\mathcal{I}O(X) \cap \mu P\mathcal{I}O(X)$.

(iv) Let $A \in \mu S\mathcal{I}O(X)$. Then $A \subseteq c^*_\mu(i_\mu(A)) \subseteq c_\mu(i_\mu(A))$ (since $c^*_\mu(A) \subseteq c_\mu(A)$) $\subseteq c_\mu(i_\mu(A \cup A^*_\mu)) = c_\mu(i_\mu(c^*_\mu(A)))$. Hence $A \in \mu\beta\mathcal{I}O(X)$.

(v) Let $A \in \mu R\mathcal{I}O(X)$. Then $A = i_\mu(c^*_\mu(A)) \subseteq i_\mu(c^*_\mu(A))$. Hence $A \in \mu P\mathcal{I}O(X)$, this implies that $A \subseteq i_\mu(c^*_\mu(A)) \subseteq c_\mu(i_\mu(c^*_\mu(A)))$. Thus $A \in \mu\beta\mathcal{I}O(X)$.

(vi) Follows from (iv) and (v). □

Note that the following examples shows that the converse of the Theorem 3.1 need not be true.

Example 3.1 Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then (i) the set $\{a, b\} \in \mu P\mathcal{I}O(X)$, but $\{a, b\} \notin \mu\mathcal{I}O(X)$.

(ii) The set $\{a, b\} \in \mu\beta\mathcal{I}O(X)$, but $\{a, b\} \notin A\mu\mathcal{I}O(X)$. Also the set $\{a, d\} \in A\mu\mathcal{I}O(X)$, but $\{a, d\} \notin \mu\mathcal{I}O(X)$.

(iii) The set $\{a, d\} \in \mu S\mathcal{I}O(X)$, but $\{a, d\} \notin \mu\alpha\mathcal{I}O(X)$. As well as the set $\{a, c\} \in \mu P\mathcal{I}O(X)$, but $\{a, c\} \notin \mu\alpha\mathcal{I}O(X)$.

(iv) The set $\{a, c\} \in \mu\beta\mathcal{I}O(X)$, but $\{a, c\} \notin \mu S\mathcal{I}O(X)$.

(v) The set $\{a, d\} \in \mu\beta\mathcal{I}O(X)$, but $\{a, d\} \notin \mu P\mathcal{I}O(X)$. Also $\{a, c\} \in \mu P\mathcal{I}O(X)$, but $\{a, c\} \notin \mu R\mathcal{I}O(X)$.

(vi) The set $\{b, c, d\} \in \mu\beta\mathcal{I}O(X)$, but $\{b, c, d\} \notin \mu S\mathcal{I}O(X) \cup \mu P\mathcal{I}O(X)$.

Example 3.2 Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{a, c, d\} \in \mu\alpha\mathcal{I}O(X)$, but $\{a, c, d\} \notin \mu$.

Remark 3.1 *(i) The concepts of μ (resp. $\mu R\mathcal{I}O(X)$) and $\mu\mathcal{I}O(X)$ (resp. $A\mu\mathcal{I}O(X)$) are independent.* *(ii) The concepts of $\mu S\mathcal{I}O(X)$ and $\mu P\mathcal{I}O(X)$ are independent.*

(i) In Example 3.1, the set $\{a, b\} \in \mu \cap \mu R\mathcal{I}O(X)$, but $\{a, b\} \notin \mu\mathcal{I}O(X) \cup A\mu\mathcal{I}O(X)$. Also the set $\{a, c\} \in \mu\mathcal{I}O(X) \cap A\mu\mathcal{I}O(X)$, but $\{a, c\} \notin \mu \cup \mu R\mathcal{I}O(X)$.

(ii) In Example 3.1, the set $\{a, d\} \in \mu S\mathcal{I}O(X))$, but $\{a, d\} \notin \mu P\mathcal{I}O(X))$. Also the set $\{a, c\} \in \mu P\mathcal{I}O(X))$, but $\{a, c\} \notin \mu S\mathcal{I}O(X))$.

Theorem 3.2 *Let $(X, \mu, \mathcal{I})$ be an IGTS. Then $\mu\xi\mathcal{I}O(X) \subseteq \mu\xi O(X)$, (where "$\xi$" stands for α, semi, pre and β; $\mu\xi\mathcal{I}O(X)$ stands for $\mu\alpha\mathcal{I}O(X)$, $\mu S\mathcal{I}O(X)$, $\mu P\mathcal{I}O(X)$, $\mu\beta\mathcal{I}O(X)$).*

Proof: Since $c^*_\mu(A) \subseteq c_\mu(A)$ and by the Definition 3.1, we have that $\mu\xi\mathcal{I}O(X) \subseteq \mu\xi O(X)$. □

Remark 3.2 *The converse of the Theorem 3.2 need not be true.*

In Example 3.1, the set $\{a, b, d\} \in \mu\alpha O(X)$, but $\{a, b, d\} \notin \mu\alpha\mathcal{I}O(X)$. Also $\{b\} \in \mu PO(X) \cap \mu\beta O(X)$, but $\{b\} \notin \mu P\mathcal{I}O(X) \cup \mu\beta\mathcal{I}O(X)$. In Example 3.2, the set $\{a, c\} \in \mu SO(X)$, but $\{a, c\} \notin \mu S\mathcal{I}O(X)$.

Proposition 3.1 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. If A is μ^*-dense, then (i) $\mu R\mathcal{I}O(X) \subseteq \mu\mathcal{I}O(X) = \mu P\mathcal{I}O(X)$; (ii) $A\mu\mathcal{I}O(X) = \mu\beta\mathcal{I}O(X)$.*

Proof: Follows from the Definitions 2.1 and 3.1. □

Proposition 3.2 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. If A is μ^*-perfect, then (i) $\mu O(X) = \mu\mathcal{I}O(X) = \mu P\mathcal{I}O(X) = \mu R\mathcal{I}O(X)$; (ii) $A\mu\mathcal{I}O(X) = \mu\beta\mathcal{I}O(X) = \mu SO(X)$.*

Proof: Follows from the Definitions 2.1 and 3.1. □

Theorem 3.3 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X.*
(i) If $\mathcal{I} = \emptyset$, then $\mu\beta\mathcal{I}O(X) = \mu\beta O(X)$;
(ii) If $\mathcal{I} = \mathcal{P}(X)$, then $\mu\beta\mathcal{I}O(X) = \mu SO(X)$.

Proof: Follows from the Proposition 2.1. □

Theorem 3.4 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. If $U \in \mu P\mathcal{I}O(X)$ such that $U \subseteq A \subseteq c^*_\mu(U)$, then $A \in \mu\beta\mathcal{I}O(X)$.*

Proof: Suppose that $U \in \mu P\mathcal{I}O(X)$ such that $U \subseteq A \subseteq c^*_\mu(U)$. Then $A \subseteq c^*_\mu(U) \subseteq c^*_\mu(i_\mu(c^*_\mu(U)))) \subseteq c_\mu(i_\mu(c^*_\mu(U))) \subseteq c_\mu(i_\mu(c^*_\mu(A)))$. Hence $A \in \mu\beta\mathcal{I}O(X)$. □

Theorem 3.5 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then $A \in \mu\beta\mathcal{I}O(X)$ if and only if $c_\mu(A) = c_\mu(i_\mu(c^*_\mu(A)))$.*

Proof: Let $A \in \mu\beta\mathcal{I}O(X)$. Then $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$. This implies that $c_\mu(A) \subseteq c_\mu(i_\mu(c^*_\mu(A))) \subseteq c_\mu(i_\mu(c_\mu(A))) \subseteq c_\mu(A)$. Hence $c_\mu(A) = c_\mu(i_\mu(c^*_\mu(A)))$. Conversely, suppose that $c_\mu(A) = c_\mu(i_\mu(c^*_\mu(A)))$. Since $A \subseteq c_\mu(A)$, then $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$. Hence $A \in \mu\beta\mathcal{I}O(X)$. □

Theorem 3.6 *Let $(X, \mu, \mathcal{I})$ be an IGTS and $\{A_\alpha : \alpha \in J\}$ a collection of subsets of X, where J is an arbitrary index set.*
(i) If $\{A_\alpha : \alpha \in J\} \subseteq \mu\beta\mathcal{I}O(X)$, then $\bigcup\{A_\alpha : \alpha \in J\} \in \mu\beta\mathcal{I}O(X)$;
(ii) If $\{A_\alpha : \alpha \in J\} \subseteq \mu\xi\mathcal{I}O(X)$, then $\bigcup\{A_\alpha : \alpha \in J\} \in \mu\xi\mathcal{I}O(X)$.

Proof: (i) Since $\{A_\alpha : \alpha \in J\} \subseteq \mu\beta\mathcal{I}O(X)$, then $A_\alpha \subseteq c_\mu(i_\mu(c^*_\mu(A_\alpha)))$ for each $\alpha \in J$. Then we have $\bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} c_\mu(i_\mu(c^*_\mu(A_\alpha))) \subseteq c_\mu(i_\mu(\bigcup_{\alpha\in J} c^*_\mu(A_\alpha))) \subseteq c_\mu(i_\mu(c^*_\mu(\bigcup_{\alpha\in J} A_\alpha)))$. Hence $\bigcup_{\alpha\in J} A_\alpha \in \mu\beta\mathcal{I}O(X)$.

(ii) Obviously. □

Remark 3.3 *The intersection of any two μ-ξ-$\mathcal{I}$-open sets need not be μ-ξ-$\mathcal{I}$-open.*

In Example 3.2, the sets $\{a, b\}, \{b, c\} \in \mu\xi\mathcal{I}O(X)$, but $\{a, b\} \cap \{b, c\} = \{b\} \notin \mu\xi\mathcal{I}O(X)$.

Theorem 3.7 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then $A \in \mu\beta\mathcal{I}C(X)$ if and only if $i_\mu(c_\mu(i^*_\mu(A))) \subseteq A$.*

Proof: Let $A \in \mu\beta\mathcal{I}C(X)$. Then $X \setminus A \in \mu\beta\mathcal{I}O(X)$, this implies that $X \setminus A \subseteq c_\mu(i_\mu(c^*_\mu(X \setminus A))) = X \setminus i_\mu(c_\mu(i^*_\mu(A)))$. Therefore $i_\mu(c_\mu(i^*_\mu(A))) \subseteq A$. Conversely, suppose that $i_\mu(c_\mu(i^*_\mu(A))) \subseteq A$. Then $X \setminus A \subseteq c_\mu(i_\mu(c^*_\mu(X \setminus A)))$ and $X \setminus A \in \mu\beta\mathcal{I}O(X)$. Hence $A \in \mu\beta\mathcal{I}C(X)$. □

Theorem 3.8 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) $A \in \mu S\mathcal{I}C(X)$ if and only if $i^*_\mu(c_\mu(A)) \subseteq A$;
(ii) $A \in \mu\alpha\mathcal{I}C(X)$ if and only if $c_\mu(i^*_\mu(c_\mu(A))) \subseteq A$;
(iii) $A \in \mu P\mathcal{I}C(X)$ if and only if $c_\mu(i^*_\mu(A)) \subseteq A$;
(iv) $A \in \mu R\mathcal{I}C(X)$ if and only if $c_\mu(i^*_\mu(A)) = A$.

Proof: Obviously. □

Remark 3.4 *For an IGTS $(X, \mu, \mathcal{I})$ and $A \subseteq X$, we have $X \setminus i_\mu(c^*_\mu(i_\mu(A))) \neq c_\mu(i_\mu(c^*_\mu(X \setminus A)))$ as shown by the following example.*

Example 3.3 Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Take $A = \{a, b, d\}$, then $X \setminus i_\mu(c^*_\mu(i_\mu(A))) = \{c, d\}$ and $c_\mu(i_\mu(c^*_\mu(X \setminus A))) = \emptyset$.

Theorem 3.9 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. If $A \in \mu\beta\mathcal{I}C(X)$, then $i_\mu(c^*_\mu(i_\mu(A))) \subseteq A$.*

Proof: Let $A \in \mu\beta\mathcal{I}C(X)$. Since $\mu^* \supseteq \mu$ and $c^*_\mu(A) \subseteq c_\mu(A)$, we have that $i_\mu(c^*_\mu(i_\mu(A))) \subseteq i_\mu(c^*_\mu(i^*_\mu(A))) \subseteq i_\mu(c_\mu(i^*_\mu(A)))$. Then by Theorem 3.7, we obtain that $i_\mu(c^*_\mu(i_\mu(A))) \subseteq A$. □

Corollary 3.1 *Let A be a subset of an IGTS $(X, \mu, \mathcal{I})$ such that $X \setminus i_\mu(c^*_\mu(i_\mu(A))) = c_\mu(i_\mu(c^*_\mu(X \setminus A)))$. Then $A \in \mu\beta\mathcal{I}C(X)$ if and only if $i_\mu(c^*_\mu(i_\mu(A))) \subseteq A$.*

Proof: Follows from the Theorem 3.7. □

Theorem 3.10 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. If $V \in \mu P\mathcal{I}C(X)$ such that $i^*_\mu(V) \subseteq A \subseteq V$, then $A \in \mu\beta\mathcal{I}C(X)$.*

Proof: Suppose that $V \in \mu P\mathcal{I}C(X)$ such that $i^*_\mu(V) \subseteq A \subseteq V$. Then $i_\mu(c_\mu(i^*_\mu(A))) \subseteq i^*_\mu(c_\mu(i^*_\mu(A))) \subseteq i^*_\mu(c_\mu(i^*_\mu(V))) \subseteq i^*_\mu(V) \subseteq A$. Hence $A \in \mu\beta\mathcal{I}C(X)$. □

Theorem 3.11 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then $A \in \mu\beta\mathcal{I}C(X)$ if and only if $i_\mu(A) = i_\mu(c_\mu(i^*_\mu(A)))$.*

Proof: Let $A \in \mu\beta\mathcal{I}C(X)$. Then $i_\mu(c_\mu(i^*_\mu(A))) \subseteq A$. This implies that $i_\mu(A) \subseteq i_\mu(c_\mu(i_\mu(A))) \subseteq i_\mu(c_\mu(i^*_\mu(A))) \subseteq i_\mu(A)$. Hence $i_\mu(A) = i_\mu(c_\mu(i^*_\mu(A)))$. Conversely, suppose that $i_\mu(A) = i_\mu(c_\mu(i^*_\mu(A)))$. Since $i_\mu(A) \subseteq A$, we have that $i_\mu(c_\mu(i^*_\mu(A))) \subseteq A$. Hence $A \in \mu\beta\mathcal{I}C(X)$. □

Theorem 3.12 *Let $(X, \mu, \mathcal{I})$ be an IGTS and $\{A_\alpha : \alpha \in J\}$ a collection of subsets of X, where J is an arbitrary index set. If $\{A_\alpha : \alpha \in J\} \subseteq \mu\xi\mathcal{I}C(X)$, then $\bigcap\{A_\alpha : \alpha \in J\} \in \mu\xi\mathcal{I}C(X)$.*

Proof: Follows from the Definition 3.1 and Theorems 3.6 and 3.7. □

Remark 3.5 *The union of any two μ-ξ-$\mathcal{I}$-closed sets need not be μ-ξ-$\mathcal{I}$-closed.*

In Example 3.2, the sets $\{a\}, \{b\} \in \mu\xi\mathcal{I}C(X)$, but $\{a\} \cup \{b\} = \{a, b\} \notin \mu\xi\mathcal{I}C(X)$.

Theorem 3.13 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) $i_\mu(c_\mu(i^*_\mu(A))) \in \mu\beta\mathcal{I}C(X)$;
(ii) $c_\mu(i_\mu(c^*_\mu(A))) \in \mu\beta\mathcal{I}O(X)$.

Proof: (i) $i_\mu(c_\mu(i^*_\mu(i_\mu(c_\mu(i^*_\mu(A)))))) \subseteq i_\mu(c_\mu(i^*_\mu(c_\mu(i^*_\mu(A))))) \subseteq i_\mu(c_\mu(c_\mu(i^*_\mu(A)))) = i_\mu(c_\mu(i^*_\mu(A)))$. Hence $i_\mu(c_\mu(i^*_\mu(A))) \in \mu\beta\mathcal{I}C(X)$.

(ii) Follows from (i) and Proposition 2.1. □

Theorem 3.14 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) $i^*_\mu(c_\mu(A)) \in \mu S\mathcal{I}C(X)$;
(ii) $c^*_\mu(i_\mu(A)) \in \mu S\mathcal{I}O(X)$;
(iii) $c_\mu(i^*_\mu(c_\mu(A))) \in \mu\alpha\mathcal{I}C(X)$;
(iv) $i_\mu(c^*_\mu(i_\mu(A))) \in \mu\alpha\mathcal{I}O(X)$;
(v) $c_\mu(i^*_\mu(A)) \in \mu P\mathcal{I}C(X)$;
(vi) $i_\mu(c^*_\mu(A)) \in \mu P\mathcal{I}O(X)$.

Proof: Obviously. □

Theorem 3.15 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X.*
(i) If $A \in \mu\beta\mathcal{I}O(X) \cap \mu\alpha\mathcal{I}C(X)$, then $A \in \mu^c$;
(ii) If $A \in \mu\beta\mathcal{I}O(X) \cap (\mu^*)^c$, then $A \in \mu SO(X)$;
(iii) If $A \in \mu\beta\mathcal{I}O(X) \cap (\mu^*)^c \cap \mu PC(X)$, then $A \in \mu RC(X)$.

Proof: (i) Suppose that $A \in \mu\beta\mathcal{I}O(X) \cap \mu\alpha\mathcal{I}C(X)$. Then $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$ and $c_\mu(i^*_\mu(c_\mu(A))) \subseteq A$, this implies that $c_\mu(A) \subseteq c_\mu(i_\mu(c^*_\mu(A))) \subseteq c_\mu(i^*_\mu(c_\mu(A))) \subseteq A$. Hence $A = c_\mu(A)$. Thus $A \in \mu^c$.

(ii) Suppose that $A \in \mu\beta\mathcal{I}O(X) \cap (\mu^*)^c$. Then $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$ and $A = c^*_\mu(A)$, this implies that $A \subseteq c_\mu(i_\mu(A))$. Hence $A \in \mu SO(X)$.

(iii) Suppose that $A \in \mu\beta\mathcal{I}O(X) \cap (\mu^*)^c \cap \mu PC(X)$. Then by (ii), we have that $A \subseteq c_\mu(i_\mu(A))$ and $c_\mu(i_\mu(A)) \subseteq A$ (since $A \in \mu PC(X)$). Hence $A = c_\mu(i_\mu(A))$. Therefore $A \in \mu RC(X)$. □

Corollary 3.2 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X.*
(i) If $A \in \mu\beta\mathcal{I}C(X) \cap \mu\alpha\mathcal{I}O(X)$, then $A \in \mu$;
(ii) If $A \in \mu\beta\mathcal{I}C(X) \cap \mu^*$, then $A \in \mu SC(X)$;
(iii) If $A \in \mu\beta\mathcal{I}C(X) \cap \mu^* \cap \mu PO(X)$, then $A \in \mu RO(X)$.

Proof: Follows from the Theorem 3.15. □

Definition 3.2 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) μ-ξ-$\mathcal{I}$-interior of A is defined as union of all μ-ξ-$\mathcal{I}$-open sets contained in A. Thus $\xi\mathcal{I}i_\mu(A) = \cup\{U : U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A\}$;
(ii) μ-ξ-$\mathcal{I}$-closure of A is defined as intersection of all μ-ξ-$\mathcal{I}$-closed sets containing A. Thus $\xi\mathcal{I}c_\mu(A) = \cap\{V : X \setminus V \in \mu\xi\mathcal{I}O(X)$ and $A \subseteq V\}$, (where "μ-ξ-$\mathcal{I}$" stands for μ-β-$\mathcal{I}$, μ-semi-$\mathcal{I}$, μ-α-$\mathcal{I}$, μ-pre-$\mathcal{I}$; $\xi\mathcal{I}i_\mu(A)$ stands for $\beta\mathcal{I}i_\mu(A)$, $s\mathcal{I}i_\mu(A)$, $\alpha\mathcal{I}i_\mu(A)$, $p\mathcal{I}i_\mu(A)$; $\xi\mathcal{I}c_\mu(A)$ stands for $\beta\mathcal{I}c_\mu(A)$, $s\mathcal{I}c_\mu(A)$, $\alpha\mathcal{I}ci_\mu(A)$, $p\mathcal{I}c_\mu(A)$).

Theorem 3.16 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) $\xi\mathcal{I}i_\mu(A)$ is the largest μ-ξ-$\mathcal{I}$-open set contained in A;
(ii) $\xi\mathcal{I}c_\mu(A)$ is the smallest μ-ξ-$\mathcal{I}$-closed set containing A;
(iii) A is μ-ξ-$\mathcal{I}$-closed if and only if $\xi\mathcal{I}c_\mu(A) = A$;
(iv) A is μ-ξ-$\mathcal{I}$-open if and only if $\xi\mathcal{I}i_\mu(A) = A$;
(v) $\xi\mathcal{I}i_\mu(\xi\mathcal{I}i_\mu(A)) = \xi\mathcal{I}i_\mu(A)$;
(vi) $\xi\mathcal{I}c_\mu(\xi\mathcal{I}c_\mu(A)) = \xi\mathcal{I}c_\mu(A)$;
(vii) $X \setminus \xi\mathcal{I}i_\mu(A) = \xi\mathcal{I}c_\mu(X \setminus A)$;
(viii) $X \setminus \xi\mathcal{I}c_\mu(A) = \xi\mathcal{I}i_\mu(X \setminus A)$.

Proof: Follows from the Definitions 3.1, 3.2, Theorems 3.6, 3.12 and Proposition 2.1. □

Theorem 3.17 *Let $(X, \mu, \mathcal{I})$ be an IGTS and A be a subset of X. Then*
(i) If $A \subseteq B$, then $\xi\mathcal{I}i_\mu(A) \subseteq \xi\mathcal{I}i_\mu(B)$;
(ii) If $A \subseteq B$, then $\xi\mathcal{I}c_\mu(A) \subseteq \xi\mathcal{I}c_\mu(B)$;
(iii) $\xi\mathcal{I}i_\mu(A) \cup \xi\mathcal{I}i_\mu(B) \subseteq \xi\mathcal{I}i_\mu(A \cup B)$;
(iv) $\xi\mathcal{I}i_\mu(A \cap B) \subseteq \xi\mathcal{I}i_\mu(A) \cap \xi\mathcal{I}i_\mu(B)$;
(v) $\xi\mathcal{I}c_\mu(A) \cup \xi\mathcal{I}c_\mu(B) \subseteq \xi\mathcal{I}c_\mu(A \cup B)$;
(vi) $\xi\mathcal{I}c_\mu(A \cap B) \subseteq \xi\mathcal{I}c_\mu(A) \cap \xi\mathcal{I}c_\mu(B)$.

Proof: Follows from the Definitions 3.1, 3.2, Theorems 3.6, 3.12, 3.16 and Proposition 2.1. □

Remark 3.6 *The Example 3.3 shows that sides of the conditions (iii), (iv), (v) and(vi) in Theorem 3.17 need not be equal.*

(i) Take $A = \{a, b, d\}$ *and* $B = \{b, c\}$*. Then* $A \cup B = X$, $A \cap B = \{b\}$, $\xi\mathcal{I}i_\mu(A) = \{a, b\}$, $\xi\mathcal{I}i_\mu(B) = \{b, c\}$, $\xi\mathcal{I}i_\mu(A) \cup \xi\mathcal{I}i_\mu(B) = \{a, b, c\}$, $\xi\mathcal{I}i_\mu(A \cup B) = X$, $\xi\mathcal{I}i_\mu(A \cap B) = \emptyset$, $\xi\mathcal{I}i_\mu(A) \cap \xi\mathcal{I}i_\mu(B) = \{b\}$*. Hence* $\xi\mathcal{I}i_\mu(A) \cup \xi\mathcal{I}i_\mu(B) \subset \xi\mathcal{I}i_\mu(A \cup B)$ *and* $\xi\mathcal{I}i_\mu(A \cap B) \subset \xi\mathcal{I}i_\mu(A) \cap \xi\mathcal{I}i_\mu(B)$*.*

(ii) Take $A = \{a, d\}$ *and* $B = \{c, d\}$*. Then* $A \cup B = \{a, c, d\}$, $A \cap B = \{a\}$, $\xi\mathcal{I}c_\mu(A) = \{a, d\}$, $\xi\mathcal{I}c_\mu(B) = \{c, d\}$, $\xi\mathcal{I}c_\mu(A) \cup \xi\mathcal{I}c_\mu(B) = \{a, c, d\}$, $\xi\mathcal{I}c_\mu(A \cup B) = X$*. Hence* $\xi\mathcal{I}c_\mu(A) \cup \xi\mathcal{I}c_\mu(B) \subset \xi\mathcal{I}c_\mu(A \cup B)$*.*

(iii) Take $A = \{a\}$ *and* $B = \{b, d\}$*. Then* $A \cup B = \{a, b, d\}$, $A \cap B = \emptyset$, $\xi\mathcal{I}c_\mu(A) = \{a\}$, $\xi\mathcal{I}c_\mu(B) = X$, $\xi\mathcal{I}c_\mu(A \cap B) = \emptyset$, $\xi\mathcal{I}c_\mu(A) \cap \xi\mathcal{I}c_\mu(B) = \{a\}$*. Hence* $\xi\mathcal{I}c_\mu(A \cap B) \subset \xi\mathcal{I}c_\mu(A) \cap \xi\mathcal{I}c_\mu(B)$*.*

Theorem 3.18 *Let* $(X, \mu, \mathcal{I})$ *be an IGTS and* A *be a subset of* X*. Then*
(i) $\xi\mathcal{I}i_\mu(A) = \{x \in A : x \in U \in \mu\xi\mathcal{I}O(X)$ *and* $U \subseteq A$ *for some* $U\}$*;*
(ii) $\xi\mathcal{I}c_\mu(A) = \{x \in X : x \in V \in \mu\xi\mathcal{I}O(X)$ *and* $V \cap A \neq \emptyset$ *for all* $V\}$*.*

Proof: (i) Let $x_0 \in \xi\mathcal{I}i_\mu(A)$. Then $x_0 \in \cup\{U : U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A\}$, this implies that there exists $U \in \mu\xi\mathcal{I}O(X)$ such that $x_0 \in U \subseteq A$ and hence $x_0 \in \{x \in A : x \in U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A$ for some $U\}$. Therefore $\xi\mathcal{I}i_\mu(A) \subseteq \{x \in A : x \in U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A$ for some $U\}$. For the reverse inclusion, let $x_0 \in \{x \in A : x \in U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A$ for some $U\}$. Then there exists $U \in \mu\xi\mathcal{I}O(X)$ such that $x_0 \in U \subseteq A$, this implies that $x_0 \in \cup\{U : U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A\}$ and hence $x_0 \in \xi\mathcal{I}i_\mu(A)$. Therefore $\{x \in A : x \in U \in \mu\xi\mathcal{I}O(X)$ and $U \subseteq A$ for some $U\} \subseteq \xi\mathcal{I}i_\mu(A)$.

(ii) Let $F_0 = \{y \in X : y \in V \in \mu\xi\mathcal{I}O(X)$ and $V \cap A \neq \emptyset$ for all $V\}$. Now, we prove that $\xi\mathcal{I}c_\mu(A) = F_0$. Let us assume $x \in \xi\mathcal{I}c_\mu(A)$ and $x \notin F_0$. Then there exists $U \in \mu\xi\mathcal{I}O(X)$ and $x \in U$ such that $U \cap A = \emptyset$. This implies that $A \subseteq X \setminus U$. Therefore $\xi\mathcal{I}c_\mu(A) \subseteq X \setminus U$. Hence $x \notin \xi\mathcal{I}c_\mu(A)$. This is a contradiction. Hence $\xi\mathcal{I}c_\mu(A) \subseteq F_0$. Conversely, let F be a set such that $A \subseteq F$ and $X \setminus F \in \mu\xi\mathcal{I}O(X)$. Let $x \notin F$. Then we have that $x \in X \setminus F$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin F_0$. Therefore $F_0 \subseteq F$. Hence $F_0 \subseteq \xi\mathcal{I}c_\mu(A)$. □

Theorem 3.19 *Let* $(X, \mu, \mathcal{I})$ *be an IGTS and* A *be a subset of* X*. Then*
(i) $\beta\mathcal{I}i_\mu(A) = A \cap c_\mu(i_\mu(c^*_\mu(A)))$*;*
(ii) $\beta\mathcal{I}c_\mu(A) = A \cup i_\mu(c_\mu(i^*_\mu(A)))$*.*

Proof: (i) Clearly $A \cap c_\mu(i_\mu(c^*_\mu(A))) \subseteq c_\mu(i_\mu(c^*_\mu(A))) = c_\mu(i_\mu(i_\mu(c^*_\mu(A)))) \subseteq c_\mu(i_\mu(c^*_\mu(A) \cap i_\mu(c^*_\mu(A))) \subseteq c_\mu(i_\mu(c^*_\mu(A \cap i_\mu(c^*_\mu(A))))) \subseteq c_\mu(i_\mu(c^*_\mu(A \cap c_\mu(i_\mu(c^*_\mu(A))))))$ (by Proposition 2.1). This implies that $A \cap c_\mu(i_\mu(c^*_\mu(A)))$ is a μ-β-$\mathcal{I}$-open set contained in A and by Theorem 3.16(i), $A \cap c_\mu(i_\mu(c^*_\mu(A))) \subseteq \beta\mathcal{I}i_\mu(A)$. Since $\beta\mathcal{I}i_\mu(A) \in \mu\beta\mathcal{I}O(X)$, then $\beta\mathcal{I}i_\mu(A) \subseteq c_\mu(i_\mu(c^*_\mu(\beta\mathcal{I}i_\mu(A)))) \subseteq c_\mu(i_\mu(c^*_\mu(A)))$ and hence $\beta\mathcal{I}i_\mu(A) \subseteq A \cap c_\mu(i_\mu(c^*_\mu(A)))$. Therefore $\beta\mathcal{I}i_\mu(A) = A \cap c_\mu(i_\mu(c^*_\mu(A)))$.

(ii) Follows from (i) and Theorem 3.16(vii), (viii). □

Definition 3.3 *A subset* A *of an IGTS* $(X, \mu, \mathcal{I})$ *is said to be a* μ-ξ-$\mathcal{I}$*-neighborhood of a point* $x \in X$ *if there exists* $U_x \in \mu\xi\mathcal{I}O(X)$ *such that* $x \in U_x \subseteq A$*.*

Theorem 3.20 *Let* $(X, \mu, \mathcal{I})$ *be an IGTS and* A *be a subset of* X*. Then* $A \in \mu\xi\mathcal{I}O(X)$ *if and only if it is a* μ-ξ-$\mathcal{I}$*-neighborhood of each of its points.*

Proof: Let $A \in \mu\xi\mathcal{I}O(X)$. Then by Definition 3.3, it is clear that A is a μ-ξ-$\mathcal{I}$-neighborhood of each of its points, since for every $x \in A$, $x \in A \subseteq A$ and $A \in \mu\xi\mathcal{I}O(X)$. Conversely, suppose that A is a μ-ξ-$\mathcal{I}$-neighborhood of each of its points. Then by Definition 3.3, for each $x \in A$, there exists $U_x \in \mu\xi\mathcal{I}O(X)$ such that $x \in U_x \subseteq A$. Clearly $A = \bigcup\{U_x : U_x \in \mu\xi\mathcal{I}O(X)$ and $x \in U_x \subseteq A\}$. Hence by Theorem 3.6,

$A \in \mu\xi\mathcal{I}O(X)$.

Note that μ-ξ-$\mathcal{I}$-neighborhood of x may be replaced by μ-ξ-$\mathcal{I}$-open neighborhood of x. □

4. μ-ξ-$\mathcal{I}$, ν-continuous functions

Definition 4.1 *Let $(X, \mu, \mathcal{I})$ be an IGTS and (Y, ν) be a GT. A function $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ is said to be (μ-ξ-$\mathcal{I}$, ν)-continuous if for any $V \in \nu$, $f^{-1}(V) \in \mu\xi\mathcal{I}O(X)$, (where "$\xi$" stands for α, semi, pre and β).*

Theorem 4.1 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ be a function. Then*
(i) (μ, ν)-continuous $\Rightarrow$ (μ-α-$\mathcal{I}$, ν)-continuous $\Leftrightarrow$ (μ-semi-$\mathcal{I}$, ν)-continuous and (μ-pre-$\mathcal{I}$, ν)-continuous;
(ii) (μ-semi-$\mathcal{I}$, ν)-continuous $\Rightarrow$ (μ-β-$\mathcal{I}$, ν)-continuous;
(iii) (μ-pre-$\mathcal{I}$, ν)-continuous $\Rightarrow$ (μ-β-$\mathcal{I}$, ν)-continuous;
(iv) (μ-β-$\mathcal{I}$, ν)-continuous $\Rightarrow$ (μ-β, ν)-continuous.

Proof: Follows from the Definition 4.1 and Theorems 3.1, 3.2. □

Remark 4.1 *The following examples shows that the converse of the Theorem 4.1 need not be true.*

(i) Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $Y = \{1, 2, 3, 4\}$, $\nu = \{\emptyset, Y, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ by $f(a) = 3$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 1$. Then f is a (μ-β, ν)-continuous function but not (μ-β-$\mathcal{I}$, ν)-continuous since for the set $\{2, 3\} \in \nu$, $f^{-1}(\{2, 3\}) = \{a, b\} \notin \mu\beta\mathcal{I}O(X)$.

(ii) Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $Y = \{1, 2, 3, 4\}$, $\nu = \{\emptyset, Y, \{2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ by $f(a) = 2$, $f(b) = 1$, $f(c) = 4$ and $f(d) = 3$. Then f is a (μ-β-$\mathcal{I}$, ν)-continuous function but not (μ-pre-$\mathcal{I}$, ν)-continuous since for the set $\{2, 3\} \in \nu$, $f^{-1}(\{2, 3\}) = \{a, d\} \notin \mu P\mathcal{I}O(X)$.

(iii) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $Y = \{1, 2, 3\}$, $\nu = \{\emptyset, Y, \{1\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ by $f(a) = 2$, $f(b) = 1$, and $f(c) = 3$. Then f is a (μ-pre-$\mathcal{I}$, ν)-continuous function but not (μ-semi-$\mathcal{I}$, ν)-continuous (resp. (μ-α-$\mathcal{I}$, ν)-continuous) since for the set $\{1\} \in \nu$, $f^{-1}(\{1\}) = \{2\} \notin \mu S\mathcal{I}O(X)$ (resp. $\mu\alpha\mathcal{I}O(X)$).

(iv) Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $Y = \{1, 2, 3, 4\}$, $\nu = \{\emptyset, Y, \{1, 2, 3\}, \{2, 3, 4\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ by $f(a) = 2$, $f(b) = 3$, $f(c) = 4$ and $f(d) = 1$. Then f is a (μ-semi-$\mathcal{I}$, ν)-continuous function but not (μ-pre-$\mathcal{I}$, ν)-continuous (resp. (μ-α-$\mathcal{I}$, ν)-continuous) since for the set $\{1, 2, 3\} \in \nu$, $f^{-1}(\{1, 2, 3\}) = \{a, b, d\} \notin \mu P\mathcal{I}O(X)$ (resp. $\mu\alpha\mathcal{I}O(X)$).

Note that form the Examples (iii) and (iv), the concepts of (μ-pre-$\mathcal{I}$, ν)-continuous and (μ-semi-$\mathcal{I}$, ν)-continuous functions are independent.

(v) Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $Y = \{1, 2, 3, 4\}$, $\nu = \{\emptyset, Y, \{1, 2, 3\}, \{1, 3, 4\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then f is a (μ-α-$\mathcal{I}$, ν)-continuous function but not (μ, ν)-continuous since for the set $\{1, 3, 4\} \in \nu$, $f^{-1}(\{1, 3, 4\}) = \{a, c, d\} \notin \mu O(X)$.

Remark 4.2 *From the Theorem 3.2 and Remark 3.2, we have that every (μ-ξ-$\mathcal{I}$, ν)-continuous function is (μ-ξ, ν)-continuous, but the converse need not be true.*

Theorem 4.2 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ be a function. Then the following statements are equivalent:*
(i) f is (μ-β-$\mathcal{I}$, ν)-continuous;
(ii) for each $x \in X$ and $V \in \nu$ such that $f(x) \in V$, there exists $U \in \mu\beta\mathcal{I}O(X)$ such that $x \in U$, $f(U) \subseteq V$;

(iii) for any $F \in \nu^c$, $f^{-1}(F) \in \mu\beta\mathcal{I}C(X)$;
(iv) $f(\beta\mathcal{I}c_\mu(A)) \subseteq c_\nu(f(A))$, for each $A \subseteq X$;
(v) $\beta\mathcal{I}c_\mu(f^{-1}(B)) \subseteq f^{-1}(c_\nu(B))$, for each $B \subseteq Y$;
(vi) $f^{-1}(i_\nu(C)) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(C))$, for each $C \subseteq Y$;
(vii) $i_\mu(c_\mu(i^*_\mu(f^{-1}(B)))) \subseteq f^{-1}(c_\nu(B))$, for each $B \subseteq Y$;
(viii) $f(i_\mu(c_\mu(i^*_\mu(A)))) \subseteq c_\nu(f(A))$, for each $A \subseteq X$.

Proof: (i) $\Rightarrow$ (ii). Let $x \in X$ and $V \in \nu$ such that $f(x) \in V$. Set $U = f^{-1}(V)$, then by Definition 4.1, $U \in \mu\beta\mathcal{I}O(X)$ and $x \in U$ and $f(U) = f(f^{-1}(V)) \subseteq V$.

(ii) $\Rightarrow$ (i). Let $V \in \nu$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. By (ii), there exists $U_x \in \mu\beta\mathcal{I}O(X)$ such that $x \in U_x$ and $f(U_x) \subseteq F$. Then $x \in U_x \subseteq f^{-1}(V)$. Hence by Theorem 3.20, $f^{-1}(V) \in \mu\beta\mathcal{I}O(X)$.

(i) $\Leftrightarrow$ (iii). Follows from the fact that for any $E \in Y$, $f^{-1}(Y \setminus E) = X \setminus f^{-1}(E)$.

(iii) $\Rightarrow$ (iv). Let $A \subseteq X$. Then $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(c_\nu(f(A)))$. Since $c_\nu(f(A)) \in \nu^c$ and by (iii), we have that $f^{-1}(c_\nu(f(A))) \in \mu\beta\mathcal{I}C(X)$. Hence by Theorem 3.16(ii), $\beta\mathcal{I}c_\mu(A) \subseteq f^{-1}(c_\nu(f(A)))$ this implies that $f(\beta\mathcal{I}c_\mu(A)) \subseteq c_\nu(f(A))$.

(iv) $\Rightarrow$ (iii). Let $F \in \nu^c$. Then by (iv), $f(\beta\mathcal{I}c_\mu(f^{-1}(F))) \subseteq c_\nu(f(f^{-1}(F))) \subseteq c_\nu(F) = F$. This implies that $\beta\mathcal{I}c_\mu(f^{-1}(F)) \subseteq f^{-1}(F)$. Therefore $f^{-1}(F) \in \mu\beta\mathcal{I}C(X)$.

(iv) $\Rightarrow$ (v). Let $B \subseteq Y$. Then $f^{-1}(B)) \subseteq X$ and by (iv), $f(\beta\mathcal{I}c_\mu(f^{-1}(B))) \subseteq c_\nu(f(f^{-1}(B))) \subseteq c_\nu(B)$. This implies that $\beta\mathcal{I}c_\mu(f^{-1}(B)) \subseteq f^{-1}(c_\nu(B))$.

(v) $\Rightarrow$ (iv). Let $B = f(A)$, where $A \subseteq X$. Then by (v), $\beta\mathcal{I}c_\mu(A) \subseteq \beta\mathcal{I}c_\mu(f^{-1}(B)) \subseteq f^{-1}(c_\nu(B)) = f^{-1}(c_\nu(f(A)))$. This implies that $f(\beta\mathcal{I}c_\mu(A)) \subseteq c_\nu(f(A))$.

(i) $\Rightarrow$ (vi). Let $V = i_\nu(C)$, where $C \subseteq Y$. Then $V \in \nu$ and by (i), $f^{-1}(V) \in \mu\beta\mathcal{I}O(X)$. By Theorem 3.16(i), $f^{-1}(i_\nu(C)) = f^{-1}(V) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(V)) = \beta\mathcal{I}i_\mu(f^{-1}(i_\nu(C))) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(C))$.

(vi) $\Rightarrow$ (i). Let $V \in \nu$. Then $f^{-1}(V) = f^{-1}(i_\nu(V)) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(V))$ (by (vi). Hence $f^{-1}(V) \in \mu\beta\mathcal{I}O(X)$.

(iii) $\Rightarrow$ (vii). Let $B \subseteq Y$. Then $c_\nu(B) \in \nu^c$ and by (i), $f^{-1}(c_\nu(B)) \in \mu\beta\mathcal{I}C(X)$. This implies that $X \setminus f^{-1}(c_\nu(B)) \in \mu\beta\mathcal{I}O(X)$ and $X \setminus f^{-1}(c_\nu(B)) \subseteq c_\mu(i_\mu(c^*_\mu(X \setminus f^{-1}(c_\nu(B))))) = X \setminus i_\mu(c_\mu(i^*_\mu(f^{-1}(c_\nu(B)))))$. Hence $i_\mu(c_\mu(i^*_\mu(f^{-1}(c_\nu(B))))) \subseteq f^{-1}(c_\nu(B))$.

(vii) $\Rightarrow$ (viii). Let $A \subseteq X$. Then by (vii), $i_\mu(c_\mu(i^*_\mu(A))) \subseteq i_\mu(c_\mu(i^*_\mu(f^{-1}(f(A))))) \subseteq f^{-1}(c_\nu(f(A)))$ and hence $f(i_\mu(c_\mu(i^*_\mu(A)))) \subseteq c_\nu(f(A))$.

(viii) $\Rightarrow$ (i). Let $V \in \nu$. Then $f^{-1}(Y \setminus V) \subseteq X$ where $Y \setminus V \in \nu^c$ and by (viii), $f(i_\mu(c_\mu(i^*_\mu(f^{-1}(Y \setminus V))))) \subseteq c_\nu(f(f^{-1}(Y \setminus V))) \subseteq c_\nu(Y \setminus V) = Y \setminus V$. Therefore $X \setminus c_\mu(i_\mu(c^*_\mu(f^{-1}(V)))) = i_\mu(c_\mu(i^*_\mu(X \setminus f^{-1}(V)))) = i_\mu(c_\mu(i^*_\mu(f^{-1}(Y \setminus V)))) \subseteq f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ this implies that $f^{-1}(V) \subseteq c_\mu(i_\mu(c^*_\mu(f^{-1}(V))))$. Hence $f^{-1}(V) \in \mu\beta\mathcal{I}O(X)$. Thus f is (μ-β-$\mathcal{I}$, ν)-continuous. □

Corollary 4.1 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ be a (μ-β-$\mathcal{I}$, ν)-continuous function. Then $f(i_\mu(U)) \subseteq c_\nu(f(U))$ for every $U \in \mu S\mathcal{I}O(X)$.*

Proof: Let $U \in \mu S\mathcal{I}O(X)$. Then $U \subseteq c^*_\mu(i_\mu(U))$ and $f(i_\mu(U)) \subseteq f(i_\mu(c^*_\mu(i_\mu(U)))) \subseteq f(i_\mu(c_\mu(i^*_\mu(U)))) \subseteq c_\nu(f(U))$ (by Theorem 4.2(vii). Hence $f(i_\mu(U)) \subseteq c_\nu(f(U))$. □

Theorem 4.3 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ be a (μ-β-$\mathcal{I}$, ν)-continuous function. Then for each $V \subseteq Y$, $f^{-1}(i_\nu(V)) \subseteq c_\mu(i_\mu(c^*_\mu(f^{-1}(V))))$.*

Proof: Let $V \subseteq Y$. Then $i_\nu(V) \in \nu$. Since f is (μ-β-$\mathcal{I}$, ν)-continuous, $f^{-1}(i_\nu(V)) \in \mu\beta\mathcal{I}O(X)$. This implies that $f^{-1}(i_\nu(V)) \subseteq c_\mu(i_\mu(c^*_\mu(f^{-1}(i_\nu(V))))) \subseteq c_\mu(i_\mu(c^*_\mu(f^{-1}(V))))$. □

Theorem 4.4 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ be a bijective function. Then f is (μ-β-$\mathcal{I}$, ν)-continuous if and only if $i_\nu(f(U)) \subseteq f(\beta\mathcal{I}i_\mu(U))$ for each $U \subseteq X$.*

Proof: Let $U \subseteq X$. Then by 4.2(vi), $f^{-1}(i_\nu(f(U))) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(f(U)))$. Since f is bijective, $i_\nu(f(U)) = f(f^{-1}(i_\nu(f(U)))) \subseteq f(\beta\mathcal{I}i_\mu(U))$. Conversely, let $V \subseteq Y$. Then $i_\nu(f(f^{-1}(V))) \subseteq f(\beta\mathcal{I}i_\mu(f^{-1}(V)))$. Since f is bijective, $i_\nu(V) = i_\nu(f(f^{-1}(V))) \subseteq f(\beta\mathcal{I}i_\mu(f^{-1}(V)))$. Thus $f^{-1}(i_\nu(V)) \subseteq \beta\mathcal{I}i_\mu(f^{-1}(V))$. Hence by Theorem 4.2(vi), f is (μ-β-$\mathcal{I}$, ν)-continuous. □

Definition 4.2 *A function $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$ is said to be (almost-μ-$\mathcal{I}$, ν)-continuous (resp. (μ^*_d-$\mathcal{I}$, ν)-continuous) if for any $V \in \nu$, $f^{-1}(V) \in A\mu\mathcal{I}O(X)$ (resp. $\mu^*_d\mathcal{I}O(X)$), where $\mu^*_d\mathcal{I}O(X)$ is the set of all μ^*-dense subset of X.*

Theorem 4.5 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu)$. Then the following conditions are equivalent: (i) f is (almost-μ-$\mathcal{I}$, ν)-continuous;*
(ii) f is (μ-β-$\mathcal{I}$, ν)-continuous and (μ^*_d-$\mathcal{I}$, ν)-continuous.

Proof: Follows from the Proposition 3.1. □

5. μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$-continuous functions

Definition 5.1 *Let $(X, \mu, \mathcal{I})$ and $(Y, \nu, \mathcal{J})$ be any two IGTS. A function $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is said to be (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous if for any $V \in \nu\xi\mathcal{J}O(Y)$, $f^{-1}(V) \in \mu\xi\mathcal{I}O(X)$, (where "$\xi$" stands for α, semi, pre and β).*

Remark 5.1 *From the Definitions 4.1 and 5.1, we have that every (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous function is (μ-ξ-$\mathcal{I}$, ν)-continuous and the following examples shows that the converse need not be true.*

(i) Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $Y = \{1, 2, 3, 4\}$, $\nu = \{\emptyset, Y, \{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathcal{J} = \{\emptyset, \{2\}\}$. Define $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ by $f(a) = 1$, $f(b) = 3$, $f(c) = 4$ and $f(d) = 2$. Then f is a (μ-β-$\mathcal{I}$, ν)-continuous (resp. (μ-pre-$\mathcal{I}$, ν)-continuous, (μ-semi-$\mathcal{I}$, ν)-continuous) function but not (μ-β-$\mathcal{I}$, ν-β-$\mathcal{J}$)-continuous (resp. (μ-pre-$\mathcal{I}$, ν-pre-$\mathcal{J}$)-continuous, (μ-semi-$\mathcal{I}$, ν-semi-$\mathcal{J}$)-continuous) since for the set $\{1, 3, 4\} \in \nu\beta\mathcal{J}O(Y)$ (resp. $\nu P\mathcal{J}O(Y)$), $f^{-1}(\{1, 3, 4\}) = \{a, b, c\} \notin \mu\beta\mathcal{I}O(X)$ (resp. $\mu P\mathcal{I}O(X)$) and for the set $\{2, 3, 4\} \in \nu S\mathcal{J}O(Y)$, $f^{-1}(\{2, 3, 4\}) = \{b, c, d\} \notin \mu S\mathcal{I}O(X)$.

(ii) In Example (i), Let us consider the the inverse function f^{-1} as g. Then $g : (Y, \nu, \mathcal{J}) \to (X, \mu, \mathcal{I})$. Clearly g is a (ν-α-$\mathcal{J}$, μ)-continuous function but not (ν-α-$\mathcal{J}$, μ-α-$\mathcal{I}$)-continuous since for the set $\{a, c, d\} \in \mu\alpha\mathcal{I}O(X)$, $g^{-1}(\{a, c, d\}) = \{1, 2, 4\} \notin \nu\alpha\mathcal{J}O(Y)$.

Definition 5.2 *Let A be a subset of an an IGTS $(X, \mu, \mathcal{I})$ and $p \in X$. Then p is called a μ-ξ-$\mathcal{I}$-limit point of A if $U \cap (A - \{p\}) \neq \emptyset$, for any set $U \in \mu\xi\mathcal{I}O(X)$ containing p. The set of all μ-ξ-$\mathcal{I}$-limit points of A is called a μ-ξ-$\mathcal{I}$-derived set of A and is denoted by $\xi\mathcal{I}d_\mu(A)$. Clearly if $A \subseteq B$ then $\xi\mathcal{I}d_\mu(A) \subseteq \xi\mathcal{I}d_\mu(B)$.*

Remark 5.2 *From the Definition 5.2, it follows that p is a μ-ξ-$\mathcal{I}$-limit point of A if and only if $p \in \xi\mathcal{I}c_\mu(A - \{p\})$.*

Theorem 5.1 *Let $(X, \mu, \mathcal{I})$ be an IGTS. For any $A, B \subseteq X$, the μ-ξ-$\mathcal{I}$-derived sets have the following properties:*
(i) $\xi\mathcal{I}c_\mu(A) \supseteq A \cup \xi\mathcal{I}d_\mu(A)$;
(ii) $\cup_i\xi\mathcal{I}d_\mu(A_i) = \xi\mathcal{I}d_\mu(\cup_i A_i)$;
(iii) $\xi\mathcal{I}d_\mu(\xi\mathcal{I}d_\mu(A)) \subseteq \xi\mathcal{I}d_\mu(A)$;
(iv) $\xi\mathcal{I}c_\mu(\xi\mathcal{I}d_\mu(A)) = \xi\mathcal{I}d_\mu(A)$.

Proof: Follows from the Definition 5.2 and Remark 5.2. □

Theorem 5.2 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be a function. Then the following statements are equivalent:*
(i) f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous;
(ii) for each point x in X, the inverse of every ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$ is a μ-ξ-$\mathcal{I}$-neighborhood of x;
(iii) for each point x in X and each ν-ξ-$\mathcal{J}$-neighborhood B of $f(x)$, there is a μ-ξ-$\mathcal{I}$-neighborhood A of x such that $f(A) \subseteq B$;
(iv) for each $x \in X$ and each set $B \in \nu\xi\mathcal{J}O(Y)$ contains $f(x)$, there is a set $A \in \mu\xi\mathcal{I}O(X)$ containing x such that $f(A) \subseteq B$;
(v) $f(\xi\mathcal{I}c_\mu(A)) \subseteq \xi\mathcal{J}c_\nu(f(A))$ holds for every subset A of X;
(vi) for any set $H \in \nu\xi\mathcal{J}C(Y)$, $f^{-1}(H) \in \mu\xi\mathcal{I}C(X)$.

Proof: (i) ⇒ (ii). Let $x \in X$ and B be a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. By Definition 3.3, there exist $V \in \nu\xi\mathcal{J}O(Y)$ such that $f(x) \in V \subseteq B$. This implies that $x \in f^{-1}(V) \subseteq f^{-1}(B)$. Since f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous, so $f^{-1}(V) \in \mu\xi\mathcal{I}O(X)$. Hence $f^{-1}(B)$ is a μ-ξ-$\mathcal{I}$-neighborhood of x.

(ii) ⇒ (i). Let $B \in \nu\xi\mathcal{J}O(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. Clearly, B (being ν-ξ-$\mathcal{J}$-open) is a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. By (ii), $A = f^{-1}(B)$ is a μ-ξ-$\mathcal{I}$-neighborhood of x. Hence by Definition 3.3, there exist $A_x \in \mu\xi\mathcal{I}O(X)$ such that $x \in A_x \subseteq A$. This implies that $A = \cup_{x\in A} A_x$. By Theorem 3.6, we have that $A \in \mu\xi\mathcal{I}O(X)$. Therefore f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous.

(i) ⇒ (iii). Let $x \in X$ and B be a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. Then, there exist $O_{f(x)} \in \nu\xi\mathcal{J}O(Y)$ such that $f(x) \in O_{f(x)} \subseteq B$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(B)$. By (i), $f^{-1}(O_{f(x)}) \in \mu\xi\mathcal{I}O(X)$. Let $A = f^{-1}(B)$. Then it follows that A is μ-ξ-$\mathcal{I}$-neighborhood of x and $f(A) = f(f^{-1}(B)) \subseteq B$.

(iii) ⇒ (i). Let $V \in \nu\xi\mathcal{J}O(Y)$. Take $W = f^{-1}(V)$. Let $x \in W$. Then $f(x) \in V$. Thus V is a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. By (iii), there exist a μ-ξ-$\mathcal{I}$-neighborhood U_x of x such that $f(U_x) \subseteq V$. Thus it follows that $x \in U_x \subseteq f^{-1}(f(U_x)) \subseteq f^{-1}(V) = W$. Since U_x is a μ-ξ-$\mathcal{I}$-neighborhood of x, which implies that there exist a $W_x \in \mu\xi\mathcal{I}O(X)$ such that $x \in W_x \subseteq W$. This implies that $W = \cup_{x\in W} W_x$. By Theorem 3.6, $W \in \mu\xi\mathcal{I}O(X)$. Thus f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous.

(iii) ⇒ (iv). We may replace the μ-ξ-$\mathcal{I}$-neighborhood of x as μ-ξ-$\mathcal{I}$-open neighborhood of x in condition (iii). Straightforward.

(iv) ⇒ (v). Let $y \in f(\xi\mathcal{I}c_\mu(A))$ and any set $V \in \nu\xi\mathcal{J}O(Y)$ containing y. Then, there exist a point $x \in X$ and a set $U \in \mu\xi\mathcal{I}O(X)$ such that $x \in U$ with $f(x) = y$ and $f(U) \subseteq V$. Since $x \in \xi\mathcal{I}c_\mu(A)$, we have that $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $y \in \xi\mathcal{J}c_\nu(f(A))$. Therefore, we have that $f(\xi\mathcal{I}c_\mu(A)) \subseteq \xi\mathcal{J}c_\nu(f(A))$.

(v) ⇒ (vi). Let $H \in \nu\xi\mathcal{J}C(Y)$. Then $\xi\mathcal{J}c_\nu(H) = H$. By (v), $f(\xi\mathcal{I}c_\mu(f^{-1}(H))) \subseteq \xi\mathcal{J}c_\nu(f(f^{-1}(H))) \subseteq \xi\mathcal{J}c_\nu(H) = H$ holds. Therefore $\xi\mathcal{I}c_\mu(f^{-1}(H)) \subseteq f^{-1}(H)$ and thus $f^{-1}(H) = \xi\mathcal{I}c_\mu(f^{-1}(H))$. Hence $f^{-1}(H) \in \mu\xi\mathcal{I}C(X)$.

(vi) ⇒ (i). Let $B \in \nu\xi\mathcal{J}O(Y)$. We take $H = Y \setminus B$. Then $H \in \nu\xi\mathcal{J}C(Y)$. By (vi), $f^{-1}(H) \in \mu\xi\mathcal{I}C(X)$. Hence $f^{-1}(B) = X \setminus f^{-1}(Y \setminus B) = X \setminus f^{-1}(H) \in \mu\xi\mathcal{I}O(X)$. □

Theorem 5.3 *A function $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous if and only if $f(\xi\mathcal{I}d_\mu(A)) \subseteq \xi\mathcal{J}c_\nu(f(A))$, for all $A \subseteq X$.*

Proof: Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous. Let $A \subseteq X$ and $x \in \xi\mathcal{I}d_\mu(A)$. Assume that $f(x) \notin f(A)$ and let V denote a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. Since f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous, so by Theorem 5.2(iii), there exist a μ-ξ-$\mathcal{I}$-neighborhood U of x such that $f(U) \subseteq V$. From $x \in \xi\mathcal{I}d_\mu(A)$, it follows that $U \cap A \neq \emptyset$; there exist, therefore, at least one element $a \in U \cap A$ such that $f(a) \in f(A)$

and $f(a) \in V$. Since $f(x) \notin f(A)$, we have that $f(a) \neq f(x)$. Thus every ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \xi\mathcal{J}d_\nu(f(A))$. Conversely, suppose that f is not (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous. Then by Theorem 5.2(iii), there exist $x \in X$ and a ν-ξ-$\mathcal{J}$-neighborhood V of $f(x)$ such that every μ-ξ-$\mathcal{I}$-neighborhood U of x contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Since $f(x) \in V$, therefore $x \notin A$ and hence $f(x) \notin f(A)$. Since $f(A) \cap (V \setminus \{f(x)\}) = \emptyset$, therefore $f(x) \notin \xi\mathcal{J}d_\nu(f(A))$. It follows that $f(x) \in f(\xi\mathcal{I}d_\mu(A)) \setminus (f(A) \cup \xi\mathcal{J}d_\nu(f(A))) \neq \emptyset$, which is a contradiction to the given condition. □

Theorem 5.4 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be an injective function. Then f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous if and only if $f(\xi\mathcal{I}d_\mu(A)) \subseteq \xi\mathcal{J}d_\nu(f(A))$, for all $A \subseteq X$.*

Proof: Let $A \subseteq X$, $x \in \xi\mathcal{I}d_\mu(A)$ and V be a ν-ξ-$\mathcal{J}$-neighborhood of $f(x)$. Since f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous, so by Theorem 5.2(iii), there exist a μ-ξ-$\mathcal{I}$-neighborhood U of x such that $f(U) \subseteq V$. But $x \in \xi\mathcal{I}d_\mu(A)$ gives there exist an element $a \in U \cap A$ such that $a \neq x$. Clearly $f(a) \in f(A)$ and since f is injective, $f(a) \neq f(x)$. Thus every ν-ξ-$\mathcal{J}$-neighborhood V of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \xi\mathcal{J}d_\nu(f(A))$. Therefore, we have that $f(\xi\mathcal{I}d_\mu(A)) \subseteq \xi\mathcal{J}d_\nu(f(A))$. Converse follows from the Theorem 5.3. □

Definition 5.3 *An IGTS $(X, \mu, \mathcal{I})$ is called a μ-ξ-$\mathcal{I}$-T_2 space if for each pair of distinct points $x, y \in X$, there exists sets $U, V \in \mu\xi\mathcal{I}O(X)$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.*

Theorem 5.5 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be a (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous and injective function. If Y is ν-ξ-$\mathcal{J}$-T_2, then X is μ-ξ-$\mathcal{I}$-T_2.*

Proof: Suppose Y is ν-ξ-$\mathcal{J}$-T_2. Let x and y be two distinct points of X. Then, there exist two sets $U, V \in \nu\xi\mathcal{I}O(Y)$ such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \emptyset$. Since f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous, for U and V, there exist two sets $W, S \in \mu\xi\mathcal{I}O(X)$ such that $x \in W$ and $y \in S$, $f(W) \subseteq U$ and $f(S) \subseteq V$, implies that $W \cap S = \emptyset$. Hence X is μ-ξ-$\mathcal{I}$-T_2. □

Theorem 5.6 *Let $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ and $g : (Y, \nu, \mathcal{J}) \to (Z, \lambda)$ be two functions.*
(i) If f is (μ-ξ-$\mathcal{I}$, ν)-continuous and g is (ν, λ)-continuous, then gof is (μ-ξ-$\mathcal{I}$, λ)-continuous;
(ii) If f is (μ-ξ-$\mathcal{I}$, ν-ξ-$\mathcal{J}$)-continuous and g is (ν-ξ-$\mathcal{J}$, λ)-continuous, then gof is (μ-ξ-$\mathcal{I}$, λ)-continuous.

Proof: Obviously. □

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