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Reverse Topology and Translated Topology of Semi-Linear Topological Spaces *

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ABSTRACT: On topological semi-linear spaces there are multiple ways to define distinct convergences starting from the basic semi-linear topology, such as translated convergence, convergence in difference, reverse convergence. Translated convergence comes from translated topology and it has many good properties. The aim of this paper is to show that reverse convergence is also topological. New connection properties between these convergences have also been obtained. We study this problem in general framework or using neighborhoods of the origin which are totally bounded by nets. Finally they are examined in the case of semi-metrizable semi-linear spaces. An important tool of our research is the concept of translated Cauchy net, through which we study the "completeness" of topological semi-linear spaces.

Key Words: Semilinear space, semilinear topology, translated topology, reverse topology.

Contents

1. Introduction

In the study of many current mathematical problems the linear context has proven to be very restrictive. Thus, for some concrete requirements, various types of spaces with more general properties have been introduced (for e.g., [2], [5], [6], [16], [21] etc.). Such an example is the semilinear space, introduced by the author (also studied in [5], [6], [14] etc.) and topological semilinear space, together with its translated topology ([2]- [5]).

Under certain conditions, the information of the translated topology leads to properties of the initial semilinear topology.

The goal of this paper is to obtain a new topology on a topological semi-linear space L, complementary to the translated topology, named reverse topology, and some of its properties.

In Section 1 we prepare the framework of the paper by introducing the notions of semilinear space, semilinear topology, translated topology, and Cauchy nets in this context.

In Section 2 we build the reverse topology - a "complementary topology" to the translated one.

Section 3 investigates totally bounded sets and compact sets in reverse topology. The metrizable case through a semi-invariant metric with respect to translations is also studied.

Some significant examples are formulated in Section 4. We will focus on those types of sets that are of interest in some areas of mathematics: intervals (Interval Analysis, integral of multifunctions and convergence of algorithms), convex cones (Convex Analysis, optimization in linear normed spaces), subspaces and hyperplanes (Functional Analysis, elements of best approximation).

Now consider a non-void set L endowed with two operations: sum and multiplication by real scalars. The axioms from the definition of the linear space, excluding the existence of invertible element and the distributivity with respect to sum of scalars, lead to the notion of semilinear space (s.s):

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Definition 1.1 *([4]) We say that the set L, endowed with sum and multiplication by real scalars:*

$$"+" : L \times L \to L \text{ and } "\cdot" : \mathbb{R} \times L \to L$$

is a semi-linear space (s.s.) if the following axioms are verified:

S1) $(x+y)+z = x+(y+z)$, $\forall x, y, z \in L$;
S2) there exists an element $0 \in L$ *such that* $x + 0 = 0 + x = x$, $\forall x \in L$;
S3) $x + y = y + x$, $\forall x, y \in L$;
S4) $\lambda(\mu x) = (\lambda\mu) x$, $\forall \lambda, \mu \in \mathbb{R}$, $\forall x \in L$;
S5) $1 \cdot x = x$, $\forall x \in L$;
S6) $\lambda(x+y) = \lambda x + \lambda y$, $\forall \lambda \in \mathbb{R}$, $\forall x, y \in L$;
S7) $0 \cdot x = 0$, $\forall x \in L$.

Definition 1.2 *If* $(L, +, \cdot)$ *is a semilinear space, a semilinear subspace of* L *is a subset* $L_1 \subset L$ *such that* $(L_1, +, \cdot)$ *is a semilinear space, too.*

Remark 1.1 $L_1 \subset L$ is a semilinear subspace if and only if $\lambda x \in L_1$ and $x_1 + x_2 \in L_1$ for any $x, x_1, x_2 \in L_1$, $\lambda \in \mathbb{R}$.

Remark 1.2 For a s.s. $(L, +, \cdot)$ we denote by L_{in} the set of invertible elements:

$$L_{in} = \{x \in L; \exists x' \in L \text{ such that. } x + x\prime = 0\}$$

and by L_{dif} the set of the elements with 0 difference

$$L_{dif} = \{x \in L; x - x = 0\}.$$

Obviously, L_{in} and L_{dif} are semilinear subspaces of L and $L_{dif} \subset L_{in}$. They indicate how different is L compared to a linear space. More exactly, L is linear space iff $L_{in} = L$ (and in this case $L_{in} = L_{dif}$).

Example 1.1 **1)** Consider $(X, \|\cdot\|)$ a linear normed space and
$\mathcal{P}(X), \mathcal{B}(X), \mathcal{C}l(X), \mathcal{T}b(X), \mathcal{P}b(X), \mathcal{K}(X), \mathcal{F}(X)$ and $\mathcal{S}(X)$
the families of non-void subsets, bounded subsets, closed subsets, totally bounded subsets, closed and bounded subsets, compact subsets, finite subsets and singletons of X, respectively.

Define $A + B = \{a + b; a \in A, b \in B\}$ and $\lambda \cdot A = \{\lambda a; a \in A\}$.

Then $(\mathcal{P}(X), +, \cdot)$ is a s.s. and $\mathcal{B}(X)$, $\mathcal{C}l(X)$, $\mathcal{T}b(X)$, $\mathcal{P}b(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$, $\mathcal{S}(X)$ are semilinear subspaces.

2) If $(L, +, \cdot)$ is a s.s. and T is a non-void set then the family of functions $(\mathcal{L}, \oplus, \bullet)$ is also s.s., where $\mathcal{L} = \{f; f : T \to L\}$ is endowed by usual sum and multiplication by real scalars:

$(f \oplus g)(t) = f(t) + g(t)$, $(\lambda \bullet f)(t) = \lambda \cdot f(t)$ for all $t \in T$, $\lambda \in \mathbb{R}$.

Now let L be one of the families $\mathcal{B}(X), \mathcal{T}b(X), \mathcal{P}b(X), \mathcal{K}(X), \mathcal{F}(X)$ or $\mathcal{S}(X)$ from the Example 1.1, 1).

Define $\mathcal{M} = \{F; F : T \to L, F \text{ is sn-bounded}\}$, where the set-valued function $F : T \to L$ is *sn-bounded* iff there exists $\alpha \in [0, +\infty)$ such that $\sup\limits_{x \in F(t)} \|x\| \leq \alpha$ for every $t \in T$, where $\|\cdot\| : L \to [0, +\infty)$ is a function with the same properties as a norm.

Then $(\mathcal{M}, \oplus, \bullet)$ is a semilinear subspace of $(\mathcal{L}, \oplus, \bullet)$.

Definition 1.3 *([4]) A topology* σ *on a s.s.* L *is a semilinear topology iff the specific operations of sum and multiplication by scalars are continuous in the product topologies. Then* (L, σ) *is called semilinear topological space (shortly, s.t.s.).*

To give some examples of s.t.s., we recall the definitions of some well-known topologies on spaces of sets. Important informations related to these hypertopologies and some applications can be found in [5], [17]- [13], [15], [17]- [20], [22]- [24].

Definition 1.4 ***1)*** *Let* $(X,+,\cdot)$ *be a linear normed space and* $A \in \mathcal{P}(X)$. *Denote* $S_\varepsilon(A) = \{x \in X;$ *there exists* $a \in A$ *such that* $\|x-a\| < \varepsilon\}$.

The lower Hausdorff topology τ_H^- *is defined on* $\mathcal{P}(X)$, *where a basic neighbourhood of a set* $A_0 \in \mathcal{P}(X)$ *is:*

$B_H^-(A_0,\varepsilon) = \{A \in \mathcal{P}(X);\ A_0 \subseteq S_\varepsilon(A)\}$, *with* $\varepsilon > 0$,

For the upper Hausdorff topology τ_H^+ *on* $\mathcal{P}(X)$ *a basic neighbourhood of a set* $A_0 \in \mathcal{P}(X)$ *is:*
$B_H^+(A_0,\varepsilon) = \{A \in \mathcal{P}(X);\ A \subseteq S_\varepsilon(A_0)\}$, *with* $\varepsilon > 0$.

Hausdorff topology τ_H *on* $\mathcal{P}(X)$ *is the supremum of these two topologies defined above:* $\tau_H = \tau_H^- \vee \tau_H^+$
This topology is also induced by the extended-valued semi-metric H_d *(Pompeiu-Hausdorff semi-metric) on* $\mathcal{P}(X)$,

$$H_d(A,B) = \max\{e(A,B), e(B,A)\}, \tag{1.1}$$

where

$$e(A,B) = \sup\{d(a,B); a \in A\} \tag{1.2}$$

is the Hausdorff excess of A *with respect to* B.

2) *If* X *is a linear topological space under the scalar field* $\mathbb{R}$, *then the lower Vietoris topology* τ_V^- *on* $\mathcal{P}(X)$ *is given by the following subbase:*

$$V^- = \{A \in \mathcal{P}(X); A \cap V \neq \varnothing\}, \tag{1.3}$$

where V *is any open subset of* X.

Example 1.2 **1)** If $(X,\|\cdot\|)$ is a linear normed space and $\mathcal{A}$ is one of families defined in Example 1.1, 1), then $(\mathcal{A},\sigma)$ is a s.t.s., where σ is τ_H^- or τ_H^+.

2) If X is a linear topological space then $\left(\mathcal{P}(X), \tau_V^-\right)$ is s.t.s.

3) Consider the space of set-valued functions from the Example 1.1, 2).

If $F, G \in \mathcal{M}$ denote

$$\|F-G\| = \sup_{t\in T}\left(\sup_{x\in F(t)-G(t)} \|x\|\right).$$

Define $B_1(F,\varepsilon) = \{G \in \mathcal{M};\ \|F-G\| < \varepsilon\} \cup \{F\}$.

For every $F \in \mathcal{M}$ the sets

$$\mathcal{V}_1(F) = \{V \subseteq \mathcal{M};\ \text{there exists } \varepsilon > 0 \text{ such that } B_1(F,\varepsilon) \subseteq V\}$$

form a fundamental system of neighborhoods for F in a semilinear topology.

If L is a linear space and $\mathcal{V}(0)$ is a fundamental system of neighborhoods of the origin in a linear topology σ on L, then for any $x \in L$ and $V \in \mathcal{V}(0)$, the sets $x+V$ form a fundamental system of neighborhoods of x. Many proofs of the properties of linear spaces is based on this observation.

But this fact is not valid in the case of s.s. because, generally, the elements $x \in L$ doesn't admit a symmetric, and so the translations are not necessarily invertible. However:

Definition 1.5 *The family*
$\mathcal{U}_t(x) = \{U \subset L; \exists V \in \mathcal{V}(0)$ *such that* $U \subset x+V\}$
forms a fundamental system of neighbourhoods for x *in other topology, named the translation of the topology* σ *(or the translated topology). We denote it by* σ_t.

Remark 1.3 **1**) This new topology is the coarsest topology on L for which the translations are all continuous (or, equivalent, all translations are continuous in the origin). It is generally different from the initial topology σ on L and commonly it isn't semilinear.

The advantage of the topology σ_t is that many of the properties of linear topologies can be adapted to it: T_1-separation, metrizability, totally boundedness ([2], [3]) etc.

2) The net $(x_i)_{i\in I}$ is convergent to x in the translated topology iff:

for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x_i \in x+V$ for all $i \geq i_V$.

Definition 1.6 *([4]) **1)** We say that $(x_i)_{i\in I}$ is called a translated Cauchy net iff:*
$\forall V \in \mathcal{V}(0)\ \exists i_V \in I$ such that $\forall i,j \geq i_V \Longrightarrow x_i \in x_j + V$.
***2)** $(x_i)_{i\in I}$ is convergent in difference to x iff:*
$\forall V \in \mathcal{V}(0)\ \exists i_V \in I$ such that $\forall i \geq i_V \Longrightarrow x_i - x \in V$.
***3)** A net $(x_i)_{i\in I}$ has small autodifferences iff*
for every $V \in \mathcal{V}(0)$ there exists $i_V \in I$ with the property $x_i - x_i \in V$ for all $i \geq i_V$.

***4)** We say that $(x_i)_{i\in I}$ is closed-translated iff:*

$$\overline{x_i + V}^{\sigma_t} = x_i + \overline{V}^{\sigma},\ \text{for all } V \in \mathcal{V}(0) \text{ and all } i \in I\ .$$

where σ_t is the translated topology of σ.

In [4] the author shows that, generally, there are no relations between the convergence of the nets in translated topology and the corresponding Cauchy condition. In this approach, several types of convergences and Cauchy conditions are defined on s.t.s. and are compared to each other.

To make the transition from convergence in difference to convergence in translated topology we need "to get x from one member to another" in the relations from Remark 1.3, 2) and Definitions 1.6, 2).

There are several ways to substitute this property, for e.g. by using limits of nets or subnets from L_{dif} (see Remark 1.4 below). Another "good" property for a nets is to have small autodifferences (see Definition 1.6, 3) and Proposition 1.1 below).

Remark 1.4 If (L,σ) is a s.t.s. and $x \in L_{dif} = \{x \in L; x - x = 0\}$ then the convergent nets in difference to x coincide with nets convergent to x in translated topology. In fact, $(x_i)_{i\in I}$ is convergent in difference to $x \in L_{in}$ if and only if $(x_i)_{i\in I}$ is convergent in translation topology to x' (where x' is the opposite element of x).

Proposition 1.1 *Let $(x_i)_{i\in I}$ be a net from a s.t.s. (L,σ), which is convergent in translated topology to x. If it is also convergent in difference to x, then $(x_i)_{\ i\in I}$ has small autodiferences.*

Proof: Consider $\mathcal{V}(0)$ a fundamental system of σ-neighbourhoods of the origin formed only by balanced sets (i.e. $V = -V$ for all $V \in \mathcal{V}(0)$). We can find a neighbourhood $V_1 \in \mathcal{V}(0)$ with $V_1 + V_1 \subset V$ and an index $i_V \in I$ such that the conditions from Remark 1.3, 2) and Definition 6, 1) are both satisfied: $x_i - x \in V_1$ and $-x_i \in -x - V_1$ for all $i \geq i_V$. It results

$$x_i - x_i \in x_i - x - V_1 \subset V_1 + V_1 \subset V.$$

□

2. The construction of the reverse topology

Suppose that $\mathcal{V}(0)$ is a fundamental system of neighborhoods of the origin of a s.t.s. L. Because the relations $x \in y + V$ and $y \in x + V$ are not equivalent not even for balanced sets V, we can define neighborhoods of y through both relations. One of the conditions leads us to convergence in the translated topology, the other to another notion, namely σ-reverse convergence:

Definition 2.1 *([4], Definition 4.3.) A net $(x_i)_{i\in I}$ from a s.t.s. L is called σ-reverse convergent net to x if:*

for any $V \in \mathcal{V}(0)$ there exists $i_V \in I$ such that $x \in x_i + V$ for all $i \geq i_V$.

We present below some connections between reverse convergence of nets and those from Definitions 6:

Proposition 2.1 *([4], Proposition 4.4) Let (L, σ) be a semilinear topological space. Suppose that $0 \in (L, \sigma)$ has a fundamental system of neighbourhoods formed only by σ-closed sets. If $(x_i)_{i \in I}$ is a closed-translated net which is also a translated Cauchy net and a convergent net in the translated topology (to an element x), then it is σ-reverse convergent (to x).*

Remark 2.1 If $(x_i)_{i \in I} \subset L$ is a Cauchy net translated and it contains a subnet $\left(x_{\phi(k)}\right)_{k \in K}$ which is convergent in translated topology to x then $(x_i)_{i \in I}$ is convergent in translated topology to x.

Now we establish a link between convergence in translation topology, σ-reverse convergence and small autodifferences.

Proposition 2.2 *Suppose σ_t is the translated topology on a s.t.s. L. If a net $(x_i)_{i \in I} \subset L$ with small autodifferences is σ-reverse convergent to $x \in L_{in}$ then $(x_i)_{i \in I}$ is convergent in difference to x and it is convergent in translated topology to x.*

Proof: Let $\mathcal{V}(0)$ be a fundamental system of σ-neighbourhoods of the origin. Without restricting the generality we can work with a fundamental system of neighbourhoods of origin formed only by balanced sets. For any $V \in \mathcal{V}(0)$ there exists a $V_1 \in \mathcal{V}(0)$ such that $V_1 + V_1 \subset V$. We write the conditions from Definition 1.6, 4) and Definition 2.1 using the neighbourhood V_1. Then there exists $i_{V_1} = i_V \in I$ such that

$$x - x_i \in x_i - x_i + V_1 \subset V_1 + V_1 \subset V, \text{ for all } i \geq i_V,$$

so

$$x - x_i \in V, \text{ for all } i \geq i_V.$$

Let x' be the inverse of x.

Because V is balanced set we obtain

$$x_i = x_i - (x + x') \in -x' - V = x + V \text{ for all } i \geq i_V,$$

meaning that $(x_i)_{i \in I}$ is convergent in translated topology to x. □

We saw in Proposition 2.1 and Proposition 2.1 that an important connection between different types of nets is the σ-reverse convergence. So we will further investigate it. We show that this convergence is a topological one, i.e. it is given by a topology on L.

So let (L, σ) be a s.t.s. and let construct a "complementary" topology on L, namely the σ-reverse topology on L.

Consider $\mathcal{V}_\sigma(0)$ a fundamental system of neighbourhoods of the origin for the topology σ and for any $V \in \mathcal{V}_\sigma(0)$ we define

$$V_r(a) = \{b \in L; \text{ such that } a \in b + V\}. \tag{2.1}$$

Denote by $\mathcal{V}_r(a)$ the family of all subsets of L of the form $V_r(a)$ given by relation (2.1). Now we take

$$\mathcal{U}_r(a) = \{U \subset L; \exists V_r(a) \in \mathcal{V}_r(a) \text{ such that } V_r(a) \subset U\}. \tag{2.2}$$

Theorem 2.1 *Suppose that (L, σ) a s.t.s. The family $\mathcal{U}_r(a)$ described by (2.2) and (2.1) forms a system of neighbourhoods for $a \in L$ (and $\mathcal{V}_r(a)$ is a fundamental system of neighbourhoods of a in the topology given by $\mathcal{U}_r(a)$). The convergence of nets in the induced topology is even the σ_t-reverse convergence.*

Proof: Obviously $a \in U(a)$. Also $U(a) \in \mathcal{U}_r(a)$ and $W \supset U(a)$ imply $W \in \mathcal{U}_r(a)$.

For $U_1(a)$,$U_2(a) \in \mathcal{U}_r(a)$ there exist $V_1(a)$,$V_2(a) \in \mathcal{V}_r(a)$ such that $V_k(a) \subset U_k(a)$, where $V_k(a) = \{b \in L;\ a \in b + W_k\}$ with $W_k \in \mathcal{V}_\sigma(0), k = 1, 2$. Consider $W_3 \in \mathcal{V}_\sigma(0)$, where $W_3 \subset W_1 \cap W_2$. Denote $V_3(a) = \{b \in L;\ a \in b + W_3\}$. If $b \in L$ satisfies $a \in b + W_3$ then $a \in (b + W_1) \cap (b + W_2)$ and thus $V_3(a) \subset U_1(a) \cap U_2(a) \in \mathcal{U}_r(a)$.

Consider now $U(a) \in \mathcal{U}_r(a)$ with $U(a) \supset \{b \in L;\ a \in b+V\}$, $V \in \mathcal{V}_\sigma(0)$. For V there exists $V_1 \in \mathcal{V}_\sigma(0)$ such that $V_1 + V_1 \subset V$ because σ is a semilinear topology. We put $W = \{d \in L;\ a \in d + V_1\}$ and then $U(a) \in \mathcal{U}_r(c)$ for every $c \in W$. Indeed, if $c \in W$, then $a \in c + V_1$. We take $U_1(c) = \{d \in L;\ c \in d + V_1\}$ and we have $a \in d + V_1 + V_1 \subset d + V$, so $d \in \{b \in L;\ a \in b + V\} \subset U(a)$.

Obviously, the convergence induced by σ-reverse topology is that from Definition 2.1. □

Definition 2.2 *We call σ-reverse topology (or, simple, the reverse topology, when topology σ is fixed) the topology given by* (2.2).

Let denote it by σ_r.

Remark 2.2 For any $a \in L$, the set $V_r(a)$ contains only elements which are sum between an element from L and an element from a neighbourhood of the origin.

Particularly, if $a = 0$, for every $V \in \mathcal{V}_\sigma(0)$, the condition (2.1) becomes

$$V_r(0) = \{b \in L; 0 \in b + V\}, \tag{2.3}$$

hence each $b \in V_r(0)$ is invertible. Thus $V_r(0) \subset L_{in}$.

But it is possible that $L_{in} = \{0\}$, so $V_r(0) = \{0\}$. This situation is rather restrictive. In this case, the nets with limit 0 are only the constant nets. The reverse topology would thus seems close to the discrete topology.

Fortunately, things are not exactly like that. If $a \neq 0$, the neighborhoods $V_r(a)$ can be very different and the convergent nets and sequences are of many types, as we will see in Section 4 dedicated to examples. On the other hand, for example, if $(X, \|\cdot\|)$ is a linear normed space and τ is a semilinear hypertopology on $\mathcal{P}b(X)$, the subspace L_{in} is not $\{0\}$, but the family $\mathcal{S}(X)$ of singletons of X. So $V_r(0) \subset \mathcal{S}(X)$.

In Section 4, we will show that the basic semilinear topology, the translated one, and the reverse one are distinct two by two, but there are also convergent sequences in all three. Also in the Section 4 are given some answers to the question "What is this topology useful for?".

3. Totally boundedness and compactness conditions

The notion of totally bounded set is usually presented on metric spaces, but also in linear topological spaces (see, for e.g., [16] or [21]). Let define it in s.t.s.:

Definition 3.1 *([3]) A set M from a s.t.s. $(L.\sigma)$ is totally bounded if*

$$\forall V \in \mathcal{V}(0)\, \exists x_1, x_2, ..., x_n \in L \text{ such that } M \subset \cup_{i=1}^n (x_i + V).$$

We observe that this notion is essential connected by the translated topology σ_t because the sets $x_i + V$ are neighbourhoods of x_i in topology σ_t.

On linear topological spaces one can characterize the totally bounded sets by nets. Now we adapt on s.t.s. this notion:

Definition 3.2 *([3]) A set M from a s.t.s. (L, σ) is totally bounded by nets iff any net from M contains a translated Cauchy subnet.*

The relationship between this two notions is given in the following theorem:

Theorem 3.1 *([3], Theorem 3.1) Let (L, σ) be a s.t.s. and $M \subset L$. If M is totally bounded by nets then M is totally bounded.*

In the following we intend to define something similar for the reverse space:

Definition 3.3 *A set M from a s.l.s. (L, σ) is reverse totally bounded (or totally bounded in reverse topology) if*

$\forall V \in \mathcal{V}(0)\ \exists a_1,\ a_2,\ ...,\ a_n \in L$ such that $M \subseteq \bigcup\limits_{k+1}^{n} V_r(a_k)$, where $V_r(a_k) = \{x \in L; a_k \in x + V\}$ for all $k = \overline{1,n}$.

Remark 3.1 Consider the space (L, σ_r). If we intend to define a notion of Cauchy net then we find the same notion as in the case of the translated space (L, σ_t). So the notions of totally bounded sets by nets coincide in (L, σ_r) and (L, σ_t).

The link between this two types of totally boundedness and compactness is given by the following theorem:

Theorem 3.2 *Consider (L, σ) a s.t.s. and $M \subseteq (L, \sigma)$.*

1) If any sequence of M has a translated Cauchy subnet then M is reverse totally bounded. Consequently, any set which is totally bounded by nets is also reverse totally bounded.

2) If M is σ-compact then M is σ_r-totally bounded.

Proof: 1) Suppose by contrary that M isn't reverse totally bounded. So there exists $V_0 \in \mathcal{V}(0)$ such that $M \nsubseteq \bigcup\limits_{k+1}^{n} V_r(a_k)$ for any finite set $F = \{a_1, a_2, ..., a_n\} \subset L$, where $V_r(a_k) = \{x \in L; a_k \in x + V_0\}$. This implies the existence of an element $x_F \in M$ such that $x_F \notin \bigcup\limits_{k+1}^{n} V_r(a_k)$. So $a_k \notin x_F + V_0$ for all $k = \overline{1,n}$ or, equivalently,

$$F \cap (x_F + V_0) = \varnothing \text{ for any } F \in \mathcal{F}, \tag{3.1}$$

where $\mathcal{F}$ denote the family of non-void finite parts of L.

The family $\mathcal{F}$ can be ordered by the inclusion "$\subseteq$" and thus $\mathcal{F}$ is a directed set: for any $F_1, F_2 \in \mathcal{F}$ there exists $F_3 = F_1 \cup F_2$ such that $F_k \subseteq F_3$, $k = 1.2$.

So we obtained a net $(x_F)_{F \in \mathcal{F}}$ from M.

Now consider an arbitrary element $x_0 \in M$ and denote $F_0 = \{x_0\}$.

From (3.1) we find $x_{F_0} \in M$ such that $F_0 \cap (x_{F_0} + V_0) = \varnothing$.

We put $F1 = \{x_0, x_{F_0}\} \in \mathcal{F}$ and we find $x_{F1} \in M$ such that $F_1 \cap (x_{F_1} + V_0) = \varnothing$.

Inductively we can construct a sequence $(x_{F_n})_{n \in \mathbb{N}} \subset M$ such that $F_n \cap (x_{F_n} + V_0) = \varnothing$ for all $n \in \mathbb{N}$, where $F_n = F_{n-1} \cup \{x_{F_{n-1}}\}$.

Or, equivalently,

$$x_{F_k} \notin x_{F_n} + V_0 \text{ for all } n \in \mathbb{N} \text{ and } k = \overline{0, n-1}. \tag{3.2}$$

Denote by $\widetilde{\mathcal{F}} = \{F_n; n \in \mathbb{N}\}$. Obviously, $\widetilde{\mathcal{F}}$ is a directed set (even a chain): $F_n \subset F_{n+1}$.

But every sequence of M has a translated Cauchy subnets, so there exists a directed set J and an application $\phi : J \to \widetilde{\mathcal{F}}$ such that

a) $\phi(j_1) \geq \phi(j_2)$ for all $j_1 j_2 \in J$, $j_1 \geq j_2$;

b) for every $n \in \mathbb{N}$ there exists $j_n \in J$ with $\phi(j_n) \supseteq F_n$;

c) $x_{\phi(j_0)} \in x_{\phi(j)} + V_0$ for all $j \geq j_0$.

Because $\phi(j_0), \phi(j) \in \widetilde{\mathcal{F}}$ then there exists $F_{n_0}, F_{n_j} \in \widetilde{\mathcal{F}}$ such that $\phi(j_0) = F_{n_0}$, $\phi(j) = F_{n_j}$ and $F_{n_j} \supseteq F_{n_0}$.

Hence

$$x_{F_{n_0}} \in x_{F_{n_j}} + V_0,$$

in contradiction with (3.2).

2) Consider $\mathcal{V}(0)$ a fundamental system of σ-open neighbourhoods of the origin. If $V \in \mathcal{V}(0)$ and $V_r(x)$ is a σ_r-neighborhood of x then $M \subseteq \bigcup\limits_{x \in M} V_r(x)$.

From (2.1) we can write $V_r(x) = \bigcup\limits_{y \in V_r(x)} (y + V)$, so $M \subseteq \bigcup\limits_{x \in M} \bigcup\limits_{y \in V_r(x)} (y + V)$, where $y + V$ are σ-open sets because the translations are σ_t-continuous.

Because M is σ-compact then there exists $x_1, x_2, ..., x_k \in M$ and $y_j^1, y_j^2, ..., y_j^{m_j} \in V_r(x_j)$, $j = \overline{1,k}$, such that $M \subseteq \bigcup_{j=1}^{k} \bigcup_{l=1}^{m_j} (y_l^{m_l} + V)$. But $\bigcup_{l=1}^{m_j} (y_l^{m_l} + V) \subseteq V_r(x_j)$. Hence M is σ_r-totally bounded. □

Remark 3.2 The sum preserve the totally boundedness by nets: if M is totally bounded by nets then $x + M$ is also totally bounded by nets.

Theorem 3.3 *Let (L, σ) be a semilinear topological space which has a fundamental system of neighbourhoods formed only by translated totally bounded and σ-closed sets. If every totally bounded set is also σ_t-closed, then any σ_t-convergent net contains a subnet which is σ_t-reverse convergent.*

Proof: Let be $\mathcal{V}(0)$ a fundamental system of neighbourhoods as in hypothesis. Using Remark 3.2, $x+V$ is totally bounded, thus it is σ_t-closed. Then $\overline{x+V}^{\sigma_t} = x+V = x+\overline{V}^{\sigma}$ and every net is closed-translated.

Consider $(x_i)_{i\in I}$ a net σ_t-convergent to an x and V an arbitrary neighbourhood of origin. Then $x_i \in x+V$ for $i \in I$ sufficiently large. Because $x+V$ is totally bounded, there exists a subnet $\left(x_{\phi(j)}\right)_{j\in J}$ which is translated Cauchy net. Because it is also σ_t-convergent to x, then (from Proposition 2.1) it is σ_t-reverse convergent to x. □

Now we investigate the particular case of metrizable topological semilinear spaces.

Definition 3.4 *A semi-metric ρ on a s.s., $\rho : L \times L \to \mathbb{R}$, is called semi-invariant with respect to translations iff:*

$\rho(a+c, b+c) \leq \rho(a,b)$, for all $a, b, c \in L$.

An example of a semi-metric with such property is the Pompeiu-Hausdorff semi-metric on $\mathcal{P}b(X)$, where $(X, \|\cdot\|)$ is a linear normed space.

Denote by $B_\rho(x, \varepsilon)$ the ρ-ball of center $x \in L$ and radius $\varepsilon > 0$. A fundamental system of neighbourhoods of x in translated topology on L is given by $(x + B_\rho(0,\varepsilon))_{\varepsilon>0}$.

Theorem 3.4 *Let (L, σ) be a semilinear topological space such that its translation topology is metrizable by a metric ρ semi-invariant with respect to translation. Then:*

1. Any net σ_t-reverse convergent to x is also convergent in the translated topology to x.

2. Suppose in addition that all balls $B_\rho(0, \varepsilon)$, $\varepsilon > 0$, are σ-compact. Then any net convergent to x in translation topology is σ_t-reverse convergent to x.

Proof: 1. Let be an arbitrary $\varepsilon > 0$ and ρ a metric (semi-invariant with respect to translations) on L which induces the topology σ_t. If $(x_i)_{i\in I}$ is σ_t-reverse convergent to x then $x = x_i + y_{i\varepsilon}$ such that $\rho(y_{i\varepsilon}, 0) < \varepsilon$ for i sufficiently large. Thus $\rho(x_i, x) = \rho(x_i, x_i + y_{i\varepsilon}) \leq \rho(0, y_{i\varepsilon}) < \varepsilon$.

2. Because all balls $B_\rho(0,\varepsilon)$, $\varepsilon > 0$, are σ-compact, they are also σ-closed. Then

$$x + \overline{B_\rho(0,\varepsilon)}^{\sigma} = x + B_\rho(0,\varepsilon) \subset \overline{x + B_\rho(0,\varepsilon)}^{\sigma_t}$$

for all $x \in L$.

Let show that $\overline{x + B_\rho(0,\varepsilon)}^{\sigma_t} \subset x + B_\rho(0,\varepsilon)$. Consider $u \in \overline{x + B_\rho(0,\varepsilon)}^{\sigma_t}$ and denote by $u_n \in x + B_\rho(0,\varepsilon)$, $n \in \mathbb{N}$, a sequence such that $u_n \overset{\sigma_t}{\to} u$. Then $u_n = x + y_n$, where $y_n \in B_\rho(0,\varepsilon)$. Because every σ-compact set is also σ_t-compact the ball $B_\rho(0,\varepsilon)$ is σ_t-compact. Thus there exists a subsequence $(y_{n_k})_k$ which is σ_t-convergent to some $y \in B_\rho(0,\varepsilon)$. Then $u_{n_k} = x + y_{n_k} \overset{\sigma_t}{\to} x + y$ since the translations are σ_t-continuous. It results $u = x + y$, so $u \in x + B_\rho(0,\varepsilon)$.

The condition

$$x + \overline{B_\rho(0,\varepsilon)}^{\sigma} = \overline{x + B_\rho(0,\varepsilon)}^{\sigma_t}$$

says that all nets from L are closed-translated nets. If $(x_i)_{i\in I}$ is a σ_t-convergent net, it is also a translated Cauchy net because the translated topology σ_t is given by a metric. Using Proposition 2.1, it follows that $(x_i)_{i\in I}$ is σ_t-reverse convergent to x. □

Remark 3.3 Suppose that the metric ρ is semi-invariant with respect to translation and complete. If we work from the hypothesis of Theorem 3.3 then, using Remark 2.1, we deduce the same conclusion like in Theorem 3.4.

4. Examples

The purpose of this Section is to show that the notion of limit in reverse topology admits enough examples, even starting from a more restrictive topology such as the upper Hausdorff topology: for $\left(\mathcal{P}(X), \tau_H^+\right)$ we highlighted several classes of elements in $\mathcal{P}(X)$ which are the limits of some sequences in the reverse topology. The examples may also be useful because the Hausdorff topology is one of the most widely used topologies. We thus show that the reverse topology is a consistent notion. Where it was possible, we have tried to formulate the examples in the most general way possible.

We will apply the notions of Sections 1 and 2 to sequences of sets.

Let $(X, \|\cdot\|)$ be a linear normed space and $\mathcal{P}(X)$ is the family of non-void sets of X. Using Definition 4, 1) we find a fundamental system of neighborhoods of the origin in the reverse topology of τ_H^+. First we see that $S_\varepsilon(\{0\})$ is the sphere of center O and radius $\varepsilon > 0$ denoted by $S(0, \varepsilon) = \{x \in X;\ \|x\| < \varepsilon\}$. We deduce that $B_H^+(\{0\}, \varepsilon) = \{A \in \mathcal{P}(X);\ A \subseteq S(0, \varepsilon)\}$.

Hence $A_n \overset{(\tau_H^+)_r}{\longrightarrow} A$ if and only if

$$\forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N}\ \exists P_n \subseteq S(0, \varepsilon) \text{ such that } A = A_n + P_n \text{ for any } n \geq n_\varepsilon. \tag{4.1}$$

So for the sequences of sets from $\mathcal{P}(X)$, the limits in the sense of reverse topology of upper Hausdorff topology are those sets that can be decomposed into a sum as above, where P_n are rest sets that are "uniformly small". Thus the sets A can be "uniformly approximated" by the sets A_n.

Condition (4.1) is more easily verified if P_n contains few elements, for example when P_n is a singleton.

In the sequel we intend to find classes of sets with this property. Denote $\mathcal{RL}\left(\tau_H^+\right) = \left\{A \in \mathcal{P}(X); \exists A_n \in \mathcal{P}(X) \text{ such that } A_n \overset{(\tau_H^+)_r}{\longrightarrow} A\right\}$.

All the singletons are in $\mathcal{RL}\left(\tau_H^+\right)$:

Proposition 4.1 *The singleton $A = \{a\}$ satisfied* (4.1) *if and only if $P_n = \{p_n\}$ with $p_n \to 0$ and $A_n = \{a - p_n\}$.*

Proof: Suppose $A \in \mathcal{S}(X)$, let's say $A = \{a\}$. Then A_n and P_n have only one element, each. We fix an arbitrary $\varepsilon > 0$.

Obviously, the sequence $A_n = \{a - p_n\}$ checks relation (4.1) for $P_n = \{p_n\} \in B_H^+(\{0\}, \varepsilon)$.

Conversely, if $A_n = \{a_n\}$ and $P_n = \{p_n\}$, where $|p_n| < \varepsilon$ for n sufficiently large, then $a_n + p_n = a$, so $a_n = a - p_n$. □

Also, all finite sets are in $\mathcal{RL}\left(\tau_H^+\right)$. We denote by $card(M)$ the cardinality of the set M.

Proposition 4.2 *Suppose $A \in \mathcal{F}(X)$, let's say $A = \left\{a^1, a^2, ..., a^k\right\}$ and $A_n \overset{(\tau_H^+)_r}{\longrightarrow} A$. The condition* (4.1) *occurs if and only if*

$$card(A_n) = k \tag{4.2}$$

and

$$A_n = \left\{a^1 - p_n, a^2 - p_n, ..., a^k - p_n\right\}, P_n = \{p_n\}, \text{ where } p_n \to 0. \tag{4.3}$$

Proof: First we prove that if $(A_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ verify relation (4.1), then $card(A_n) = k$ for any $n \geq n_\varepsilon$.

Suppose by contrary that there exists an increasing sequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $card(A_{n_l}) \neq k$. Denote $card(A_{n_l}) = j$, let's say $A_{n_l} = \left\{a_{n_l}^1, a_{n_l}^2, ..., a_{n_l}^j\right\}$.

1) If $j \leq k-1$ then $card\,(P_{n_l}) \geq 2$, let's say $P_n \supseteq \{p_n, q_n\}$, where $p_n, q_n < \varepsilon$ for every $n \geq n_\varepsilon$.

Using (4.1), the distinct elements two by two

$a^1_{n_l} + p_{n_l}, a^2_{n_l} + p_{n_l}, ..., a^j_{n_l} + p_{n_l}$ are in A. To make a choice we assume that $a^i = a^i_{n_l} + p_{n_l}$ for $j = \overline{1, k-1}$. So $a^i_{n_l} \to a^i$ for every $j = \overline{1, k-1}$.

Now consider $a^{j+1} \in A$. Then there exists $m_j \in \{1, 2, ..., j\}$ such that $a^{m_j}_{n_l} \in A_{n_l}$ and $a^{j+1} = a^{m_j}_{n_l} + q_{n_l}$. Hence $a^{m_j}_{n_l} \to a^{j+1}$, in contradiction with $a^{m_j}_{n_l} \to a^{m_j}$.

2) If $j \geq k+1$ then $a^1_{n_l} + p_{n_l}, a^2_{n_l} + p_{n_l}, ..., a^j_{n_l} + p_{n_l} \in A$ are distinct elements two by two for any $p_{n_l} \in P_{n_l}$. So

$card\,(A) = card\,(A_{n_l} + P_{n_l}) \geq card\left\{a^1_{n_l} + p_{n_l}, a^2_{n_l} + p_{n_l}, ..., a^j_{n_l} + p_{n_l}\right\} = j > k = card\,(A)$, a contradiction.

Obviously, property (4.3) implies (4.1). Conversely, if $A = A_n + P_n$ for any $n \geq n_\varepsilon$, $card\,(A) = card\,(A_n) = k$ then $card\,(P_n) = 1$. Suppose $P_n = \{p_n\}$, where $p_n \to 0$.

Possibly renumbering the elements of A we can assume that $a^i = a^i_n + p_n$ for all $i = \overline{1, k}$, so (4.3) take place. □

Regarding the class of intervals, we have the following result:

Proposition 4.3 *Any interval* $[\alpha, \beta] \subset \mathbb{R}$ *can be decomposed according to condition* (4.1) *either by* $card\,(P_n) \in \mathbb{N}^*$, *or by* $card\,(P_n) = c$*:*

a) If $card\,(P_n) = 1$ *then the decomposition* (4.1) *is unique:*

$[\alpha, \beta] = [\alpha - p_n, \beta - p_n] + \{p_n\}$, *where* $P_n = \{p_n\} \subset S\,(0; \varepsilon)$ *for all* $n \geq n_\varepsilon$.

b) If $card\,(P_n) = k$ *then there exists* $p^1_n < p^2_n < ... < p^k_n$, *such that*

$p^k_n - p^1_n \leq \frac{k-1}{k}(\beta - \alpha)$ *and* $[\alpha, \beta] = \left[\alpha - p^1_n, \beta - p^k_n\right] + P_n$,

where $P_n = \left\{p^1_n, p^2_n, ..., p^k_n\right\} \subset S\,(0; \varepsilon)$ *for all* $n \geq n_\varepsilon$.

c) If $card\,(P_n) = c$ *then there exists* $p_n, q_n \in S\,(0; \varepsilon)$, *with* $p_n < q_n$ *and* $q_n - p_n < \beta - \alpha$ *for any* $n \geq n_\varepsilon$, *such that* $[\alpha, \beta] = [\alpha - p_n, \beta - q_n] + [p_n, q_n]$.

Proof: a) Obviously, for $A = [\alpha, \beta]$, $P_n = \{p_n\} \subset S\,(0; \varepsilon)$ we find $A_n = [\alpha - p_n, \beta - p_n]$.

b) For $P_n = \left\{p^1_n, p^2_n, ..., p^k_n\right\} \subset S\,(0; \varepsilon)$ we can suppose $p^1_n < p^2_n < ... < p^k_n$. We will look for the decomposition (4.1) with $A_n = [\alpha_n, \beta_n]$. We can write

$$[\alpha, \beta] = \bigcup_{j=1}^{k} \left[\alpha_n + p^j_n, \alpha_n + p^j_n\right]. \tag{4.4}$$

From (4.4) we deduce

$\begin{cases} \alpha = \alpha_n + p^1_n, \\ \beta = \beta_n + p^k_n, \\ \beta_n + p^j_n \leq \alpha_n + p^{j+1}_n, \end{cases}$ for all $j = \overline{1, k-1}$. Giving to j all values from 1 to $k-1$ and summing the inequalities term by term we obtain $p^k_n - p^1_n \leq \frac{k-1}{k}(\beta - \alpha)$ (condition that is possible because $p^j_n \in S\,(0; \varepsilon)$) and

$[\alpha, \beta] = \left[\alpha - p^1_n, \beta - p^k_n\right] + \left\{p^1_n, p^2_n, ..., p^k_n\right\}$.

c) In the case $card\,(P_n) = c$ we are looking for P_n as an interval $[p_n, q_n] \subset S\,(0; \varepsilon)$. If $A_n = [\alpha_n, \beta_n]$ then

$\begin{cases} \alpha = \alpha_n + p_n, \\ \beta = \beta_n + q_n, \\ \beta_n + q_n \leq \alpha_n + p_n. \end{cases}$

Now we observe that $card\,(P_n) \neq \aleph_0$: suppose by contrary that $P_n = \left\{p^1_n, p^2_n, ..., p^k_n, ...\right\}$, where $p^1_n < p^2_n < ... < p^k_n < ...$ and $P_n \subset S\,(0; \varepsilon)$. If $A_n = [\alpha_n, \beta_n]$ then $\alpha = \alpha_n + p^1_n$ and $\beta_n + p^j_n \leq \alpha_n + p^{j+1}_n$ for all $j \in \mathbb{N}^*$. We affirm that $\beta \notin [\alpha_n, \beta_n] + P_n$: if there exists $p^m_n \in P_n$ such that $\beta = \beta_n + p^m_n$ then $\beta_n + p^{m+1}_n \in [\alpha_n, \beta_n] + P_n$, but $\beta < \beta_n + p^{m+1}_n$, so $\beta_n + p^{m+1}_n \notin [\alpha, \beta]$. □

Remark 4.1 Proposition 4.3 can be easily generalized to intervals of the form $[\alpha e_i, \beta e_i]$, where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and $B = (e_i)_{i \in I}$ is an algebraic base in real linear normed space X.

Proposition 4.4 *a) Any convex cone is in* $\mathcal{RL}\left(\tau_H^+\right)$.

b) Every linear subspace of X *is from* $\mathcal{RL}\left(\tau_H^+\right)$.

c) All hyperplanes are from $\mathcal{RL}\left(\tau_H^+\right)$.

Proof: Consider $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}\cap(0,+\infty)$, $\lambda_n \to 0$. For $\varepsilon > 0$ we choose $n_\varepsilon \in \mathbb{N}$ such that $\lambda_n < \varepsilon$ for every $n \geq n_\varepsilon$.

a) Let $C \subset X$ be a convex cone.

Then for every $n \geq n_\varepsilon$ we put $P_n = \{p \in C; \|p\| \leq \lambda_n\}$ and $A_n = \{a \in C; \|a\| > \lambda_n\} \cup \{0\}$. Observe that $P_n \subset S(0;\varepsilon)$.

Consider $x \in C$.

If $x = 0$ then $x = 0 + 0 \in A_n + P_n$.

Now we take $x \in C \setminus \{0\}$. If $\|x\| \leq \lambda_n$ then $x = 0 + x \in A_n + P_n$. If $\|x\| > \lambda_n$ then $x = x + 0 \in A_n + P_n$.

Now consider $y = a + p \in A_n + P_n$. Since C is a convex cone then $y = a + p \in C$.

b) Let X_s be a linear subspace of X.

If $X_s = \{0\}$ then $A_n = P_n = \{0\}$.

Suppose $X_s \neq \{0\}$. For every $n \geq n_\varepsilon$ we take $P_n = \{p \in X_s; \|p\| \leq \lambda_n\}$ and $A_n = \{a \in X_s; \|a\| \geq \lambda_n\}$.

Let x be an element from X_s.

If $x = 0$ then consider $a \in X_s \setminus \{0\}$ and $b = \frac{a}{\|a\|}$. It results that $a_n = \lambda_n b \in A_n$, $p_n = -\lambda_n b \in P_n$, so $0 = a_n + p_n \in A_n + P_n$.

If $x \neq 0$ and $\|x\| < \lambda_n$ then we can choose $a_n = \left(1 + \frac{\lambda_n}{\|x\|}\right) x \in A_n$, $p_n = -\frac{\lambda_n}{\|x\|} x \in P_n$ because $\|a_n\| = \left(1 + \frac{\lambda_n}{\|x\|}\right) \|x\| > \lambda_n$, $\|p_n\| = \lambda_n$ and $x = a_n + p_n$.

If $x \neq 0$ and $\|x\| \geq \lambda_n$ then we can take $\mu = \min\left\{1 - \frac{\lambda_n}{\|x\|}, \frac{\lambda_n}{\|x\|}\right\} > 0$, $a_n = (1-\mu)x \in A_n$, $p_n = \mu x \in P_n$ because $\|a_n\| = (1-\mu)\|x\| \geq \lambda_n$, $\|p_n\| = \mu\|x\| \leq \lambda_n$ and $x = a_n + p_n$.

Now consider $a_n \in A_n$, $p_n \in P_n$. Then $a_n + p_n \in X_s$ because $A_n, P_n \subset X_s$ and X_s is a linear subspace of X.

c) If H is an arbitrary hyperplane from X then H can be write $H = H_0 + h$, where H_0 is a hyperplane which passes through the origin and h is an arbitrary element from H. From b) every hyperplane which passes through the origin is in $\mathcal{RL}\left(\tau_H^+\right)$. Hence $H_0 = A_n + P_n$, where $P_n \subset S(0;\varepsilon)$. Then $H = (A_n + h) + P_n$ is the decomposition of H that verifies relation (4.1). □

Now we compare semi-linear convergence, translated and reverse convergences. Consider A_n, $A \subset \mathcal{P}b(X)$ for any $n \in \mathbb{N}$.

Example 4.1 There exist sequences for which the 3 convergences coincide:

a) Let $(X, \|\cdot\|)$ be a linear normed space. According to Proposition 4.1 and (1.2), in the case of singletons $A_n = \{a_n\}$, $A = \{a\}$, the relations $A_n \xrightarrow{\left(\tau_H^+\right)_t} A$, $A_n \xrightarrow{\left(\tau_H^+\right)_r} A$, $A_n \xrightarrow{\tau_H^+} A$ are all equivalent with $a_n \xrightarrow{\|\cdot\|} a$.

b) As we see in Proposition 11, a), if $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $p_n \to 0$ and $A_n = [\alpha - p_n, \beta - p_n]$ then $(A_n)_{n\in\mathbb{N}}$ is convergent at $A = [\alpha, \beta]$ in $\left(\tau_H^+\right)_r$.

Also $A_n = [\alpha - p_n, \beta - p_n] = [\alpha, \beta] + \{-p_n\}$ and $e(A_n, A) = \sup\limits_{b\in A_n} \inf\limits_{a\in A} \|b - a\| \leq p_n$ for all n sufficiently large, so $A_n \to A$ in the sense of $\left(\tau_H^+\right)_t$ and τ_H^+, respectively.

Example 4.2 There exist sequences which are convergent in the translated topology, but not in the reverse topology:

In $\mathcal{K}(\mathbb{R})$ the sequence $A_n = [\alpha - p_n, \alpha + p_n]$ is convergent in $\left(\tau_H^+\right)_t$ to $A = \{\alpha\}$, where $p_n > 0$, $p_n \to 0$, because for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $M_n = [-p_n, p_n] \subset [-\varepsilon, \varepsilon]$ such that $A_n = A + M_n$, for all $n \geq n_\varepsilon$.

But $(A_n)_{n\in\mathbb{N}}$ isn't convergent in $\left(\tau_H^+\right)_r$: we suppose by contrary that there exists $B \in \mathcal{K}(\mathbb{R})$ such that

$$\forall\varepsilon > 0 \text{ there exists } n_\varepsilon \in \mathbb{N} \text{ such that } B = A_n + P_n \text{ and } |p_n| < \varepsilon \text{ for any } n \geq n_\varepsilon, \tag{4.5}$$

where $P_n \in B_H^+(\{0\}, \varepsilon)$.

We fix $\varepsilon > 0$, $\varepsilon < |\alpha|$. One observe that the sets $B = \{0\}$, $B = \{\alpha\}$ and $B = \{0, \alpha\}$ do not verify the relation (4.1) for any $P_n \in B_H^+(\{0\}, \varepsilon)$.

Suppose now that there exists $\beta \in B$, $\beta \neq 0, \alpha$.

Choose $\varepsilon = |\beta - \alpha|/3$ and adjust n_ε such that $p_n < \varepsilon$ for all $n \geq n_\varepsilon$.

If $\beta > \alpha$ then $P_n \subseteq [-\varepsilon, \varepsilon]$ and we obtain:

$A_n + P_n \subseteq [\alpha - p_n, \alpha + p_n] + [-\varepsilon, \varepsilon] \subseteq [\alpha - 2\varepsilon, \alpha + 2\varepsilon] = \left[\frac{5\alpha-2\beta}{3}, \frac{\alpha+2\beta}{3}\right]$. Since $\beta > \frac{\alpha+2\beta}{3}$, (4.5) is not occurred.

If $\beta < \alpha$ then $A_n + P_n \subseteq [\alpha - 2\varepsilon, \alpha + 2\varepsilon] = \left[\frac{2\alpha+\beta}{3}, \frac{5\alpha-2\beta}{3}\right]$. But $\beta < \frac{2\alpha+\beta}{3}$ and (4.5) is not verified.

Example 4.3 There exist sequences which are convergent in both the translated and reverse topology, but the limits are different.

Consider $l_2 = \left\{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R};\ \sum\limits_{n=1}^{\infty} |x_n|^2 < +\infty\right\}$ and $\left(\mathcal{K}(l_2), \tau_V^-\right)$.

On the family $\mathcal{P}(l_2)$ of non-void subsets of l_2 we use the topology τ_V^- (see Definition 1.4, 2)); a neighbourhood of $0 \in l_2$ in τ_V^- is given by $U = U_1^- \cap ... \cap U_l^-$, where U_k are open sets in l_2 such that $0 \in U_k$, $k = \overline{1,l}$ (so there exist $r_1, r_2, ..., r_l > 0$ such that $S(0; r_k) \subseteq U_k$ for all $k = \overline{1,l}$).

Let $e_1, e_2, ..., e_n, ...$ be the vectors of canonical base in l_2. We take $A = l_2$, $A_n = \mathcal{I}(e_1, e_2, ..., e_n) = \{\lambda_1 e_1 + ... + \lambda_n e_n; \lambda_1, .., \lambda_n \in \mathbb{R}\}$ and $B = \{0\}$. Denote $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. Then:

a) $A_n \overset{(\tau_V^-)_r}{\to} A$:

We put $P_n = \mathcal{I}(e_{n+1}, e_{n+2}, ...)$ for all $n \in \mathbb{N}$ and $n_U = 1$. For all $n \geq n_U$ we have $P_n \in U$ (because $0 \in P_n$.) and $A = A_n + P_n$.

Another choice is $P_n = l_2$, $n_U = 1$.

b) But $A_n \overset{(\tau_V^-)_t}{\nrightarrow} A$:

If U_1,..., U_p are open sets with $0 \in U_1 \cap ... \cap U_p$, $U = U_1^- \cap ... \cap U_p^-$ and $M \in U$ then $M \cap U_k \neq \varnothing$, for any $k = \overline{1,p}$.

Suppose by contrary that $A_n \overset{(\tau_V^-)_t}{\to} A$. So there exist $n_U \in \mathbb{N}$ and $M_n \in U$ such that $A_n = A + M_n$ for any $n \geq n_U$. But $A + M_n = l_2$, a contradiction.

Now we show that (A_n) is convergent to B in $\left(\tau_V^-\right)_t$: for $n_U = 1$ and $M_n = \mathcal{I}(e_1, e_2, ..., e_n) \in U$ we have $A_n = B + M_n$ for any $n \geq n_U$.

Of course, $A_n \overset{(\tau_V^-)_r}{\nrightarrow} B$ because $\{0\} \neq \mathcal{I}(e_1, e_2, ..., e_n) + \mathcal{I}(e_{n+1}, e_{n+2}, ...)$.

c) It should be noted that $A_n \overset{\tau_V^-}{\to} A$: let $(U_k)_{k=\overline{1,p}}$, $p \in \mathbb{N}^*$ be an arbitrary non-empty open set from l_2 with $A \cap U_k \neq \varnothing$ for all $k = \overline{1,p}$. Consider $x_k \in U_k$ and $\rho_k > 0$ such that $S(x_k, \rho_k) \subset U_k$ for all $k = \overline{1,p}$. Here we denote by $S(x_k, \rho_k)$ the sphere of center x_k and radius ρ_k.

If $x_k = \lambda_1^k e_1 + \lambda_2^k e_2 + ... + \lambda_n^k e_n + ...$, we choose $y_k = \lambda_1^k e_1 + \lambda_2^k e_2 + ... + \lambda_n^k e_n \in A_n$. Then $\|x_k - y_k\| = \left\|\lambda_{n+1}^k e_{n+1} + \lambda_{n+2}^k e_{n+2} + ...\right\| = \sum\limits_{m=n+1}^{\infty} \left|\lambda_m^k\right|^2 < \rho_k$ for n sufficiently large, so $A_n \cap S(x_k, \rho_k) \neq \varnothing$, i.e. $A_n \cap U_k \neq \varnothing$.

Example 4.4 There exist sequences that are $\left(\tau_H^+\right)_r$-convergent, but not $\left(\tau_H^+\right)_t$-convergent:

Let $q = (q_n)_{n\in\mathbb{N}^*} \in l_2$ be a fixed sequence (i.e. $\sum\limits_{n=1}^{\infty} |q_n|^2 < +\infty$). Consider $A = \left\{x = (x_n)_{n\in\mathbb{N}^*}; |x_n| \leq |q_n| \text{ for all } n \in \mathbb{N}^*\right\}$, so $A \in l_2$.

We write Cauchy condition for q: $\forall\varepsilon > 0\ \exists k_\varepsilon \in \mathbb{N}$ such that $\sum\limits_{n=k}^{\infty} |q_n|^2 < \varepsilon$ for all $k \geq k_\varepsilon$.

Let $e_k = (0, 0, ..., 0, 1, 0, ...0)$, $k \in \mathbb{N}^*$, be the vectors of canonical base in l_2 and $A_k = \mathcal{I}(e_1, e_2, ..., e_k) \cap A$, $P_k = \mathcal{I}(e_{k+1}, e_{k+2}, ...) \cap A$.

a) Then $A_k \overset{(\tau_H^+)_r}{\to} A$:

Show that $A = A_k + P_k$, where $P_k \subset S(0;\varepsilon)$ for all $k \geq k_\varepsilon$:

• Every $x = (x_n)_{n\in\mathbb{N}^*} \in A$ can be written $x = (x_1e_1 + x_2e_2 + ...x_ke_k) + (x_{k+1}e_{k+1} + x_{k+2}e_{k+2} + ...) \in A_k + P_k$ for all $k \geq k_\varepsilon$ (because $\sum\limits_{n=k+1}^{\infty} |x_n|^2 \leq \sum\limits_{n=k+1}^{\infty} |q_n|^2 < \varepsilon$).

• If $y = (y_n)_{n\geq 1} \in A_k$ and $z = (z_n)_{n\in\mathbb{N}^*} \in P_k$, $y_{k+l} = 0$ for $\forall l \geq 1$ and $z_1 = z_2 = ..z_k = 0$, then $y = y_1e_1 + ... + y_ke_k$ and $z = z_{k+1}e_{k+1} + z_{k+2}e_{k+2} + ...$, where $\sum\limits_{n=k+1}^{\infty} |z_n|^2 \leq \sum\limits_{n=k+1}^{\infty} |q_n|^2 < \varepsilon$.

Hence $x = y + z = y_1e_1 + ... + y_ke_k + z_{k+1}e_{k+1} + z_{k+2}e_{k+2} + ... \in A$.

b) We argue that $(A_k)_{k\in\mathbb{N}^*}$ is not $\left(\tau_H^+\right)_t$-convergent.

Suppose, by contrary, that there exists $B \in \mathcal{P}(X)$ such that $A_k \overset{\left(\tau_H^+\right)_t}{\to} B$.

We can write:

$$\forall\varepsilon > 0 \exists k_\varepsilon \in \mathbb{N}^* \exists M_k \subset S(0;\varepsilon) \text{ such that } A_k = B + M_k \text{ for all } k \geq k_\varepsilon. \tag{4.6}$$

We fix $\varepsilon > 0$ and $k \geq k_\varepsilon$.

From (4.6) it results that any $b \in B$ can be written in canonical base like $b = b_1e_1 + b_2e_2 + ... + b_ke_k - m_{k+1}^k e_{k+1} - m_{k+2}^k e_{k+2} + ...$, where b_j is an arbitrary scalar in $\mathbb{R}$, $\forall j = \overline{1,k}$, and $\left(m_l^k\right)_{l\in\mathbb{N}^*} \subset S(0;\varepsilon)$ (i.e. $\sum\limits_{l=1}^{\infty} \left|m_l^k\right|^2 < \varepsilon$).

Applied now (4.6) for $k+1$ it results that any $b \in B$ can be written in canonical base like $b = b_1e_1 + b_2e_2 + ... + b_ke_k + b_{k+1}e_{k+1} - m_{k+2}^{k+1}e_{k+2} - m_{k+3}^{k+1}e_{k+3} + ...$, where b_j is an arbitrary scalar in $\mathbb{R}$, $\forall j = \overline{1,k+1}$, and $\left(m_l^{k+1}\right)_{l\in\mathbb{N}^*} \subset S(0;\varepsilon)$

We observe that the two writings of the elements of B are contradictory: for example e_{k+1} checks the second writing, but does not check the first because it should $m_{k+1}^k = -1$, but in this case $\sum\limits_{l=1}^{\infty} \left|m_l^k\right|^2 \geq 1$.

The contradiction proves the $\left(\tau_H^+\right)_t$-divergence of the sequence $(A_k)_{k\in\mathbb{N}^*}$.

Thus, through these examples, we showed that, generally, the reverse topology and the translate topology are not comparable.

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