Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 9** : 1–6.
ISSN-0037-8712
doi:10.5269/bspm.79799

Arithmetic of General Partition Functions $p_r(n)$ Modulo Primes

B. R. Srivatsa Kumar, H. S. Manu Banakar and R. G. Veeresha*

ABSTRACT: In the present investigation, we establish several new infinite families of congruences for the generalized partition function $p_r(n)$. Our emphasis throughout this paper is on demonstrating how classical and modern q-identities can be effectively employed to derive these congruences. By systematically applying these identities, we uncover congruences modulo primes such as 19, 23 and 29, valid for all positive integers λ. This approach not only yields elegant arithmetic results but also highlights the deep interplay between partition theory and the analytic properties of q-series.

Keywords: Congruences, general partitions, Theta-functions, q-identities.

Contents

1. Introduction

For $|xy| < 1$, Ramanujan's general theta function $f(x, y)$ is given by

$$f(x,y) := \sum_{k=-\infty}^{\infty} x^{\frac{k(k+1)}{2}} y^{\frac{k(k-1)}{2}}.$$

The function $f(x, y)$ enjoys the well-known Jacobi triple product identity [5, p.35],

$$f(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty,$$

where here and throughout the paper, we will utilize the following definition of the q-shifted factorial

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k) \qquad |q| < 1.$$

It generalizes the classical factorial and appears frequently in identities and transformations involving q-analogues. One of the special case of $f(x, y)$ as defined by S. Ramanujan [5] is as follows

$$\begin{aligned} f(-q) := f(-q,-q^2) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \\ &= (q;q)_\infty. \end{aligned}$$

Due to Euler, we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where $p(n)$ is the number of partitions of n. Srinivasa Ramanujan introduced the general partition function $p_r(n)$ to count the number of partitions of a positive integer n into parts not divisible by r.

---

[*] Corresponding author.

2020 *Mathematics Subject Classification:* Primary 05A17, Secondary 11P83.
Submitted October 29, 2025. Published March 12, 2026

This function generalizes the classical partition function $p_r(n)$, which counts all partitions of n without restriction. Formally, Ramanujan defined $p_r(n)$ through its generating function:

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r \tag{1.1}$$

for non-zero integer r. This product excludes terms where n is divisible by r, effectively removing those parts from the partition count. For partition function $p(n)$, Ramanujan's so called "most beautiful identity" is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

which readily implies

$$p(5n+4) \equiv 0 \pmod{5}.$$

The generalization of Ramanujan-type congruences, specifically those modulo powers of 5 and 7, to all partition functions $p_r(n)$ was established by [16]. Their work extended the classical congruences discovered by Ramanujan for the ordinary partition function $p(n)$ to a broader family of multipartition functions. At a later stage, Atkin [2] established that the proof proposed by Ramanathan was not valid. Further [13,14,15] studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. The functions $p_r(n)$ have been the subject of extensive investigation by various mathematicians. For the wonderful work one can see [1,2,3,4,6,8,9,10,12,17,18,19,20,21,22,23]. Recently, [11] proved that

$$p_{-2}(5n+\ell) \equiv 0 \pmod{5},$$

where $\ell \in \{2,3,4\}$. Also [7] proved

$$p_{-2}(25n+23) \equiv 0 \pmod{25}$$

by using modular forms. More recently [24] proved some congruences modulo powers of 5 for $p_r(n)$ with $r \in \{2,6,7\}$. For example,

$$\begin{aligned} &p_{-2}\left(5^{2\delta-1}n + \frac{7\times 5^{2\delta-1}+1}{12}\right)\\ &\equiv p_{-6}\left(5^{2\delta}n + \frac{3\times 5^{\delta}+1}{4}\right)\\ &\equiv p_{-7}\left(5^{2\delta-1}n + \frac{13\times 5^{2\delta-1}+7}{24}\right)\\ &\equiv 0 \pmod{5^\delta}. \end{aligned}$$

Motivated by the aforementioned work, we deduce new infinite families of congruences modulo 19, 23 and 29 for the generalized partition function $p_r(n)$. These congruences are derived through the application of suitable q-series identities and hold for any positive integer λ. Specifically, we establish that for each modulus $m \in \{19,23,29\}$ there exist arithmetic progressions of the form $n = m\lambda k + \delta$ (for some fixed δ) such that

$$p_r(n) \equiv 0 \pmod{n}$$

for all integers $k \geq 0$. The proofs rely on dissecting generating functions and exploiting the modular properties of associated q-series.

Using the well-known binomial theorem, one can easily deduce the following congruence, which will be employed repeatedly throughout our proofs without explicit reference:

$$(1+q)^p \equiv 1+q^p \pmod{p} \tag{1.2}$$

for any prime p, assuming q is an indeterminate. This congruence follows directly from the binomial expansion and the fact that all intermediate binomial coefficients $\binom{p}{k}$ for $1 \leq k \leq p-1$ are divisible by p.

2. Main Results

Theorem 2.1 *For any non-negative integer λ and $k \neq 1, 3, 6, 10, 15, 16$ we have,*

$$p_{-(19\lambda+3)}(19n+k) \equiv 0 \pmod{19}.$$

Proof On setting $r = -(19\lambda + 3)$ in (1.1), we have

$$\sum_{n=0}^{\infty} p_{(19\lambda+3)}(n)q^n \equiv (q;q)_\infty^{(19\lambda+3)} = (q;q)_\infty^{19\lambda}(q;q)_\infty^3.$$

Using equation (1.2) in the above equation, we have

$$\sum_{n=0}^{\infty} p_{(19\lambda+3)}(n)q^n \equiv (q^{19};q^{19})_\infty^{\lambda}(q;q)_\infty^3 \pmod{19}. \tag{2.1}$$

From [5, p. 39, Entry 24(ii)], we have

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{2.2}$$

Now, on expanding (2.2) for various powers of q, on aligning the terms and after some simplification, we deduce

$$\begin{aligned}
(q;q)_\infty^3 = {} & \left(1 - q^{190} + q^{711} - \ldots\right) - 3q\left(1 - q^{209} + q^{779} - \ldots\right) \\
& + 5q^3\left(1 - q^{228} + q^{817} - \ldots\right) - 7q^6\left(1 - q^{247} + q^{855} - \ldots\right) \\
& + 9q^{10}\left(1 - q^{266} + q^{893} - \ldots\right) - 11q^{15}\left(1 - q^{285} + q^{931} - \ldots\right) \\
& + 13q^{21}\left(1 - q^{304} + q^{969} - \ldots\right) - 15q^{28}\left(1 - q^{323} + q^{1012} - \ldots\right) \\
& + 17q^{36}\left(1 - q^{342} + q^{1049} - \ldots\right) + 0 + 2q^{55}\left(1 - q^{380} + q^{1121} - \ldots\right) \\
& - 4q^{66}\left(1 - q^{399} + q^{1159} - \ldots\right) + 6q^{78}\left(1 - q^{418} + q^{1197} - \ldots\right) \\
& - 8q^{91}\left(1 - q^{437} + q^{1235} - \ldots\right) + 10q^{105}\left(1 - q^{456} + q^{1273} - \ldots\right) \\
& - 12q^{120}\left(1 - q^{475} + q^{1311} - \ldots\right) + 14q^{136}\left(1 - q^{494} + q^{1349} - \ldots\right) \\
& - 16q^{153}\left(1 - q^{513} + q^{1387} - \ldots\right) + 18q^{171}\left(1 - q^{532} + q^{1425} - \ldots\right)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(q;q)_\infty^3 \equiv {} & J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 \\
& - 11q^{15}J_5 + 13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8 \pmod{19}.
\end{aligned} \tag{2.3}$$

where J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7 and J_8 are the series with integral powers of 19.
Employing (2.3) in the (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{(19\lambda+3)}(n)q^n \equiv {} & (q^{19};q^{19})_\infty^{\lambda}\left(J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 - 11q^{15}J_5\right. \\
& \left. + 13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8\right) \pmod{19}.
\end{aligned}$$

On extracting the terms involving q^{19n+k} for $k \neq 1, 3, 6, 10, 15, 16$ in the above, we obtain the desired result.

Theorem 2.2 *For any non-negative integer λ and $k \neq 1, 3, 6, 10, 15, 16, 21, 23$ we have,*

$$p_{-(23\lambda+3)}(23n+k) \equiv 0 \pmod{23}.$$

Proof On setting $r = -(23\lambda + 3)$ in (1.1), we have

$$\sum_{n=0}^{\infty} p_{(23\lambda+3)}(n)q^n \equiv (q;q)_\infty^{(23\lambda+3)} = (q;q)_\infty^{23\lambda}(q;q)_\infty^3. \tag{2.4}$$

On expanding from (2.2) for modulo 23, on the similar manner, we can deduce

$$\begin{aligned}(q;q)_\infty^3 \equiv & J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 - 11q^{15}J_5\\ & + 13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8 - 19q^{45}J_9 + 21q^{55}J_{10} \pmod{23}.\end{aligned} \tag{2.5}$$

Using equation (1.2) in equation (2.4), we have

$$\sum_{n=0}^{\infty} p_{(23\lambda+3)}(n)q^n \equiv (q^{23};q^{23})_\infty^{\lambda}(q;q)_\infty^3 \pmod{23}.$$

Employing (2.5) in the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_{(23\lambda+3)}(n)q^n \equiv & (q^{23};q^{23})_\infty^{\lambda}\left(J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 - 11q^{15}J_5\right.\\ & \left.+13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8 - 19q^{45}J_9 + 21q^{55}J_{10}\right) \pmod{23}.\end{aligned}$$

On extracting the terms involving q^{23n+k} for $k \neq 1, 3, 6, 10, 15, 16, 21, 23$ in the above we obtain the desired result.

Theorem 2.3 *For any non-negative integer λ and $k \neq 1, 3, 6, 10, 15, 16, 21, 23, 26, 27, 28, 29$ we have,*

$$p_{-(29\lambda+3)}(29n + k) \equiv 0 \pmod{29}.$$

Proof On setting $r = -(29\lambda + 3)$ in (1.1), we have

$$\sum_{n=0}^{\infty} p_{(29\lambda+3)}(n)q^n \equiv (q;q)_\infty^{(29\lambda+3)} = (q;q)_\infty^{29\lambda}(q;q)_\infty^3. \tag{2.6}$$

Similarly, for modulo 29 from (2.2), we can deduce

$$\begin{aligned}(q;q)_\infty^3 \equiv & J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 - 11q^{15}J_5 + 13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8\\ & - 19q^{45}J_9 + 21q^{55}J_{10} - 23q^{66}J_{11} + q^{78}J_{12} - 27q^{91}J_{13} \pmod{29}.\end{aligned} \tag{2.7}$$

Using equation (1.2) in the equation (2.6), we have

$$\sum_{n=0}^{\infty} p_{(29\lambda+3)}(n)q^n \equiv (q^{29};q^{29})_\infty^{\lambda}(q;q)_\infty^3 \pmod{29}.$$

Employing (2.7) in the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_{(29\lambda+3)}(n)q^n \equiv & (q^{29};q^{29})_\infty^{\lambda}\left(J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 - 11q^{15}J_5\right.\\ & + 13q^{21}J_6 - 15q^{28}J_7 + 17q^{36}J_8 - 19q^{45}J_9 + 21q^{55}J_{10} - 23q^{66}J_{11}\\ & \left.+q^{78}J_{12} - 27q^{91}J_{13}\right) \pmod{29}.\end{aligned}$$

On extracting the terms involving q^{29n+k} for $k \neq 1, 3, 6, 10, 15, 16, 21, 23, 26, 27, 28, 29$ in the above we obtain the desired result.

3. Conclusion

In this investigation, we established three congruences for the general partition functions modulo 19, 23, and 29 through the application of a q-series identity. These results not only extend the landscape of known partition congruences but also demonstrate the competence of the employed technique. We anticipate that this approach can be further generalized to derive similar congruences for larger moduli, thereby opening avenues for future exploration in the theory of partitions and modular forms.

References

1. Andrews, G. E., in: G.-C. Rota(Ed.), The Theory of Partitions, in: Encyclopedia of Mathematics and its Applications, vol. 2, Addison-Wesley, Reading, 1976 (Reprinted: Cambridge Univ. Press, London and New York, 1984).
2. Atkin, A. O. L., *Ramanujan congruences for* $p_k(n)$, Canad. J. Math., 20, 67-78 (1968).
3. Baruah, N. D. and Ojah, K. K., *Some congruences deducible from Ramanujan's cubic continued fraction*, Int. J. Number Theory, 7, 1331-134 (2011).
4. Baruah, N. D. and Sarmah, B. K., *Identities and congruences for the General partition and Ramanujan* τ *functions*, Indian J. of Pure and Appl. Math., 44(5), 643-671 (2013).
5. Berndt, B. C., Ramanujan's Notebooks, Part III, Springer, New York (1991).
6. Boylan, M., *Exceptional congruences for powers of the partition functions*, Acta Arith., 111, 187-203 (2004).
7. Chen, W. Y. C., Du, D. K., Hou, Q. H. and Sun, L. H., *Congruences of multi-partition functions modulo powers of primes*, Ramanujan J., 35, 1-19 (2014).
8. Farkas, H. M. and Kra, I. *Ramanujan's partition identities*, Contemporary Math., 240, 111-130 (1999).
9. Gandhi, J. M., *Congruences for* $p_k(n)$ *and Ramanujan's* τ *function*, Amer. Math. Mon., 70, 265-274 (1963).
10. Gordon, B., *Ramanujan congruences for* $p_k \pmod{11^r}$, Glasgow Math. J., 24, 107-123 (1983).
11. Hammond, P. and Lewis, R., *Congruences in ordered pairs of partitions*, Int. J. Math. Math. Sci., 45-48, 2509-2512 (2004).
12. Kiming, I. and Olsson, J. B., *Congruences like Ramanujan's for powers of the partition function*, Arch. Math., (Basel), 59(4), 348-360 (1992).
13. Newmann, M., *An identity for the coefficients of certain modular forms*, J. Lond. Math. Soc., 30, 488-493 (1955).
14. Newmann, M., *Congruence for the coefficients of modular forms and some new congruences for the partition function*, Canad. J. Math., 9, 549-552 (1957).
15. Newmann, M., *Some theorems about* $p_k(n)$, Canad. J. Math., 9, 68-70 (1957).
16. Ramanathan, K. G., *Identities and congruences of the Ramanujan type*, Canad. J. Math., 2, 168-178 (1950).
17. Ramanujan, S., *Some properties of* $p(n)$, *the number of partitions of* n, Proc. Cambridge Phillos. Soc., 19, 207-210 (1919).
18. Ramanujan, S. *Congruence properties of partitions*, Proc. Lond. Math. Soc., 18, Records of 13 March 1919. (1920).
19. Ramanujan, S., *Congruence properties of partitions*, Math. Z., 147-153 (1921).
20. Saikia, N. and Chetry, J., *Infinite families of congruences modulo* 7 *for Ramanujan's general partition function*, Ann. Math. Quebec., 42(1), 127–132 (2018).
21. Shruthi and Srivatsa Kumar B. R., *Arithmetic identities of Ramanujan's general partition function for modulo 17, Proceedings of the Jangjeon Mathematical Society, 22(4), 625-630 (2019).*
22. *Srivatsa Kumar, B. R., Shruthi, and Ranganatha, D.,* Some new congruences modulo 5 for general partition function*, Russian Mathematics, 64(7), 73-78 (2020).*
23. *Srivatsa Kumar, B. R., Narendra, R., and Rajanna, K. R.,* Arithmetic properties of Ramanujan's general partition function for modulo 11, Kuwait Journal of Science, 48(1), 134-137 (2021).
24. Tang, D., *Congruences modulo powers of* 5 *for* k*-colored partitions*, J. Number Theory, 187, 198-214 (2018).

B. R. Srivatsa Kumar,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal - 576 104,
India
ORCID 0000-0002-5684-9834
E-mail address: `srivatsa.kumar@manipal.edu`

and

H. S. Manu Banakar,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal - 576 104,
India.
E-mail address: `manubanakar786@gmail.com`

and

R. G. Veeresha,
Department of Mathematics,
Sri Jayachamarajendra College of Engineering,
JSS Science and Technology University,
Manasagangotri, Mysuru - 570 006,
India
ORCID 0000-0003-1297-0568
E-mail address: `veeru.rg@gmail.com`