Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 3** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.79805

Eta Quotients of Level 18 and Weight 1: Classification and Applications

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ABSTRACT: We classify all eta quotients in the space $M_1\left(\Gamma_0(18), \left(\frac{-3}{*}\right)\right)$ of modular forms and explicitly compute their Fourier coefficients, where $\left(\frac{d}{*}\right)$ denotes the Legendre–Jacobi–Kronecker symbol, viewed as a Dirichlet character modulo 18 taking values in $\mathbb{Q}$.

Key Words: Eta quotients, modular forms, Fourier coefficients of cusp forms, Eta function.

Contents

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, and complex numbers, respectively. Let $N \in \mathbb{N}$, $k \in \mathbb{Z}$, and let χ be a Dirichlet character whose modulus divides N. We define the congruence subgroup $\Gamma_0(N)$ of level N by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k and character χ on the congruence subgroup $\Gamma_0(N)$. The subspaces of Eisenstein series and cusp forms are denoted by $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. It is well-known that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{1.1}$$

For instance, (see [7, p. 83]).

The Dedekind eta function $\eta(z)$ is a holomorphic function on the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

defined by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right).$$

A product of the form

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z), \tag{1.2}$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an eta quotient.

2020 *Mathematics Subject Classification:* 11F11, 11F20, 11F30.
Submitted October 29, 2025. Published January 22, 2026

Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$, we define the generalized divisor sum $\sigma_{(\chi,\psi)}(n)$ by

$$\sigma_{(\chi,\psi)}(n) = \sum_{1 \leq m | n} \chi(m)\psi(n/m). \tag{1.3}$$

If $n \notin \mathbb{N}$, we set $\sigma_{(\chi,\psi)}(n) = 0$.

Let χ_0 denote the trivial character, *i.e.*, $\chi_0(m) = 1$ for all $m \in \mathbb{Z}$. In particular, we observe that $\sigma_{(\chi_0,\chi_0)}(n)$ coincides with the classical divisor-counting function

$$\sigma_0(n) = \sum_{1 \leq m | n} 1.$$

We now define two Dirichlet character modulo 18 by

$$\chi_1(m) = \left(\frac{-3}{m}\right), \qquad (m \in \mathbb{Z}),$$

where $\left(\frac{d}{\cdot}\right)$ denotes the Legendre-Jacobi-Kronecker symbol considered as a Dirichlet character modulo 18 with values in the field of rational numbers. These are precisely nontrivial Dirichlet characters modulo 18.

The cusps of $\Gamma_0(N)$ can be represented by rational numbers $\frac{a}{c}$, where $a \in \mathbb{Z}$, $c \in \mathbb{N}$, $c \mid N$, and $\gcd(a,c) = 1$; (see [6, p. 320]).

For the group $\Gamma_0(18)$, we can chose a set of representatives of the cusps as

$$\frac{1}{18}, \quad \frac{1}{9}, \quad \frac{1}{6}, \quad \frac{1}{3}, \quad \frac{2}{3}, \quad \frac{5}{6}, \quad \frac{1}{2}, \quad \frac{1}{1}. \tag{1.4}$$

Let $f(z)$ be an eta quotient as defined in (1.2). A formula for the order $v_{a/c}(f)$ of f at the cusp $\frac{a}{c}$ (see [6, p. 320]) is given by

$$v_{a/c}(f) = \frac{N}{24 \gcd(c^2, N)} \sum_{1 \leq \delta | N} \frac{\gcd(\delta, c)^2 \, r_\delta}{\delta}. \tag{1.5}$$

It follows from the dimension formulae [7, Section 6.3] that the only nontrivial modular form spaces of level 18 with trivial cuspidal subspaces are

$$M_1(\Gamma_0(18), \chi_1).$$

Moreover, we have that

$$\dim\big(M_1(\Gamma_0(18), \chi_1)\big) = \dim\big(E_1(\Gamma_0(18), \chi_1)\big) = 4. \tag{1.6}$$

In this paper, we find all the eta quotients in $M_1(\Gamma_0(18), \chi_1)$ and we determine their Fourier coefficients.

2. Preliminary results

Throughout the remainder of this paper we use the notation $q = e^{2\pi i z}$ with $z \in \mathbb{H}$. We define the Eisenstein serie $E_{\chi_1,\chi_0}(q)$ by

$$E_{\chi_1,\chi_0}(q) = \frac{1}{3} + \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_0)}(n) q^n.$$

In view of (2.1) for $N = 18$, we define an eta quotient $f(z)$ by

$$f(z) = \eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_3}(3z)\eta^{r_6}(6z)\eta^{r_9}(9z)\eta^{r_{18}}(18z). \tag{2.1}$$

Theorem 2.1 *Let $f(z) \in M_1(\Gamma_0(18), \chi_1)$ be an eta quotient given by (2.1), and let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be its Fourier series expansion. Then*

$$f(z) = b_1 E_{\chi_1,\chi_0}(q) + b_2 E_{\chi_1,\chi_0}(q^2) + b_3 E_{\chi_1,\chi_0}(q^3) + b_3 E_{\chi_1,\chi_0}(q^6)$$

for unique scalars $b_1, b_2, b_3, b_4 \in \mathbb{C}$, and the Fourier coefficients a_n are given by

$$a_n = b_1\sigma_{\chi_1,\chi_0}(n) + b_2\sigma_{\chi_1,\chi_0}(n/2) + b_3\sigma_{\chi_1,\chi_0}(n/3) + b_4\sigma_{\chi_1,\chi_0}(n/6) \text{ for } n \geq 1,$$
$$a_0 = \frac{1}{4}(b_1 + b_2 + b_3 + b_4).$$

Proof: It follows from (1.6) and [7, Theorem 5.9] that the set of Eisenstein series $\{E_{\chi_1,\chi_0}(q),\ E_{\chi_1,\chi_0}(q^2),\ E_{\chi_1,\chi_0}(q^3),\ E_{\chi_1,\chi_0}(q^6)\}$ is a basis for $M_1(\Gamma_0(18), \chi_1)$. Thus,

$$f(z) = b_1 E_{\chi_1,\chi_0}(q) + b_2 E_{\chi_1,\chi_0}(q^2) + b_3 E_{\chi_1,\chi_0}(q^3) + b_4 E_{\chi_1,\chi_0}(q^6)$$

for some unique scalars $b_1, b_2, b_3, b_4 \in \mathbb{C}$, from which the assertion follows by equating the coefficients of q^n on both sides. □

We use the following lemma to determine if certain eta quotients are modular forms. See [3, Theorem 5.7, p. 99], [4, Corollary 2.3, p. 37], [2, p. 174], and [5].

Lemma 2.2 *Let $f(z)$ be an eta quotient given by (2.1), and let $k = \frac{1}{2}\sum_{1\leq\delta|N} r_\delta$ and $s = \prod_{1\leq\delta|N} \delta^{r_\delta}$. Suppose that the following conditions are satisfied:*

(i) $\sum_{1\leq\delta|N} \delta \cdot r_\delta \equiv 0\,(mod\ 24)$,

(ii) $\sum_{1\leq\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0\,(mod\ 24)$,

(iii) $\sum_{1\leq\delta|N} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta} \geq 0$ *for each positive divisor d of N,*

(iv) k is an integer.

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where the character χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right).$$

We utilize the following notation for eta-quotients

$$[\delta_1, r_{\delta_1}; \delta_2, r_{\delta_2}; \ldots, \delta_{d-1}, r_{\delta_{d-1}}; \delta_d, r_{\delta_d}] = \prod_{i=1}^{d} \eta^{r_{\delta_i}}(\delta_i z),$$

We take $N = 18$ and $k = 1$ in Lemma 2.2 to obtain the following theorem.

Theorem 2.3 *Let $f(z)$ be an eta quotient given by (2.1), which satisfies the conditions (i)-(iv) in Lemma 2.2 with*

$$r_1 + r_2 + r_3 + r_6 + r_9 + r_{18} = 2. \tag{2.2}$$

Then

$$f(z) \in M_1(\Gamma_0(18), \chi_1), \quad \text{if } r_1 + r_6 + r_9 \equiv 1 \pmod 2. \tag{2.3}$$

Proof: For $N = 18$, we have

$$s = \prod_{1\leq\delta|32} \delta^{r_\delta} = 1^{r_1}2^{r_2}3^{r_3}6^{r_6}9^{r_9}18^{r_{18}} = 2^{r_2+r_6+r_{18}}3^{r_3+r_6+2r_9+2r_{18}}.$$

The conditions (i) and (ii) in Lemma 2.2 becomes

$$r_1 + 2r_2 + 3r_3 + 6r_6 + 9r_9 + 18r_{18} \equiv 0 \pmod{24},$$
$$18r_1 + 9r_2 + 6r_3 + 3r_6 + 2r_9 + r_{18} \equiv 0 \pmod{24}.$$

Let

$$\nu_3(s) = \sum_{\delta|18} r_\delta \cdot \nu_3(\delta) = r_3 + r_6 + 2r_9 + 2r_{18}. \tag{2.4}$$

Reducing (2.4) modulo 2, we obtain

$$\nu_3(s) \equiv r_3 + r_6 \pmod{2}. \tag{2.5}$$

Let us denote:

$$\begin{aligned} &r_1 + r_2 + r_3 + r_6 + r_9 + r_{18} = 2, \\ &r_1 + 2r_2 + 3r_3 + 6r_6 + 9r_9 + 18r_{18} \equiv 0 \pmod{24}, \\ &18r_1 + 9r_2 + 6r_3 + 3r_6 + 2r_9 + r_{18} \equiv 0 \pmod{24}. \end{aligned} \tag{2.6}$$

Reducing (2.6) modulo 2, we find:

$$r_1 + r_3 + r_9 \equiv 0 \pmod{2}, \qquad r_2 + r_6 + r_{18} \equiv 0 \pmod{2}. \tag{2.7}$$

Since, the character associated to $f(z)$ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right) = \left(\frac{-s}{m}\right).$$

Then, from (2.5), (2.6) and (2.7) we obtain

$$r_1 + r_6 + r_9 \equiv 0 \pmod{2}$$

These match the given definition of χ_1 and the result follows. □

3. Main result

Theorem 3.1 *Let $f(z)$ be an eta quotient given by (2.1). Then we have $f(z) \in M_1(\Gamma_0(18), \chi_1)$ if and only if*

$$\begin{aligned} r_1 + 2r_2 + 3r_3 + 6r_6 + 9r_9 + 18r_{18} &\equiv 0 \pmod{24}, \\ 18r_1 + 9r_2 + 6r_3 + 3r_6 + 2r_9 + r_{18} &\equiv 0 \pmod{24}, \\ 0 \le v_{1/c}(f) < 4 \text{ for } c &= \frac{1}{18}, \frac{1}{9}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \\ r_1 + r_2 + r_3 + r_6 + r_9 + r_{18} &= 2, \\ r_1 + r_6 + r_9 &\equiv 1 \pmod{2}. \end{aligned}$$

Proof: Let $f(z) \in M_1(\Gamma_0(18), \chi_1)$ be an eta quotient given by (2.1). By (1.6) we have $\dim(M_1(\Gamma_0(18), \chi_1)) = 4$. We define the eta quotients $f_1(z), f_2(z), f_3(z), f_4(z)$ by

$$\begin{aligned} f_1(z) &= [1, -3; 2, 3; 3, 7; 6, -4; 9, -2; 18, 1], \\ f_2(z) &= [1, -3; 2, 6; 3, 1; 6, -2; 9, 0; 18, 0], \\ f_3(z) &= [1, -2; 2, 1; 3, 6; 6, -3; 9, 0; 18, 0], \\ f_4(z) &= [1, -2; 2, 4; 3, 0; 6, -1; 9, 2; 18, -1]. \end{aligned}$$

By Lemma 2.2, we have $f_1(z), f_2(z), f_3(z), f_4(z) \in M_1(\Gamma_0(18), \chi_1)$. One can easily see that the set $\{f_1(z), f_2(z), f_3(z), f_4(z)\}$ is linearly independent, and so it is a basis for $M_1(\Gamma_0(18), \chi_1)$. Appealing to (1.4) and (1.5), we have

$v_{1/c}(\cdot)/c$	1	1/2	1/3	1/6	1/9	1/18
f_1	0	1	1	0	0	0
f_2	0	3	0	0	0	0
f_3	0	0	1	0	1	0
f_4	0	2	0	0	1	0

Thus, for any $b_1, b_2, b_3, b_4 \in \mathbb{C}$ we have

$$v_{1/c}(b_1 f_1 + b_2 f_2 + b_3 f_3 + b_4 f_4) \in \mathbb{N}_0.$$

As $f(z)$ can be expressed as a linear combination of $f_1(z), f_2(z), f_3(z)$, and $f_4(z)$, we have

$$v_{1/c}(f) \in \mathbb{N}_0,$$

from which the second and first assertions follow, respectively. The third assertion follows from [4, Corollary 2.3] and the fifth assertion follows from (2.3). The converse follows from Theorem 2. □

There are 28 eta quotients in $M_1(\Gamma_0(18), \chi_1)$. We found all the eta quotients with Sagemath using Theorems 3.1. We then determined their Fourier coefficients using Theorems 2.1. All these eta quotients and their Fourier coefficients are listed in the appendices (Table 1).

4. Applications

Theorem 4.1 *Let $f(z) \in M_1(\Gamma_0(18), \chi_1)$ with the Fourier series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

Then If $n \equiv 5, 11, 15, 17 \pmod{18}$, we have $a_n = 0$.

Proof: Suppose $n \equiv 5, 11, 15, 17 \pmod{18}$. Then

$$\chi_1(n) = \left(\frac{-3}{n}\right) = \left(\frac{n}{3}\right) = -1.$$

Also, for all positive divisors d of n, we have

$$\chi_1(n/d) = \left(\frac{-3}{n/d}\right) = \left(\frac{-3}{nd}\right) = \left(\frac{-3}{n}\right)\left(\frac{-3}{d}\right) = -\left(\frac{-3}{d}\right) = -\chi_1(d).$$

By pairing $\chi_1(d)$ and $\chi_1(n/d)$ for all $d \mid n$, we obtain

$$\sum_{d \mid n} \chi_1(d) = 0.$$

The assertion now follows from (1.4), (1.3), and Theorem 2. □

The following corollary follows immediately from Theorem 4.1.

Corollary 4.2 *If an eta quotient $f(z)$ given by (2.1) is a modular form of weight 1 with the Fourier series representation $f(z) = \sum_{n=0}^{\infty} a_n q^n$, then*

$$a_n = 0, \quad \text{if } n \equiv 5, 11, 15, 17 \pmod{18} \text{ and } r_1 + r_6 + r_9 \equiv 1 \pmod{2}.$$

Acknowledgments

The authors would like to thank the anonymous referee for the valuable suggestions and valuable comments which improved the quality of this paper.

Conflict of interest

The authors declare that there is no conflict of interest.

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5. Appendices

Table 1: Eta quotients in $M_1(\Gamma_0(18),\chi_1)$ and their Fourier coefficients

No.	r_1	r_2	r_3	r_6	r_9	r_{18}	(b_1,b_2,b_3,b_4)
1	-3	3	7	-4	-2	1	$(3,6,3,-9)$
2	-3	6	1	-2	0	0	$(3,3,0,-3)$
3	-2	1	6	-3	0	0	$(2,4,0,-3)$
4	-2	1	7	-4	-3	3	$(1,2,3,-6)$
5	-2	4	0	-1	2	-1	$(2,1,0,0)$
6	-2	4	1	-2	-1	2	$(1,2,0,-3)$
7	-1	2	-1	0	4	-2	$(1,0,1,1)$
8	-1	2	0	-1	1	1	$(1,1,-1,-1)$
9	-1	2	1	-2	-2	4	$(0,1,1,-2)$
10	0	0	-3	6	1	-2	$(0,0,3,0)$
11	0	0	-2	1	6	-3	$(0,0,2,1)$
12	0	0	-1	0	3	0	$(1,0,-1,0)$
13	0	0	0	-1	0	3	$(0,1,0,-1)$
14	0	0	1	-2	-3	6	$(0,0,1,-1)$
15	0	0	6	-3	-2	1	$(0,0,-6,9)$
16	0	3	0	-1	0	0	$(0,-3,0,6)$
17	1	-2	-4	7	3	-3	$(-1,1,3,0)$
18	1	-2	-3	6	0	0	$(1,-1,0,0)$
19	1	1	-1	0	2	-1	$(-1,-2,3,3)$
20	1	1	0	-1	-1	2	$(1,-1,-3,3)$
21	2	-1	-2	1	4	-2	$(-2,0,4,1)$
22	2	-1	-1	0	1	1	$(1,-2,-1,2)$
23	2	-1	0	-1	-2	4	$(0,1,-2,1)$
24	3	-3	-4	7	1	-2	$(-3,3,3,0)$
25	3	0	-1	0	0	0	$(-3,0,9,-3)$
26	4	-2	-2	1	2	-1	$(-4,4,6,-3)$
27	4	-2	-1	0	-1	2	$(1,-4,3,0)$
28	6	-3	-2	1	0	0	$(-6,12,0,-3)$

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