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On Reduced Reciprocal Randić Energy of Lexocographic Product of Graphs

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ABSTRACT: Let $G = G_1[G_2]$ represent the lexicographic product of two graphs with $m + n$ vertices and mn edges. The vertex set of G is given by $V(G) = \{w_{ij} = (u_{ij}, v_{ij}) : 1 \leq i \leq m, 1 \leq j \leq n\}$. As introduced in [4], the reduced reciprocal Randić matrix of a graph G with n vertices, denoted by $RRR(G)$, is an $n \times n$ matrix whose (i, j)th entry is defined as $\sqrt{(d_{v_i} - 1)(d_{v_j} - 1)}$ if the vertices v_i and v_j are adjacent, and 0 otherwise. The reduced reciprocal Randić energy of a graph, denoted by $RRRE(G)$, is defined as the sum of the absolute values of the eigenvalues of $RRR(G)$. In this work, we investigate the reduced reciprocal Randić energy $RRRE(G)$ for several lexicographic product graphs, including $RRR(K_m[K_n])$, $RRR(K_m[C_n])$and $RRR(C_m[C_n])$.

Keywords: Energy of graph, lexicographic product, reduced reciprocal randic energy.

Contents

1. Introduction

Let G_1 and G_2 be undirected simple graphs with m and n vertices, respectively. The vertex set of their lexicographic product $G = G_1[G_2]$ is defined as

$$V(G) = \{w_{ij} = (u_i, v_j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}.$$

Two vertices w_i and w_j are adjacent if an edge exists between them in G.

The concept of graph energy was introduced in 1978 [3], defined as the sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$ associated with the graph G.

In 2014, Gutman et al. [3] proposed the reduced reciprocal Randić matrix $RRR(G)$, given by

$$RRR(G) = \sum_{v_i v_j \in E(G)} \sqrt{(d_{v_i} - 1)(d_{v_j} - 1)},$$

where $E(G)$ denotes the edge set of G, and d_{v_i} and d_{v_j} represent the degrees of vertices v_i and v_j, respectively.

Further research on the reduced reciprocal Randić matrix and its corresponding energy can be found in [4]. In the present work, we analyze the reduced reciprocal Randić energy $RRRE(G)$ for several lexicographic product graphs, including

$$RRR(K_m[K_n]), \quad RRR(K_m[C_n]), \quad \text{and} \quad RRR(C_m[C_n]).$$

2. Preliminaries

Definition 2.1 Let G_1 and G_2 be two graphs with m and n vertices respectively. Then Lexicographic product of two graphs G_1 and G_2 is such that the vertex set is the cartesian product $V(G_1) \times V(G_2)$ and joining any two vertices (u_1, u_2) and (v_1, v_2) if and only if, either u_1 is adjacent to v_1 in G_1 or u_1=v_1, u_2 is adjacent to v_2 in G_2. This graph is represented by $G_1[G_2]$ and is also called as composition of graphs.

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Definition 2.2 [4] The reduced reciprocal randic matrix of a graph $G = G_1 \otimes G_2$, defined by

$$RRR(G) = [a_{ij}] = \begin{cases} \sqrt{(d_{v_i}-1)(d_{v_j}-1)} & if\ v_i\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

If $\lambda_1, \lambda_2, \cdots, \lambda_{mn}$ are its eigenvalues, the reduced reciprocal randic energy of G is defined as

$$RRRE(G) = \sum_{i=1}^{mn} |\lambda_i|.$$

Definition 2.3 [1] Let $B_1, B_2, \ldots B_m$ be square matrices of order n. A block circulant matrix of order mn is of the form

$$bcirc(B_1, B_2, \ldots B_m) = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ B_m & B_1 & \cdots & B_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_3 & \cdots & B_1 \end{pmatrix}$$

If each B_i for $1 \leqslant i \leqslant n$ is circulant, then we call the above matrix a circulant block matrix with circulant blocks.

Theorem 2.1 *[1] Let C be an A-factor block circulant. Then*

$$C = V_A P(D_A) V_A^{-1},$$

where V_A is block Vandermonde matrix and $P(z)$ is the representor of C. Moreover, the set of A-factor circulants coincides with the set of matrices of the form

$$V_A diag[M_1, M_2, \ldots, M_m] V_A^{-1},$$

that is,$P(D_A) = diag[M_1, M_2, \ldots, M_m]$ for a matrix polynomial

$$P(z) = C_1 + C_2 z + \ldots + C_m z^{m-1} if\ and\ only\ if\ [C_1 C_2 \ldots C_m] V_A = [M_1 M_2 \ldots M_m].$$

The following result is a consequence of the above Theorem 2.1

Corollary 2.1 *[1] The factor circulant C can also be expressed as*

$$C = \Re F_{mn}^* P(K\Omega) F_{mn} \Re^{-1}$$

where F_{mn} is a block Fourier matrix, $\Omega = diag[I, \omega I, \omega^2 I, \ldots, \omega^{m-1} I]$ $(\omega = exp(\frac{2\pi i}{m})), K$ is the principal m^{th} root of the non -singular matrix A and $\Re = diag[IKK^2 \ldots K^{m-1}]$. In particular if C is a block circulant then it can be represented as

$$C = F_{mn}^* P(\Omega) F_{mn}.$$

Motivated by the preceding discussion, Section 2 presents an analysis of the reduced reciprocal Randić energy for the lexicographic product graphs $K_m[K_n]$, $K_m[C_n]$ and $C_m[C_n]$.

3. Reduced Reciprocal Randić Energy of the Lexicographic Product of Graphs

Let G_1 and G_2 be two graphs with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. The lexicographic product of these graphs, denoted by $G = G_1[G_2]$, contains mn vertices of the form $w_{ij} = (u_i, v_j)$.
In [4], the authors examined the reduced reciprocal Randić energy for several types of graphs, including the complete graph K_m, the crown graph S_n^0, the complete bipartite graph $K_{m,n}$, and the cocktail party graph $K_{n\times 2}$.

Motivated by their work, we extend this study by determining the reduced reciprocal Randić energy of the lexicographic product of two graphs G_1 and G_2, represented as $G = G_1[G_2]$.

Theorem 3.1 *Let K_m and K_n be complete graphs with vertices m and n respectively. Then*

$$RRRE(K_m[K_n]) = 2(mn-1)^2.$$

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_m and K_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(K_m[K_n])$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(K_m[K_n]) = bcirc(A_1, A_2, \ldots, A_n)_{mn\times mn}$$

where

$$A_1 = circ(0, mn-1, mn-1, \cdots, mn-1)_{m\times m}$$

and

$$A_2 = \cdots = A_n = circ\left(mn-1, mn-1, \cdots, mn-1\right)_{m\times m}.$$

From Theorem 1.6, the diagonal form of $RRR(K_m[K_n])$ is

$$diag([\wedge_1 + (n-1)\wedge_2], [\wedge_1 - \wedge_2], \ldots, [\wedge_1 - \wedge_2])_{mn\times mn}$$

where

$$\wedge_1 = diag((m-1)(mn-1), 1-mn, \cdots, 1-mn)_{m\times m}$$

and

$$\wedge_2 = diag(m(mn-1), 0, \cdots, 0)_{m\times m}$$

are the spectra of A_1 and A_2 respectively.
Consider,

$$[\wedge_1 + (n-1)\wedge_2] = diag((mn-1)^2, (1-mn), \cdots, (1-mn))_{m\times m}$$

and

$$[\wedge_1 - \wedge_2] = diag((1-mn), (1-mn), \cdots, (1-mn))_{m\times m}.$$

Hence, spectrum of $RRR(K_m[K_n])$ is given by

$$\begin{cases} (mn-1)^2 & once \\ (1-mn) & (mn-1)\ times \end{cases}.$$

Thus,

$$RRRE(K_m[K_n]) = 2(mn-1)^2.$$

□

Theorem 3.2 *Let K_m and C_n be complete graph and cycle graph with vertices m and n respectively. Then*

$$RRRE(K_m[C_n]) = (mn-n+1)\left[(mn-n+2) + (m-1)|n-2| + 2m\sum_{k=1}^{(n-1)}\left|cos\left(\frac{2\pi k}{n}\right)\right|\right].$$

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_m and C_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(K_m[C_n])$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(K_m[C_n]) = bcirc(A_1, A_2, \ldots, A_n)_{mn\times mn}$$

where

$$A_1 = A_3 = A_4 = \cdots = A_{n-1} = circ(0, 1+(m-1)n, 1+(m-1)n, \cdots, 1+(m-1)n)_{m\times m},$$

and

$$A_2 = A_n = circ(1 + (m-1)n, 1 + (m-1)n, \cdots, 1 + (m-1)n)_{m\times m}.$$

From Theorem 1.6, the diagonal form of $RRR(K_m[C_n])$ is

$$diag\left([(n-2)\wedge_1 + 2\wedge_2], \left[2cos\left(\frac{2\pi k}{n}\right)(\wedge_2 - \wedge_1) \right] \right)_{mn\times mn}$$

where $1 \leq k \leq (n-1)$,

$$\wedge_1 = diag((m-1)(mn-n+1), (n-mn-1), \cdots, (n-mn-1))_{m\times m}$$

is the spectrum of A_1.

$$\wedge_2 = diag(m(mn-n+1), 0, \cdots, 0)_{m\times m}$$

is the spectrum of A_2.
Consider,

$$[(n-2)\wedge_1 + 2\wedge_2] = diag((mn-n+1)(mn-n+2), (n-2)(n-mn-1), \cdots, (n-2)(n-mn-1))_{m\times m}$$

and

$$\left[2cos\left(\frac{2\pi k}{n}\right)(\wedge_2 - \wedge_1) \right]$$

$$= diag\left(2(mn-n+1)cos\left(\frac{2\pi k}{n}\right), 2(mn-n+1)cos\left(\frac{2\pi k}{n}\right), \cdots, 2(mn-n+1)cos\left(\frac{2\pi k}{n}\right) \right)_{m\times m}.$$

Hence, spectrum of $RRR(K_m[C_n])$ is given by

$$\begin{cases} (mn-n+1)(mn-n+2) & once \\ (n-2)(n-mn-1) & (m-1)\ times \\ 2(mn-n+1)cos\left(\frac{2\pi k}{n}\right)\ where\ 1 \leq k \leq (n-1) & m(n-1)\ times \end{cases}.$$

Thus,

$$RRRE(K_m[C_n]) = (mn-n+1)\left[(mn-n+2) + (m-1)|n-2| + 2m\sum_{k=1}^{(n-1)} \left| cos\left(\frac{2\pi k}{n}\right) \right| \right].$$

□

Theorem 3.3 *Let C_m and C_n be cycle graphs with vertices m and n respectively. Then*

$$RRRE(C_m[C_n]) = 2(2n+1)(n+1) + 2(2n+1)\sum_{t=1}^{(m-1)} \left| 1 + ncos\left(\frac{2\pi t}{m}\right) \right|$$
$$+ 2m(2n+1)\sum_{k=1}^{(n-1)} \left| cos\left(\frac{2\pi k}{n}\right) \right|.$$

Proof: Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_m and C_n respectively. Then $\{w_{11}, w_{12}, \ldots, w_{1n}, \ldots, w_{m1}, w_{m2}, \ldots, w_{mn}\}$ is the vertex set of $(C_m[C_n])$ where $w_{ij} = (u_i, v_j)$. Then

$$RRR(C_m[C_n]) = bcirc(A_1, A_2, \ldots, A_n)_{mn\times mn}$$

where

$$A_1 = A_3 = \cdots = A_{n-1} = circ(0, (2n+1), 0, \cdots, 0, (2n+1))_{m\times m}$$

and

$$A_2 = A_n = circ((2n+1),(2n+1),0,\cdots,0,(2n+1))_{m\times m}.$$

From Theorem 1.6, the diagonal form of $RRR(C_m[C_n])$ is

$$diag\left([(n-2)\wedge_1 + 2\wedge_2],\left[2cos\left(\frac{2\pi k}{n}\right)(\wedge_2-\wedge_1)\right]\right)_{mn\times mn}$$

where $1\le k\le (n-1)$,

$$\wedge_1 = diag\left(2(2n+1), 2(2n+1)cos\left(\frac{2\pi t}{m}\right)\right)_{m\times m} \quad where\ 1\le t\le (m-1)$$

and

$$\wedge_2 = diag\left(3(2n+1),(2n+1)\left(1+2cos\left(\frac{2\pi t}{m}\right)\right)\right)_{m\times m} \quad where\ 1\le t\le (m-1)$$

are the spectra of A_1 and A_2 respectively.
Consider,

$$[(n-2)\wedge_1 + 2\wedge_2] = diag\left(2(2n+1)(n+1), 2(2n+1)\left(1+ncos\left(\frac{2\pi t}{m}\right)\right)\right)_{m\times m}$$

and

$$\left[2cos\left(\frac{2\pi k}{n}\right)(\wedge_2-\wedge_1)\right]$$

$$= diag\left(2(2n+1)cos\left(\frac{2\pi k}{n}\right),\cdots,2(2n+1)cos\left(\frac{2\pi k}{n}\right)\right)_{m\times m}.$$

Hence, spectrum of $RRR(C_m[C_n])$ is given by

$$\begin{cases} 2(2n+1)(n+1) & once \\ 2(2n+1)\left(1+ncos\left(\frac{2\pi t}{m}\right)\right)\ where\ 1\le t\le (m-1) & (m-1)\ times \\ 2(2n+1)cos\left(\frac{2\pi k}{n}\right)\ where\ 1\le k\le (n-1) & m(n-1)\ times \end{cases}.$$

Thus,

$$RRRE(C_m[C_n]) = 2(2n+1)\left[2+n\left(1+\sum_{t=1}^{m-1}\left|\cos\left(\frac{2\pi t}{m}\right)\right|\right)+m\sum_{k=1}^{n-1}\left|\cos\left(\frac{2\pi k}{n}\right)\right|\right].$$

□

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