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Analytical Solutions for Riemann-Liouville Fractional Operator Using a Generalized Power Series Framework

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ABSTRACT: In this paper, we propose a general form of the power series method to solve non-trivial fractional differential equations within the Riemann-Liouville framework . The proposed methodology extends and refines conventional series techniques, effectively overcoming their principal limitations. A rigorous theoretical foundation underpinning the approach is established, encompassing essential theorems and a comprehensive convergence analysis that guarantees its reliability. The practical efficacy of the method is demonstrated through its application to a variety of fractional models, where it is shown to yield accurate solutions and exhibit performance advantages over some existing schemes.

Keywords: Fractional differential equations, fractional power series, Riemann–Liouville derivatives.

Contents

1. Introduction

The Taylor formula within the Riemann-Liouville framework was first introduced by Riemann in [5]. The Riemann-Liouville generalized Taylor formula was later presented by the authors in [4] in the following form:

$$y(t) = \sum_{n=0}^{m} \gamma_{(n)} (t-t_0)^{(n+1)\nu-1} + R_m(t,t_0), \quad 0 < \nu \leq 1, \tag{1.1}$$

where $R_m(t,t_0) = \frac{{}^{RL}D_{t_0}^{(m+1)\nu} y(\xi)}{\Gamma((m+1)\nu+1)}(t-t_0)^{(m+1)\nu}, \quad t_0 \leq \xi \leq t$ and ${}^{RL}D_{t_0}^{(m+1)\nu}$ is the Riemann-Liouville fractional derivative of order $(m+1)\nu$, and $\gamma_{(n)}$ are the series coefficients for all $n = 0, \dots, m$.
The extension of Taylor's formula to the so-called Caputo derivative was presented in [9]. Similarly, the authors in [10] proposed solutions to fractional differential equations within the framework of the Caputo derivative, expressed as power series of the form:

$$y(t) = \sum_{n=0}^{+\infty} \gamma_{(n)} (t-t_0)^{n\nu} \tag{1.2}$$

where $0 \leq m-1 < \nu \leq m$ and $t \geq t_0$, which is referred to as an fractional power serie (FPS) centered at t_0, with t as a variable and γ_n as constants known as the coefficients of the series.
In the same spirit,many authors have generalized the form of the previous series, including the so-called generalized fractional power series within the Caputo framework [2,1], given in the following form:

$$y(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} (t-t_0)^{n\nu+m} \tag{1.3}$$

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where $n, m \in \mathbb{N}$, $t \geq t_0$, and $\gamma_{(n,m)}$ are the series coefficients.

However, we note that the previous forms of power series are not adequate for the fractional derivative wihin the framework of riemman liouville . For example, one cannot easily solve the following fractional differential equation using the most recent approaches:

$$\begin{aligned} &{}^{RL}D^{\nu}_{0+}y(t) = \rho y(t), \quad \rho \neq 0, \\ &{}^{RL}D^{\nu-k}_{0+}y(0) = b_k, \quad b_k \in \mathbb{R} \qquad k = 1, 2, \ldots, j. \end{aligned} \tag{1.4}$$

where $j - 1 < \nu \leq j$, $j \in \mathbb{N}^*$.

Motivated by this remark, we propose the following form of the power series:

$$y(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} t^{(n+1)\nu+m-j} \qquad , j - 1 < \nu \leq j \tag{1.5}$$

where $t > 0$ and $\gamma(n, m)$ are the series coefficients.

This form will be used to find solutions for several examples of fractional differential equations, particularly for the significant problem in (1.4).

The rest of this work is organized as follows. In Section 2,we review essential Definitions and Properties about fractional derivatives . In Section 3, we provide more details and explanations about our method and examine some important examples known in the literature to shed light on our present method. In the final section, we provide an overview of the major findings and advancements presented in this paper.

2. Preliminaries

This section outlines the foundational concepts and establishes the necessary notation [6]:

Definition 2.1 *For $\nu \in \mathbb{R}^+$ and $t \geq 0$, the Riemann-Liouville integral of order ν is defined as:*

$$(\mathcal{I}^{\nu}_{0+}y)(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{y(\xi)}{(t-\xi)^{1-\nu}}\, d\xi, \tag{2.1}$$

The Euler gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(\xi) = \int_0^{+\infty} t^{\xi-1} e^{-s}\, ds \quad (\mathbb{R}(\xi) > 0).$$

Definition 2.2 *For $n - 1 < \nu \leq n$ and $t \geq t_0$, the Riemann-Liouville derivative of order ν is defined as:*

$${}^{RL}D^{\nu}_{t_0+}y(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_{t_0}^t \frac{y(\xi)}{(t-\xi)^{\nu-n+1}} d\xi. \tag{2.2}$$

property 1 *If $\nu \in \mathbb{R}^+$ and $\mu > 0$, then:*

$$\left({}^{RL}D^{\nu}_{t_{0+}}(t-t_0)^{\mu-1}\right)(\xi) = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}(\xi-t_0)^{\mu-\nu-1}. \tag{2.3}$$

In particular, for the case where $\mu = 1$ and $\nu \in \mathbb{R}^+$, the Riemann-Liouville fractional derivative of a constant is generally non-zero :

$$\left({}^{RL}D^{\nu}_{t_0+}1\right)(\xi) = \frac{(\xi-t_0)^{-\nu}}{\Gamma(1-\nu)}. \tag{2.4}$$

For each integer j satisfying $1 \leq j \leq \lfloor \nu \rfloor + 1$,

$$\left({}^{RL}D^{\nu}_{t_0+}(t-t_0)^{\nu-j}\right)(\xi) = 0. \tag{2.5}$$

Definition 2.3 *For $n-1<\nu\leq n$ and $t\geq t_0$, the Caputo derivative of order ν is*

$$
{}^{C}D^{\nu}_{t_0}y(t)=\frac{1}{\Gamma(n-\nu)}\int_{t_0}^{t}\frac{y^{(n)}(\xi)}{(t-\xi)^{\nu-n+1}}d\xi. \tag{2.6}
$$

property 2 *If $\nu\in\mathbb{R}^+$ and $\mu>0$, then*

$$
\left({}^{C}D^{\nu}_{t_0}(t-t_0)^{\mu-1}\right)(\xi)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}(\xi-t_0)^{\mu-\nu-1}. \tag{2.7}
$$

In particular, if $\mu=1$ and $\nu\in\mathbb{R}^+$, the Caputo fractional derivatives of a constant are generally zero:

$$
\left({}^{C}D^{\nu}_{t_0}1\right)(\xi)=0.
$$

Moreover, for $j=0,1,\ldots,[\nu]$.

$$
\left({}^{C}D^{\nu}_{t_0}(t-t_0)^{j}\right)(\xi)=0. \tag{2.8}
$$

Definition 2.4 *If $\nu\in\mathbb{R}^+$ and $z\in\mathbb{R}$, the Mittag-Leffler function is defined as:*

$$
E_{\nu}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\nu n+1)}.
$$

Definition 2.5 *If $\nu,\mu\in\mathbb{R}^+$ and $z\in\mathbb{R}$, the generalized Mittag-Leffler function is defined as:*

$$
E_{(\nu,\mu)}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\nu n+\mu)}.
$$

3. Generalized Riemann-Liouville Fractional Power Series

In this section, we define FPS to establish and prove a generalized Taylor's formula with the Riemann-Liouville fractional derivative. Additionally, we provide illustrative examples to demonstrate the effectiveness of our approach.

Definition 3.1 *The generalized fractional power series within the riemann-liouville framework (GFPSR) is an infinite series of the form:*

$$
\begin{aligned}
y(t)&=\sum_{n\geq 0,m\geq 0}^{+\infty}\gamma_{(n,m)}(t-t_0)^{n\nu+\nu+m-k}\\
&=\sum_{n\geq 0,m\geq 0}^{+\infty}\gamma_{(n,m)}(t-t_0)^{(n+1)\nu+m-k}\qquad ,k-1\leq\nu<k
\end{aligned} \tag{3.1}
$$

where $k\in\mathbb{N}^*$, and $t>t_0$ serves as an indeterminate variable, while $\gamma(n,m)$ represents the coefficients of the series.

The choice of the exponent of GFPSR in the (3.1) based by the the property 1 and the property 2 for more explain we have the following relation:

$$
\left({}^{C}D^{\nu}_{t_0}(t-t_0)^{m}\right)(\xi)=\left({}^{RL}D^{\nu}_{t_0+}(t-t_0)^{\nu+m-k}\right)(\xi)=0,\quad \text{for } m=0,1,\ldots,[\nu].
$$

This expression highlights the connection between the Caputo fractional derivative and the Riemann-Liouville fractional derivative, illustrating how the exponent in the GFPSR is equated with the integer part of the generalized fractional power series (1.3). Specifically, for integer values of m within the range $0\leq m\leq[\nu]$, this choice proves effective in preserving the same calculus techniques for solving fractional differential equations within the Riemann-Liouville framework.

Lemma 3.1 *Let* $k-1<\nu\leq k$,$k\in\mathbb{N}^*$, *The equality* $\left(D_{0+}^{\nu}y\right)(t)=0$ *is valid if, and only if,*

$$y(t)=\sum_{m=0}^{k-1}c_m t^{\nu+m-k} \tag{3.2}$$

where $c_m\in\mathbb{R}^*$.

Proof: We begin by expressing $y(t)$ as follows :

$$y(t)=\sum_{j=0}^{n-1}c_j t^{\nu-n+j}=\sum_{j=1}^{n}c_j t^{\nu-j}.$$

Applying the riemann-liouville fractional derivative ${}^{RL}D_{0+}^{\nu}$ and using its linearity:

$${}^{RL}D_{0+}^{\nu}\ y(t)=\sum_{j=1}^{n}c_j\,{}^{RL}D_{0+}^{\nu}\left(t^{\nu-j}\right). \tag{3.3}$$

By Property 2, we know that ${}^{RL}D_{0+}^{\nu}\left(t^{\nu-j}\right)=0$ for $j=1,2,\dots,[\nu]+1$. Thus, we obtain:

$${}^{RL}D_{0+}^{\nu}\left(y(t)\right)=0.$$

□

Theorem 3.1 *Let* $0\leqslant k-1<\nu\leqslant k$ *and assume the generalized power series*

$$\sum_{n\geq0,m\geq0}^{+\infty}\gamma_{(n,m)}t^{(n+1)\nu+m-k},0\leqslant k-1<\nu\leqslant k$$

has a radius of convergence $R>0$. *For* $0<t<R$, *define the function*

$$y(t)=\sum_{n\geq0,m\geq0}^{+\infty}\gamma_{(n,m)}t^{n\nu+m-(k-\nu)},$$

Then the following properties hold:

$$\begin{aligned}{}^{RL}D_{0+}^{\nu}y(t)=&\sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)}t^{-k+m}\\&+\sum_{n\geq1,m\geq0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)}t^{n\nu+m-k}\end{aligned} \tag{3.4}$$

$$\begin{aligned}{}^{RL}D_{0+}^{\nu}\left[{}^{RL}D_{0+}^{\nu}y(t)\right]=&\sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-\nu-k+1)}t^{-k+m-\nu}\\&+\sum_{m\geq k}^{+\infty}\gamma_{(1,m)}\frac{\Gamma(2\nu+m-k+1)}{\Gamma(m-k+1)}t^{m-k}\\&+\sum_{n\geq2,m\geq0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma((n-1)\nu+m-k+1)}t^{(n-1)\nu+m-k}\end{aligned} \tag{3.5}$$

Proof:

$$\begin{aligned}
{}^{RL}D^{\nu}_{0+}y(t) &= \frac{1}{\Gamma(k-\nu)}\frac{d^k}{dt^k}\int_0^t \frac{y(\xi)}{(t-\xi)^{\nu-k+1}}d\xi \\
&= \frac{1}{\Gamma(k-\nu)}\frac{d^k}{dt^k}\int_0^t (t-\xi)^{k-\nu-1}\left(\sum_{m\geq 0}^{+\infty}\gamma_{(n,m)}t^{\nu+m-k} + \sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}t^{(n+1)\nu+m-k}\right)d\xi \\
&= \frac{1}{\Gamma(k-\nu)}\sum_{m\geq 0}^{+\infty}\gamma_{(0,m)}\frac{d^k}{dt^k}\int_0^t (t-\xi)^{k-\nu-1}\left(t^{\nu+m-k}\right)d\xi \\
&+ \frac{1}{\Gamma(k-\nu)}\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{d^k}{dt^k}\int_0^t (t-\xi)^{k-\nu-1}\left(t^{(n+1)\nu+m-k}\right)d\xi \\
&= \frac{1}{\Gamma(k-\nu)}\sum_{m\geq 0}^{+\infty}\gamma_{(0,m)}\,{}^{RL}D^{\nu}_{0+}\left(t^{\nu+m-k}\right) \\
&+ \frac{1}{\Gamma(k-\nu)}\sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\,{}^{RL}D^{\nu}_{0+}\left(t^{(n+1)\nu+m-k}\right) \\
&= \sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)}t^{-k+m} \\
&+ \sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)}t^{n\nu+m-k}. \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
,,{}^{RL}D^{\nu}_{0+}\left[{}^{RL}D^{\nu}_{0+}y(t)\right] &= {}^{RL}D^{\nu}_{0+}\Bigg(\sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)}t^{-k+m} \\
&+ \sum_{n\geq 1, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)}t^{n\nu+m-k}\Bigg) \\
&= {}^{RL}D^{\nu}_{0+}\left(\sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)}t^{-k+m}\right) \\
&+ {}^{RL}D^{\nu}_{0+}\Bigg(\sum_{m\geq 0}^{+\infty}\gamma_{(1,m)}\frac{\Gamma(2\nu+m-k+1)}{\Gamma(\nu+m-k+1)}t^{\nu+m-k} \\
&+ \sum_{n\geq 2, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)}t^{n\nu+m-k}\Bigg) \\
&= \sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-\nu-k+1)}t^{-k+m-\nu} \\
&+ \sum_{m\geq k}^{+\infty}\gamma_{(1,m)}\frac{\Gamma(2\nu+m-k+1)}{\Gamma(m-k+1)}t^{m-k} \\
&+ \sum_{n\geq 2, m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma((n-1)\nu+m-k+1)}t^{(n-1)\nu+m-k}. \qquad (3.7)
\end{aligned}$$

□

Additionally, the generalized fractional power series (14) arises naturally as the Cauchy product of two power series after a suitable rearrangement:

$$\sum_{n\geq 0, m\geq 0}^{+\infty}\gamma_{(n,m)}t^{(n+1)\nu+m-k} = t^{\nu-k}\left(\sum_{n=0}^{+\infty}c_n t^{n\nu}\right)\left(\sum_{m=0}^{+\infty}d_m t^m\right). \qquad (3.8)$$

where $\gamma_{(n,m)} = c_n d_m$.

Proposition 3.1 *([2]) A power series of the form $\sum_{k=0}^{+\infty} c_k t^{k\nu}$, convergent at some positive point $t = c$, converges absolutely for all t in the interval $(0, c)$.*

Corollary 3.1 *The convergence of the power series $\sum_{k=0}^{+\infty} d_k t^k$ at a positive point $t = d$ guarantees its absolute convergence for all t in the interval $0 < t < d$.*

Theorem 3.2 *Consider the two power series $C(t) = \sum_{k=0}^{+\infty} c_k t^{k\nu}$ and $D(t) = \sum_{k=0}^{+\infty} d_k t^k$. Suppose $C(t)$ converges absolutely to c for $t = t_c > 0$ and $D(t)$ converges to d for $t = t_d > 0$. Then the Cauchy product (21) converges to $t_m^{n-\nu} cd$ for $t = t_m > 0$, where $t_m = \min\{t_c, t_d\}$.*

Proof: The Cauchy product of the series C and D is given by cd, where $t = t_m > 0$ and $t_m = \min\{t_c, t_d\}$ (see [2]). This implies that the product $t^{n-\nu} CD$ converges to $t_m^{n-\nu} cd$, where the convergence occurs for $t = t_m$. □

To illustrate the application of the fundamental method and its essential properties, we provide several examples demonstrating its effectiveness and versatility in addressing complex mathematical problems:

Example 3.1 Let $k - 1 < \nu \leq k$,$k \in \mathbb{N}^*$ and $\rho \neq 0$. Consider the initial value problem defined by the following differential equation

$$D_{0+}^{\nu} y(t) + \rho y(t) = 0, \tag{3.9}$$

$$D_{0+}^{\nu-j} y(0) = b_j, \quad b_j \in \mathbb{R} \qquad j = 1, 2, \ldots, k. \tag{3.10}$$

has its solution given by

$$y(t) = \sum_{j=1}^{k} b_j t^{\nu-j} E_{\nu,\nu-j+1}\left(-\rho t^{\nu}\right). \tag{3.11}$$

Proof: We consider a modified generalized fractional power series, adapted for the Riemann-Liouville derivative, of the form:

$$y(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} t^{n\nu+m-(k-\nu)} \qquad , 0 \leq k - \nu < 1$$

Applying the Riemann-Liouville fractional derivative term-by-term yields:

$$D_{0+}^{\nu} y(t) = \sum_{m\geq k}^{+\infty} \gamma_{(0,m)} \frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)} t^{-k+m} + \sum_{n\geq 1, m\geq 0}^{+\infty} \gamma_{(n,m)} \frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)} t^{(n-1)\nu+m+\nu-k}. \tag{3.12}$$

We have also

$$\rho y(t) = \rho \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{n,m} t^{n\nu+m+\nu-k} = \rho \sum_{n\geq 1, m\geq 0}^{+\infty} \gamma_{n-1,m} t^{(n-1)\nu+m+\nu-k}. \tag{3.13}$$

We then substitute (3.12) and (3.13) into (3.9) to arrive at the balance equation:

$$\sum_{m\geq k}^{+\infty}\gamma_{(0,m)}\frac{\Gamma(m+\nu-k+1)}{\Gamma(m-k+1)}t^{-k+m}+\sum_{n\geq 1,m\geq 0}^{+\infty}\gamma_{(n,m)}\frac{\Gamma((n+1)\nu+m-k+1)}{\Gamma(n\nu+m-k+1)}t^{(n-1)\nu+m+\nu-k}$$
$$+\rho\sum_{n\geq 1,m\geq 0}^{+\infty}\gamma_{(n-1,m)}t^{(n-1)\nu+m+\nu-k}=0. \tag{3.14}$$

It follows directly that

$$\gamma_{(0,m)}=0 \quad m\geq k, \tag{3.15}$$
$$\gamma_{(n,m)}=-\rho\frac{\Gamma(n\nu+m-k+1)}{\Gamma((n+1)\nu+m-k+1)}\gamma_{(n-1,m)} \quad n\geq 1, m\geq 0. \tag{3.16}$$

From (3.15) and the the recurrence relation (3.16), it follows that $\gamma_{(n,m)}=0$ for all $n\geq 0$ and $m\geq k$. Consequently, the only non-zero coefficients are those with $m<k$, resulting in a solution of the form:

$$y(t)=\sum_{m=0}^{k-1}\sum_{n=0}^{+\infty}\gamma_{(n,m)}t^{n\nu+m+\nu-k}=\sum_{m=0}^{k-1}t^{m+\nu-k}\sum_{n=0}^{+\infty}\gamma_{(n,m)}t^{n\nu}$$

From (3.16) we have

$$\gamma_{(n,m)}=-\rho\frac{\Gamma(n\nu+m-k+1)}{\Gamma((n+1)\nu+m-k+1)}\gamma_{(n-1,m)} \quad n\geq 1, m\leq k-1.$$

This is equivalent to the expression:

$$\gamma_{(n,m)}=(-1)^n\frac{\rho^n\Gamma(\nu+m-k+1)}{\Gamma((n+1)\nu+m-k+1)}\gamma_{(0,m)} \quad n\geq 1, \quad m\leq k-1.$$

where $\gamma_{(0,m)}=\frac{{}^{RL}D_{0+}^{(m+\nu-k)}y(0)}{\Gamma(\nu+m-k+1)}$ and we deduce that :

$$\gamma_{(n,m)}=(-1)^n\frac{\rho^n}{\Gamma((n+1)\nu+m+1)}{}^{RL}D_{0+}^{(m+\nu-k)}y(0) \quad n\geq 1, \quad m\leq k-1.$$

Therefore, the series solution takes the form:

$$y(t)=\sum_{m=1}^{k}b_m t^{\nu-m}E_{\nu,\nu-m+1}\left(-\rho t^{\nu}\right). \tag{3.17}$$

□

The plot shows the behavior of the function $y(t)$ for different values of ν. The parameter $\rho=1$ and $k=2$, and the coefficients b_1 and b_2 are chosen as 0.5 and 0.02, respectively. The curves are plotted for $\nu\in\{1.6,1.7,1.8,2.0\}$presenting the oscillation equation within the Riemann-Liouville derivative.

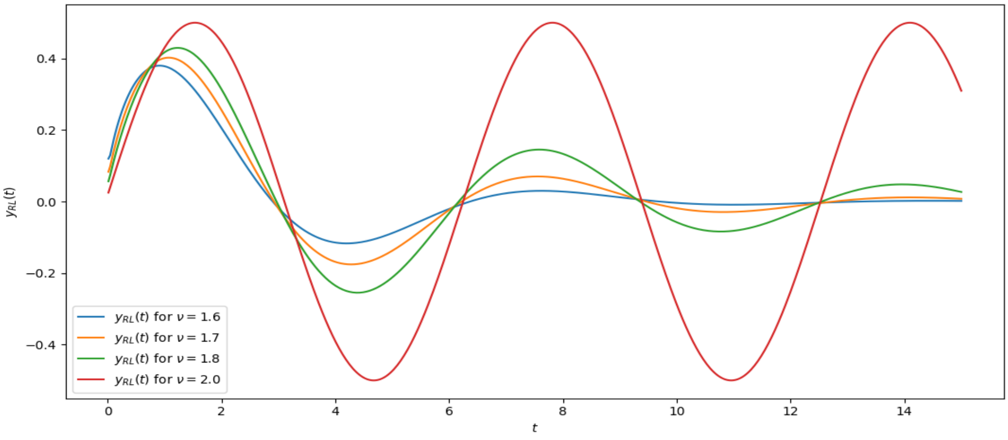

Figure 1: Plot of $y(t)$ for different values of $1 < \nu \leq 2$ in the time range $0 \leq t \leq 15$.

Let's now move on to another example, which focuses on inhomogeneous fractional differential equations to demonstrate the applicability of our approach.

Example 3.2 Consider the following fractional initial value problem:

$$D_{0+}^{\nu}\left[y(t)\right] - \rho y(t) = \frac{t}{2}, \qquad D_{0+}^{\nu-1} y(0) = y_0. \tag{3.18}$$

Given $0 < \nu \leq 1$ and $t > 0$, the preceding discussion and the initial condition lead to the following form for the generalized fractional power series solution of (3.18):

$$y(t) = \sum_{n\geq 0, m\geq 0}^{+\infty} \gamma_{(n,m)} t^{(n+1)\nu+m-1}. \tag{3.19}$$

Applying the Riemann-Liouville fractional derivative term-by-term yields:

$$D_{0+}^{\nu} y(t) = \sum_{m\geq 1}^{+\infty} \gamma_{(0,m)} \frac{\Gamma(m+\nu)}{\Gamma(m)} t^{-1+m} + \sum_{n\geq 1, m\geq 0}^{+\infty} \gamma_{(n,m)} \frac{\Gamma((n+1)\nu+m)}{\Gamma(n\nu+m)} t^{(n-1)\nu+m+\nu-1}. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18) and equating coefficients of like powers of t yields:

$$\gamma_{(0,2)} = \frac{1}{\Gamma(\nu+2)}, \tag{3.21}$$

$$\gamma_{(0,m)} = 0 \quad m \geq 1, \tag{3.22}$$

$$\gamma_{(n,m)} = -\rho \frac{\Gamma(n\nu+m)}{\Gamma((n+1)\nu+m)} \gamma_{(n-1,m)} \quad n \geq 1, m \geq 0. \tag{3.23}$$

This consequently implies that for $n \geq 1$:

$$\gamma_{(n,0)} = \frac{\rho^n}{\Gamma(n\nu+\nu)} y_0, \quad \gamma_{(n,2)} = \frac{\rho^n}{\Gamma(n\nu+\nu+2)}, \quad \text{and} \quad \gamma_{(n,m)} = 0 \quad \text{, otherwise.}$$

Consequently, the exact solution to (3.18)is

$$y(t) = y_0 \sum_{n=0}^{+\infty} \frac{\rho^n}{\Gamma(n\nu+\nu)} t^{(n+1)\nu-1} + \sum_{n=0}^{+\infty} \frac{\rho^n}{\Gamma(n\nu+2+\nu)} t^{(n+1)\nu+1} \tag{3.24}$$

$$= y_0 t^{\nu-1} E_{\nu,\nu}\left(\rho t^{\nu}\right) + t^{\nu+1} E_{\nu,\nu+2}\left(\rho t^{\nu}\right) \tag{3.25}$$

In the specific case $\nu = 1$, the solution reduces to the classical exact solution of (3.18):

$$y(t) = \left(y_0 + \frac{1}{\rho^2}\right) e^{\rho t} - \frac{1}{\rho^2}(\rho t + 1)$$

It is important to remark that existing FPS expansions [2,3,4,10] are incapable of producing the solution given in (3.18) .

Example 3.3 Consider the following fractional differential equation:

$$ {}^{RL}D^{\nu}_{0+}\left[{}^{RL}D^{\nu}_{0+} y(t)\right] - 2\,{}^{RL}\mathcal{D}^{\nu}_{t}[y(t)] + y(t) = t^{1-\nu} \frac{\Gamma(2+\nu)}{2\Gamma(2-\nu)}, \quad 0 < \nu \leq 1, t > 0 \tag{3.26}$$

Substituting Eqs. (3.1), (3.4), and (3.5) into (3.26) and equating the coefficients of like powers of t, we get $\gamma_{(0,2)} = \frac{1}{2}, \gamma_{(0,m)} = 0$ for $m \notin \{0, 2\}, \gamma_{(1,2)} = \frac{\Gamma(2+\nu)}{\Gamma(2\nu+2)}, \gamma_{(1,m)} = 0$ for $m \notin \{0, 2\}$, and the following recursive relation

$$\gamma_{(n+2,m)} = \frac{2\Gamma((n+2)\nu+m)\gamma_{(n+1,m)} - \Gamma((n+1)\nu+m)\gamma_{(n,m)}}{\Gamma((n+3)\nu+m)}.$$

This yields that

$$\begin{aligned}
\gamma_{(n,0)} &= \frac{n\Gamma(2\nu)\gamma_{(1,0)} - (n-1)\Gamma(\nu)\gamma_{(0,0)}}{\Gamma((n+1)\nu)}, && n \geq 2 \\
\gamma_{(n,2)} &= \frac{\frac{1}{2}\Gamma(\nu+2)(n+1)}{\Gamma((n+1)\nu+2)}, && n \geq 0 \\
\gamma_{(n,m)} &= 0, && \text{otherwise.}
\end{aligned}$$

Then, the exact solution is :

$$\begin{aligned}
y(t) =& \gamma_{(1,0)}\Gamma(2\nu) \sum_{n=0}^{+\infty} \frac{n}{\Gamma((n+1)\nu)} t^{(n+1)\nu-1} - \gamma_{(0,0)}\Gamma(\nu) \sum_{n=0}^{+\infty} \frac{(n-1)}{\Gamma((n+1)\nu)} t^{(n+1)\nu-1} \\
&+ \frac{1}{2}\Gamma(\nu+2) \sum_{n=0}^{+\infty} \frac{(n+1)}{\Gamma((n+1)\nu+2)} t^{(n+1)\nu+1}
\end{aligned} \tag{3.27}$$

We note that the solution in (3.27) is not attainable through other established FPS methods [2,3,4,10]. Furthermore, in the classical limit $\nu = 1$, the solution reduces to the known exact solution for (3.26):

$$y(t) = \underbrace{\gamma_{(0,0)} e^t + \left(\gamma_{(1,0)} - \gamma_{(0,0)}\right) t e^t}_{\text{homogenous solution}} + \underbrace{t e^t - e^t + 1}_{\text{particular solution}} \quad .$$

The provided figure depicts the 5-term approximate "memory" solutions, denoted as $y_5(t, \nu)$, for equation (3.27) under different values of $0 < \nu \leq 1$ within the interval $I = (0, \frac{3}{2})$, given that $\gamma_{(0,0)} = 2$ and $\gamma_{(1,0)} = 3$. Clearly, the approximate solution $y_5(t, 1)$ aligns well with the exact solution in I. Furthermore, the solution exhibits certain memory and heredity characteristics, suggesting that an appropriate solution can be obtained for different values of ν.

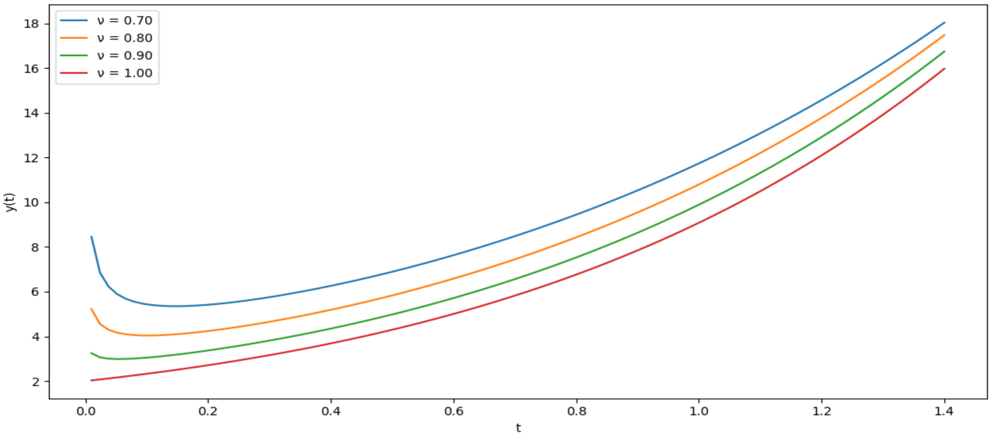

Figure 2: Plot of $y(t)$ for different values of $0 < \nu \leq 1$ in the time range $0 \leq t \leq 1,4$.

4. Conclusion and Comments

In conclusion, the proposed methodology provides a robust and adaptable framework for solving fractional differential equations within the Riemann-Liouville framework. By introducing the modified generalized fractional power series, this approach refines traditional fractional power series methods, addressing their limitations. The framework's adaptive nature enhances solution accuracy and flexibility, making it particularly effective for fractional differential equations that cannot be solved by other power series techniques. This provides a precise and efficient tool for tackling a wide range of problems in fractional calculus.

Looking ahead, this work paves the way for several promising research directions. A key future objective is the development of analogous power series methods tailored for other prominent fractional operators, such as the Atangana-Baleanu and Caputo-Fabrizio derivatives. These operators, defined with non-singular kernels, have demonstrated significant potential in modeling complex phenomena across various scientific and engineering fields [16,17,18,19], particularly in capturing material heterogeneities and processes with memory effects. Establishing corresponding power series solutions would unlock new analytical and numerical opportunities, further expanding the applicability of fractional calculus and providing powerful new tools for researchers.

Data Availability

No data were used to support this study.

Declaration of competing interest

The authors state that there are no financial interests or personal relationships that could have influenced the work presented in this paper.

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References

1. Angstmann, Christopher N., and Bruce Ian Henry. "Generalized fractional power series solutions for fractional differential equations." Applied Mathematics Letters 102 (2020): 106107.

2. Jaradat, I., et al. "Theory and applications of a more general form for fractional power series expansion." Chaos, Solitons & Fractals 108 (2018): 107-110.

3. Assebbane, Y., Echchehira, M., Atraoui, M., Hannabou, M., & Bouaouid, M. "Fractional power series methods for solving fractional differential equations." Kragujevac Journal of Mathematics 51, no. 1 (2027): 95-111.

4. Trujillo, J. J., M. Rivero, and B. Bonilla. "On a Riemann–Liouville generalized Taylor's formula." Journal of Mathematical Analysis and Applications 231.1 (1999): 255-265.

5. Riemann, Bernhard. "Versuch einer allgemeinen Auffassung der Integration und Differentiation." Gesammelte Werke 62.1876 (1876).

6. Kilbas, Anatoliĭ Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. Theory and applications of fractional differential equations. Vol. 204. elsevier, 2006.

7. Hardy, G. H. "Riemann's form of Taylor's series." Journal of the London Mathematical Society 1.1 (1945): 48-57.

8. Watanabe, Y. "Notes on the generalized derivative of Riemann–Liouville and its application to Leibniz's formula. I and II." Tohoku Math. J 34 (1931): 8-14.

9. Odibat, Zaid M., and Nabil T. Shawagfeh. "Generalized Taylor's formula." Applied Mathematics and computation 186.1 (2007): 286-293.

10. El-Ajou, Ahmad, et al. "New results on fractional power series: theories and applications." Entropy 15.12 (2013): 5305-5323.

11. Saigo, Megumi, and Anatolii Aleksandrovich Kilbas. "The solution of a class of linear differential equations via functions of the Mittag-Leffler type." Differential Equations 36 (2000): 193-202.

12. de Oliveira, Edmundo Capelas, Francesco Mainardi, and Jayme Vaz Jr. "Fractional models of anomalous relaxation based on the Kilbas and Saigo function." Meccanica 49.9 (2014): 2049-2060.

13. Arikoglu, Aytac, and Ibrahim Ozkol. "Solution of difference equations by using differential transform method." Applied mathematics and computation 174.2 (2006): 1216-1228.

14. Lin, Shy-Der, and Chia-Hung Lu. "Laplace transform for solving some families of fractional differential equations and its applications." Advances in Difference Equations

15. Knopfmacher, Arnold, Toufik Mansour, and Augustine Munagi. "Recurrence relation with two indices and plane compositions." Journal of Difference Equations and Applications 17.1 (2011): 115-127.

16. Assebbane, Y., Hannabou, M., Atraoui, M., & Bouaouid, M. (2025). Existence and uniqueness of boundary value problems for nonlinear hybrid differential equations with ABC-fractional derivative. Filomat, 39(11), 3759-3768.

17. Echchehira, M., Hannabou, M., Atraoui, M., & Bouaouid, M. (2025). On an image denoising model based on Atangana–Baleanu fractional derivative. Physica A: Statistical Mechanics and its Applications, 130633.

18. Hannabou, M., Echchehira, M., Atraoui, M., & Bouaouid, M. (2025). Existence and Uniqueness of Boundary Value Problems involving Nonlinear Hybrid Differential Equations with Caputo–Fabrizio Fractional Derivative. Proyecciones (Antofagasta), 44(3), 516-527.

19. Echchehira, M., Assebbane, Y., Atraoui, M., & Bouaouid, M. (2024). A Predictor-Corrector Algorithm in the Framework of Conformable Fractional Differential Equations. arXiv preprint arXiv:2406.16216.

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