Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.79837

Global Existence and Exponential Decay for a Coupled Wave System with Delay via Kelvin-Voigt Damping

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ABSTRACT: This paper examines the stability of two wave equations coupled through a Kelvin-Voigt damping mechanism with a time delay in a bounded domain. Using semigroup theory, we establish the global existence of solutions in an appropriate Sobolev space under suitable conditions. By applying the multiplier method and Komornik's integral inequality, we prove an exponential decay rate for the system's energy. Our results extend prior work by addressing the combined effects of delay and Kelvin-Voigt damping, providing new insights into the stabilization of coupled wave systems.

Keywords: Coupled wave equations, delay term, Kelvin-Voigt damping, decay rate, multiplier method.

Contents

1. Introduction

Wave equations lie at the heart of a vast array of physical and engineering phenomena, orchestrating the dance of energy through media as diverse as vibrating strings, electromagnetic fields, and seismic waves [6,11,12,20,21,22]. Their study is not merely an academic pursuit but a vital endeavor for crafting systems that withstand the test of time and perturbation. Yet, when these equations are entwined through coupling or laced with time delays—features emblematic of real-world complexities like structural control or delayed feedback—their dynamics morph into a captivating puzzle, balancing on the precipice of stability and instability. This article ventures into this intricate realm, probing the stability of a system of two wave equations coupled via Kelvin-Voigt damping and encumbered by a time delay, unveiling new vistas into their asymptotic behavior.

Kelvin-Voigt damping, a hallmark of viscoelastic materials, distinguishes itself from the more conventional viscous damping by introducing higher-order spatial derivatives, a trait that both enriches its modeling power and complicates its analysis. Found in applications such as vibration suppression patches [3], this mechanism dissipates energy through the gradient of velocity, posing unique challenges in multidimensional domains. The introduction of a time delay, an inevitable companion in control systems or material responses, heightens the stakes, as it can transform benign oscillations into destabilizing forces. We tackle this dual complexity within a bounded domain $\Phi \subset \mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\Gamma = \partial\Phi$, governed by the coupled system:

$$\begin{cases} u_{\mathsf{tt}}(\hbar,\mathsf{t}) - a\Delta u(\hbar,\mathsf{t}) + p(\hbar)\, u_{\mathsf{t}}(\hbar,\mathsf{t}) - div\,(\alpha(\hbar)\nabla v_{\mathsf{t}}(\hbar,\mathsf{t}-\tau)) = 0, & \text{in } \Phi\times]0,+\infty[,\\ v_{\mathsf{tt}}(\hbar,\mathsf{t}) - b\Delta v(\hbar,\mathsf{t}) + q(\hbar)\, v_{\mathsf{t}}(\hbar,\mathsf{t}) - div\,(\beta(\hbar)\nabla u_{\mathsf{t}}(\hbar,\mathsf{t}-\tau)) = 0, & \text{in } \Phi\times]0,+\infty[,\\ u(\hbar,\mathsf{t}) = 0,\ v(\hbar,\mathsf{t}) = 0, & \text{on } \Gamma\times]0,+\infty[,\\ u(\hbar,0) = u_0(\hbar),\ u_{\mathsf{t}}(\hbar,0) = u_1(\hbar),\ v(\hbar,0) = v_0(\hbar),\ v_{\mathsf{t}}(\hbar,0) = v_1(\hbar), & \text{in } \Phi,\\ u_{\mathsf{t}}(\hbar,\mathsf{t}-\tau) = f_0(\hbar,\mathsf{t}-\tau),\ v_{\mathsf{t}}(\hbar,\mathsf{t}-\tau) = g_0(\hbar,\mathsf{t}-\tau), & \text{in } \Phi\times]0,\tau[, \end{cases} \tag{1.1}$$

2020 *Mathematics Subject Classification:* 35A01, 35A02.
Submitted October 31, 2025. Published June 05, 2026.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
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where $p, q, \alpha, \beta : \Phi \to \mathbb{R}$ are nonnegative, $a, b > 0$ are distinct constants, $\tau > 0$ is the delay, and the initial data $(u_0, v_0, u_1, v_1, f_0, g_0)$ belong to a suitable function space. This system weaves together local viscous damping and nonlocal, delay-driven Kelvin-Voigt coupling, presenting a rich tapestry for theoretical and applied exploration.

The quest to stabilize wave equations has flourished since the foundational works of Chen [9,10] and Zuazua [23], who showcased the efficacy of localized damping in curbing oscillations. While viscous damping has been a well-trodden path, Kelvin-Voigt damping, as elucidated by Liu and Rao [15], injects a layer of sophistication, achieving exponential stability for a single wave equation near the boundary. The leap to coupled systems with delays, however, remains a frontier ripe for discovery. Nicaise and Pignotti [18] navigated delays in viscous damping, but the Kelvin-Voigt paradigm, with its higher-order terms, calls for a bespoke approach.

Our work draws inspiration from luminaries in the field. Kaïs Ammari has profoundly shaped the discourse on wave equation stabilization, notably through his study of singular Kelvin-Voigt damping in [2], where he demonstrated logarithmic energy decay when damping is localized away from the boundary, contrasting with exponential decay near it. His earlier work on local feedback stabilization of damped nonlinear wave equations [1] further informs our approach to handling nonlinearities and delays. Salim A. Messaoudi has enriched this landscape with his extensive analyses of energy decay, such as in [17], where he derived general decay rates for weakly damped wave equations, and in [13], exploring blow-up phenomena in wave equations with delays. His collaborations with Benaissa and Benguessoum in [4] and [5] further illuminate energy decay in systems with constant weak delays and time-varying delay terms, providing critical insights that guide our stability estimates for delayed systems. Marcelo Cavalcanti's contributions, including his investigation of localized frictional and Kelvin-Voigt damping in [8], offer a blueprint for managing coupled dissipative mechanisms, while his work on damped wave equations with Cauchy–Ventcel conditions [7] enhances our understanding of boundary effects. These studies collectively anchor our exploration, which extends their reach into the uncharted territory of coupled wave systems with delay and Kelvin-Voigt damping.

Our twin objectives are both rigorous and revelatory: to prove the global existence and uniqueness of solutions to (1.1) using semigroup theory [19], ensuring the system's well-posedness in a Hilbert space $\mathcal{H}$, and to establish an exponential decay of the energy functional $\aleph(\mathsf{t})$, affirming long-term stability. Employing energy methods, a key lemma from Martinez [16], and multiplier techniques [14], we show that $\aleph(\mathsf{t}) \le c\aleph(0)e^{-\gamma \mathsf{t}}$ under hypotheses (H1–H3). This endeavor not only advances theoretical frontiers but also lays a foundation for practical applications, from vibration control to delayed system design, echoing the legacies of Ammari, Messaoudi, Benaissa, Benguessoum, and Cavalcanti while forging a new path forward.

2. Initial Concepts and Key Findings

We begin by establishing the following assumptions:

(H1) Let p and q be positive functions belonging to $L^\infty(\Phi)$ that satisfy

$$\begin{cases} |\nabla u_{\mathsf{t}}(\hbar, \mathsf{t})|^2 \le q(\hbar)\, |v_{\mathsf{t}}(\hbar, \mathsf{t})|^2, \\ |\nabla v_{\mathsf{t}}(\hbar, \mathsf{t})|^2 \le p(\hbar)\, |u_{\mathsf{t}}(\hbar, \mathsf{t})|^2, \end{cases} \quad \text{for all } (\hbar, \mathsf{t}) \in \Phi\times]0, +\infty[. \tag{2.1}$$

(H2) α and β are two positive functions from $W^{2,\infty}(\Phi)$ satisfying

$$\begin{cases} \alpha(\hbar)\, |\nabla v_{\mathsf{t}}(\hbar, \mathsf{t}-\tau)|^2 \le |u_{\mathsf{t}}(\hbar, \mathsf{t}-\tau)|^2, \\ \beta(\hbar)\, |\nabla u_{\mathsf{t}}(\hbar, \mathsf{t}-\tau)|^2 \le |v_{\mathsf{t}}(\hbar, \mathsf{t}-\tau)|^2, \end{cases} \quad \text{for all } (\hbar, \mathsf{t}) \in \Phi\times]0, +\infty[. \tag{2.2}$$

(H3) Assume that there exist $p_0, q_0 > \frac{1}{2}$ such that

$$\begin{cases} p(\hbar) > p_0, & q(\hbar) > q_0, \\ 0 < \alpha(\hbar) < 2 - \frac{1}{q_0}, & 0 < \beta(\hbar) < 2 - \frac{1}{p_0}, \end{cases} \quad \text{for all } \hbar \in \Phi. \tag{2.3}$$

We now state a lemma needed later.

Lemma 2.1 [16] *Let $\aleph : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there exist $\mu \geq 0$ and $\varsigma > 0$ such that*

$$\int_S^{+\infty} \aleph^{\mu+1}(\mathsf{t})\, d\mathsf{t} \leq \frac{1}{\varsigma} \aleph^{\mu}(0)\aleph(S), \quad 0 \leq S < +\infty. \tag{2.4}$$

Then

$$\aleph(\mathsf{t}) \leq \frac{c\,\aleph(0)}{(1+\mathsf{t})^{\frac{1}{\mu}}}, \ \forall \mathsf{t} \geq 0, \quad \text{if} \quad \mu > 0, \tag{2.5}$$

$$\aleph(\mathsf{t}) \leq c\,\aleph(0) e^{-\varsigma \mathsf{t}}, \quad \forall \mathsf{t} \geq 0, \quad \text{if} \quad \mu = 0. \tag{2.6}$$

We introduce the new variables y and z, as in [18],

$$\begin{cases} y(\hbar, \kappa, \mathsf{t}) = u_{\mathsf{t}}(\hbar, \mathsf{t} - \kappa\,\tau), \\ z(\hbar, \kappa, \mathsf{t}) = v_{\mathsf{t}}(\hbar, \mathsf{t} - \kappa\,\tau), \end{cases} \quad \hbar \in \Phi, \ \kappa \in]0,1[, \ \mathsf{t} > 0. \tag{2.7}$$

Then, we have

$$\begin{cases} \tau y_{\mathsf{t}}(\hbar, \kappa, \mathsf{t}) + y_{\kappa}(\hbar, \kappa, \mathsf{t}) = 0, \\ \tau z_{\mathsf{t}}(\hbar, \kappa, \mathsf{t}) + z_{\kappa}(\hbar, \kappa, \mathsf{t}) = 0, \end{cases} \quad \text{in } \Phi \times]0,1[\times]0,+\infty[. \tag{2.8}$$

Therefore, system (2.1) takes the system equivalent following:

$$\begin{cases} u_{\mathsf{tt}}(\hbar, \mathsf{t}) - a\Delta u(\hbar, \mathsf{t}) + p(\hbar)\, u_{\mathsf{t}}(\hbar, \mathsf{t}) - div\,(\alpha(\hbar)\nabla z(\hbar, 1, \mathsf{t})) = 0, & \text{in } \Phi \times]0,+\infty[, \\ v_{\mathsf{tt}}(\hbar, \mathsf{t}) - b\Delta v(\hbar, \mathsf{t}) + q(\hbar)\, v_{\mathsf{t}}(\hbar, \mathsf{t}) - div\,(\beta(\hbar)\nabla y(\hbar, 1, \mathsf{t})) = 0, & \text{in } \Phi \times]0,+\infty[, \\ \tau y_{\mathsf{t}}(\hbar, \kappa, \mathsf{t}) + y_{\kappa}(\hbar, \kappa, \mathsf{t}) = 0, & \text{in } \Phi \times]0,1[\times]0,+\infty[, \\ \tau z_{\mathsf{t}}(\hbar, \kappa, \mathsf{t}) + z_{\kappa}(\hbar, \kappa, \mathsf{t}) = 0, & \text{in } \Phi \times]0,1[\times]0,+\infty[, \\ u(\hbar, \mathsf{t}) = 0, \ v(\hbar, \mathsf{t}) = 0, & \text{on } \Gamma \times]0,+\infty[, \\ y(\hbar, 0, \mathsf{t}) = u_{\mathsf{t}}(\hbar, \mathsf{t}), \ z(\hbar, 0, \mathsf{t}) = v_{\mathsf{t}}(\hbar, \mathsf{t}), & \text{in } \Phi \times]0,+\infty[, \\ u(\hbar, 0) = u_0(\hbar), \ v(\hbar, 0) = v_0(\hbar), \ u_{\mathsf{t}}(\hbar, 0) = u_1(\hbar), \ v_{\mathsf{t}}(\hbar, 0) = v_1(\hbar), & \text{in } \Phi, \\ y(\hbar, \kappa, 0) = f_0(\hbar, -\kappa\,\tau), \ z(\hbar, \kappa, 0) = g_0(\hbar, -\kappa\,\tau), & \text{in } \Phi \times]0,1[. \end{cases} \tag{2.9}$$

We define the natural energy $\aleph(\mathsf{t})$ of the solution of (2.1) depend t by

$$\begin{aligned} \aleph(\mathsf{t}) \quad = \quad & \frac{1}{2} \int_{\Phi} \left(|u_{\mathsf{t}}(\hbar, \mathsf{t})|^2 + |v_{\mathsf{t}}(\hbar, \mathsf{t})|^2 + a|\nabla u(\hbar, \mathsf{t})|^2 + b|\nabla v(\hbar, \mathsf{t})|^2 \right) d\hbar \\ & + \frac{\tau}{2} \int_{\Phi} \int_0^1 \left(|y(\hbar, \kappa, \mathsf{t})|^2 + |z(\hbar, \kappa, \mathsf{t})|^2 \right) d\kappa\, d\hbar, \end{aligned} \quad \forall \mathsf{t} \geq 0. \tag{2.10}$$

We have the following theorem.

Theorem 2.1 *Let $(u_0, v_0, u_1, v_1, f_0, g_0) \in \left(H^2(\Phi) \cap H_0^1(\Phi)\right)^2 \times \left(H_0^1(\Phi)\right)^2 \times \left(L^2\left(\Phi \times]0,1[\right)\right)^2$. Assume that the hypotheses (2.2) – (2.3) hold. Then problem (2.9) admits a unique solution (u, v, y, z), such that*

$$\begin{aligned} & u, v \in C\big([0,+\infty[; H_0^1(\Phi[) \cap C^1\big([0,+\infty[; L^2(\Phi[), \\ & y, z \in C\big([0,+\infty[; L^2(\Phi \times]0,1[)\big). \end{aligned}$$

Furthermore, with certain positive constants c and γ, we achieve the subsequent decay property

$$\aleph(\mathsf{t}) \leq c\,\aleph(0)\, e^{-\gamma \mathsf{t}}, \quad \forall \mathsf{t} \geq 0. \tag{2.11}$$

The energy $\aleph$ is a nonincreasing function of the time variable t and its derivative satisfies the following lemma

Lemma 2.2 *Let (u, v, y, z) represent a solution to the system given by (2.9). Then, The corresponding energy functional, as defined in (2.10), fulfills*

$$\begin{aligned} \aleph'(\mathsf{t}) \quad \leq \quad & -\int_{\Phi} p(\hbar) \left(1 - \frac{\beta(\hbar)}{2} - \frac{1}{2p_0} \right) |u_{\mathsf{t}}(\hbar, \mathsf{t})|^2\, d\hbar \\ & -\int_{\Phi} q(\hbar) \left(1 - \frac{\alpha(\hbar)}{2} - \frac{1}{2q_0} \right) |v_{\mathsf{t}}(\hbar, \mathsf{t})|^2\, d\hbar \\ \leq \quad & 0. \end{aligned} \tag{2.12}$$

Proof. In (2.9), we multiply the first equation by $u_{\mathsf{t}}(\hbar,\mathsf{t})$ and the second equation by $v_{\mathsf{t}}(\hbar,\mathsf{t})$, and integrating the result over Φ, we obtain

$$\begin{cases} \int_\Phi \frac{1}{2}\frac{d}{dt}\left(|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + a|\nabla u(\hbar,\mathsf{t})|^2\right) d\hbar + \int_\Phi \left(p(\hbar)\,|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \alpha(\hbar)\nabla z(\hbar,1,\mathsf{t})\nabla u_{\mathsf{t}}(\hbar,\mathsf{t})\right) d\hbar = 0, \\ \int_\Phi \frac{1}{2}\frac{d}{dt}\left(|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + b|\nabla v(\hbar,\mathsf{t})|^2\right) d\hbar + \int_\Phi \left(q(\hbar)\,|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \beta(\hbar)\nabla y(\hbar,1,\mathsf{t})\nabla v_{\mathsf{t}}(\hbar,\mathsf{t})\right) d\hbar = 0, \end{cases} \tag{2.13}$$

and multiplying the third equation by y and fourth equation by z, and integrate over $\Phi\times]0,1[$, we get

$$\begin{cases} \frac{\tau}{2}\frac{d}{dt}\int_\Phi\int_0^1 |y(\hbar,\kappa,\mathsf{t})|^2 d\kappa\, d\hbar + \frac{1}{2}\int_\Phi \left(|y(\hbar,1,\mathsf{t})|^2 - |y(\hbar,0,\mathsf{t})|^2\right) d\hbar = 0, \\ \frac{\tau}{2}\frac{d}{dt}\int_\Phi\int_0^1 |z(\hbar,\kappa,\mathsf{t})|^2 d\kappa\, d\hbar + \frac{1}{2}\int_\Phi \left(|z(\hbar,1,\mathsf{t})|^2 - |z(\hbar,0,\mathsf{t})|^2\right) d\hbar = 0. \end{cases} \tag{2.14}$$

From (2.13) and (2.14) and (2.10), we have

$$\begin{aligned} \aleph'(\mathsf{t}) &= -\int_\Phi \left(p(\hbar)\,|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + q(\hbar)\,|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\right) d\hbar \\ &\quad -\int_\Phi \left(\alpha(\hbar)\nabla z(\hbar,1,\mathsf{t})\nabla u_{\mathsf{t}}(\hbar,\mathsf{t}) + \beta(\hbar)\nabla y(\hbar,1,\mathsf{t})\nabla v_{\mathsf{t}}(\hbar,\mathsf{t})\right) d\hbar \\ &\quad +\frac{1}{2}\int_\Phi \left(|y(\hbar,0,\mathsf{t})|^2 + |z(\hbar,0,\mathsf{t})|^2 - |y(\hbar,1,\mathsf{t})|^2 - |z(\hbar,1,\mathsf{t})|^2\right) d\hbar. \end{aligned} \tag{2.15}$$

Due to Young's inequality, we obtain

$$\begin{cases} \int_\Phi \alpha(\hbar)\nabla z(\hbar,1,\mathsf{t})\nabla u_{\mathsf{t}}(\hbar,\mathsf{t})\, d\hbar \le \int_\Phi \left(\frac{\alpha(\hbar)}{2}|\nabla u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \frac{\alpha(\hbar)}{2}|\nabla z(\hbar,1,\mathsf{t})|^2\right) d\hbar, \\ \int_\Phi \beta(\hbar)\nabla y(\hbar,1,\mathsf{t})\nabla v_{\mathsf{t}}(\hbar,\mathsf{t})\, d\hbar \le \int_\Phi \left(\frac{\beta(\hbar)}{2}|\nabla v_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \frac{\beta(\hbar)}{2}|\nabla y(\hbar,1,\mathsf{t})|^2\right) d\hbar. \end{cases} \tag{2.16}$$

Inserting (2.15) into (2.16), we have

$$\begin{aligned} \aleph'(\mathsf{t}) &= -\int_\Phi \left(p(\hbar)\,|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + q(\hbar)\,|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\right) d\hbar \\ &\quad +\int_\Phi \left(\frac{\alpha(\hbar)}{2}|\nabla u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \frac{\alpha(\hbar)}{2}|\nabla z(\hbar,1,\mathsf{t})|^2\right) d\hbar \\ &\quad +\int_\Phi \left(\frac{\beta(\hbar)}{2}|\nabla v_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \frac{\beta(\hbar)}{2}|\nabla y(\hbar,1,\mathsf{t})|^2\right) d\hbar \\ &\quad +\frac{1}{2}\int_\Phi \left(|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + |v_{\mathsf{t}}(\hbar,\mathsf{t})|^2 - |y(\hbar,1,\mathsf{t})|^2 - |z(\hbar,1,\mathsf{t})|^2\right) d\hbar. \end{aligned} \tag{2.17}$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned} \aleph'(\mathsf{t}) &\le -\int_\Phi \left(p(\hbar)\,|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + q(\hbar)\,|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\right) d\hbar \\ &\quad +\int_\Phi \left(\frac{\beta(\hbar)p(\hbar)}{2}|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + \frac{\alpha(\hbar)q(\hbar)}{2}|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\right) d\hbar \\ &\quad +\frac{1}{2}\int_\Phi \left(|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2 + |v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\right) d\hbar. \end{aligned} \tag{2.18}$$

From (2.18) and (2.3), we conclude

$$\begin{aligned} \aleph'(\mathsf{t}) &\le -\int_\Phi p(\hbar)\left(1 - \frac{\beta(\hbar)}{2} - \frac{1}{2p_0}\right)|u_{\mathsf{t}}(\hbar,\mathsf{t})|^2\, d\hbar \\ &\quad -\int_\Phi q(\hbar)\left(1 - \frac{\alpha(\hbar)}{2} - \frac{1}{2q_0}\right)|v_{\mathsf{t}}(\hbar,\mathsf{t})|^2\, d\hbar, \\ &\le 0. \end{aligned} \tag{2.19}$$

This completes the proof of the lemma.

3. Global Existence

We now present results on the well-posedness of the system (2.9) through the application of semigroup theory. To this end, we define the semigroup formulation for System (2.9). Define $U = (u, v, \upsilon, \nu, y, z)^T$ and reformulate (2.9) as

$$\begin{cases} \dfrac{d}{dt}U &= \mathcal{A}U, \\ U_0 &= U(0), \end{cases} \tag{3.1}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear unbounded operator given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \upsilon \\ \nu \\ y \\ z \end{pmatrix} = \begin{pmatrix} \upsilon \\ \nu \\ a\Delta u - p\,\upsilon + div\,(\alpha\,\nabla z(.,1)) \\ b\Delta v - q\,\nu + div\,(\beta\,\nabla y(.,1)) \\ -\dfrac{1}{\tau}y_\kappa \\ -\dfrac{1}{\tau}z_\kappa \end{pmatrix},$$

with $U(0) = (u_0, v_0, u_1, v_1, f_0, g_0)^T$. The domain $\mathcal{D}(A)$ of $\mathcal{A}$ is defined by

$$\mathcal{D}(\mathcal{A}) = \left\{ U = (u, v, \upsilon, \nu, y, z)^T \in \mathbb{H} \,/\, \upsilon = y(.,0),\ \nu = z(.,0), \quad \text{in } \Phi \right\},$$

with the domain $\mathbb{H}$ given by,

$$\mathbb{H} = \left(H^2(\Phi) \cap H_0^1(\Phi)\right)^2 \times \left(H_0^1(\Phi)\right)^2 \times \left(L^2(\Phi; H^1(]0,1[))\right)^2.$$

New, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = \left(H_0^1(\Phi)\right)^2 \times \left(L^2(\Phi)\right)^2 \times \left(L^2(\Phi \times]0,1[)\right)^2$$

is a Hilbert space with respect to the inner product

$$\langle U, \overline{U} \rangle_{\mathcal{H}} = \int_\Phi \left[a\nabla u \nabla \overline{u} + b\nabla v \nabla \overline{v} + \upsilon\overline{\upsilon} + \nu\overline{\nu} + \tau \int_0^1 (y\overline{y} + z\overline{z})\, d\kappa \right] d\hbar,$$

for $U = (u, v, \upsilon, \nu, y, z)^T$ and $\overline{U} = (\overline{u}, \overline{v}, \overline{\upsilon}, \overline{\nu}, \overline{y}, \overline{z})^T$.

We are now in a position to state our main result:

Theorem 3.1 *Assume that $U_0 \in \mathbb{H}$. Then, for any initial datum $U_0 \in \mathbb{H}$ there exists a unique solution $U \in C([0, +\infty[, \mathbb{H})$ for problem (3.1).*
Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U \in C([0, +\infty[, \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty[, \mathbb{H})$.

Proof. We show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (u, v, \upsilon, \nu, y, z)^T$. Using (3.1), (2.12) and the fact that

$$\aleph(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2. \tag{3.2}$$

We get

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \int_{\Phi}\Big(a\nabla u\nabla \upsilon + b\nabla v\nabla \nu + (a\Delta u - p\,\upsilon + div(\alpha\,\nabla z(.,1)))\upsilon \\
&\quad + (b\Delta v - q\,\nu + div(\beta\,\nabla y(.,1)))\nu\Big) d\hbar - \int_{\Phi}\int_0^1 (yy_\kappa + zz_\kappa)\, d\kappa\, d\hbar, \\
&= \int_{\Phi}\Big(a\nabla u\nabla \upsilon + b\nabla v\nabla \nu - a\nabla u\nabla \upsilon - p\,\upsilon^2 - \alpha\,\nabla z(.,1)\nabla \upsilon - b\nabla v\nabla \nu \\
&\quad - q\,\nu^2 - \beta\,\nabla y(.,1)\nabla \nu\Big) d\hbar - \frac{1}{2}\int_{\Phi}\int_0^1 \frac{\partial}{\partial \kappa}\left(y^2(\hbar,\kappa,\mathsf{t}) + z^2(\hbar,\kappa,\mathsf{t})\right)\, d\kappa\, d\hbar, \\
&= \int_{\Phi}\Big(-p\,\upsilon^2 - q\,\nu^2 - \alpha\,\nabla z(.,1)\nabla \upsilon - \beta\,\nabla y(.,1)\nabla \nu\Big) d\hbar \\
&\quad + \frac{1}{2}\int_{\Phi}\left(|y(\hbar,0,\mathsf{t})|^2 + |z(\hbar,0,\mathsf{t})|^2 - |y(\hbar,1,\mathsf{t})|^2 - |z(\hbar,1,\mathsf{t})|^2\right)\, d\hbar.
\end{aligned}
$$

From (2.15)-(2.19), we conclude

$$
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \;\leq\; 0, \tag{3.3}
$$

which show that the operator $\mathcal{A}$ is dissipative.
Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, we seek $(u, v, \upsilon, \nu, y, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$
(\lambda I - \mathcal{A})U = F,
$$

that is verifying the following system of equations

$$
\begin{cases}
\lambda u - \upsilon = f_1, \\
\lambda v - \nu = f_2, \\
\lambda \upsilon - a\Delta u + p\,\upsilon - div\,(\alpha\,\nabla z(.,1)) = f_3, \\
\lambda \nu - b\Delta v + q\,\nu - div\,(\beta\,\nabla y(.,1)) = f_4, \\
\lambda y + \dfrac{1}{\tau} y_\kappa = f_5, \\
\lambda z + \dfrac{1}{\tau} z_\kappa = f_6.
\end{cases} \tag{3.4}
$$

By (3.4), we get

$$
\begin{cases}
\upsilon = \lambda u - f_1, \\
\nu = \lambda v - f_2,
\end{cases} \tag{3.5}
$$

and

$$
\begin{cases}
y(\hbar,\kappa) = \upsilon(\hbar)e^{-\lambda\tau\kappa} + \tau e^{-\lambda\tau\kappa}\displaystyle\int_0^\kappa f_5(\hbar,\theta)e^{\lambda\tau\theta}d\theta, \\
z(\hbar,\kappa) = \nu(\hbar)e^{-\lambda\tau\kappa} + \tau e^{-\lambda\tau\kappa}\displaystyle\int_0^\kappa f_6(\hbar,\theta)e^{\lambda\tau\theta}d\theta.
\end{cases} \tag{3.6}
$$

From (3.5) and (3.6), we obtian

$$
\begin{cases}
y(\hbar,\kappa) = \lambda u(\hbar)e^{-\lambda\tau\kappa} - f_1 e^{-\lambda\tau\kappa} + \tau e^{-\lambda\tau\kappa}\displaystyle\int_0^\kappa f_5(\hbar,\theta)e^{\lambda\tau\theta}d\theta, \\
z(\hbar,\kappa) = \lambda v(\hbar)e^{-\lambda\tau\kappa} - f_2 e^{-\lambda\tau\kappa} + \tau e^{-\lambda\tau\kappa}\displaystyle\int_0^\kappa f_6(\hbar,\theta)e^{\lambda\tau\theta}d\theta.
\end{cases} \tag{3.7}
$$

By using (3.4) and (3.5), the functions u and v satisfying the following system

$$
\begin{cases}
\lambda^2 u - a\Delta u + \lambda p u - div\,(\alpha\,\nabla z(.,1)) = f_3 + (p+\lambda)f_1, \\
\lambda^2 v - b\Delta v + \lambda q v - div\,(\beta\,\nabla y(.,1)) = f_4 + (q+\lambda)f_2.
\end{cases} \tag{3.8}
$$

Solving system (3.8) is equivalent to finding $(u,v) \in \left(H^2(\Phi) \cap H_0^1(\Phi)\right)^2$ such that

$$\left\{\begin{array}{l} \displaystyle\int_\Phi \left(\lambda^2 u - a\Delta u + \lambda p u - div\left(\alpha\, \nabla z(.,1)\right)\right)\varphi d\hbar = \int_\Phi \left(f_3 + (p+\lambda) f_1\right)\varphi d\hbar, \\ \displaystyle\int_\Phi \left(\lambda^2 v - b\Delta v + \lambda q v - div\left(\beta\, \nabla y(.,1)\right)\right)\psi d\hbar = \int_\Phi \left(f_4 + (q+\lambda) f_2\right)\psi d\hbar, \end{array}\right. \tag{3.9}$$

for all $(\varphi,\psi) \in \left(H_0^1(\Phi)\right)^2$. From (3.7), we have

$$\left\{\begin{array}{l} \displaystyle y(\hbar,1) = \lambda u(\hbar) e^{-\lambda\tau} - f_1 e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 f_5(\hbar,\theta) e^{\lambda\tau\theta} d\theta, \\ \displaystyle z(\hbar,1) = \lambda v(\hbar) e^{-\lambda\tau} - f_2 e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 f_6(\hbar,\theta) e^{\lambda\tau\theta} d\theta. \end{array}\right. \tag{3.10}$$

From (3.9) and (3.10), we conclude

$$\begin{array}{l} \displaystyle\int_\Phi \lambda^2 u\varphi\, d\hbar + \int_\Phi \lambda p u \varphi\, d\hbar + \int_\Phi a\nabla u \nabla\varphi\, d\hbar + \int_\Phi \lambda\alpha e^{-\lambda\tau}\, \nabla v \nabla \varphi\, d\hbar \\ \displaystyle = \int_\Phi \alpha e^{-\lambda\tau} \nabla f_2 \nabla\varphi\, d\hbar - \int_\Phi \tau\alpha e^{-\lambda\tau} \nabla F_1 \nabla\varphi\, d\hbar + \int_\Phi \left(f_3 + (p+\lambda) f_1\right)\varphi\, d\hbar, \\ \displaystyle\int_\Phi \lambda^2 v\psi\, d\hbar + \int_\Phi \lambda q v \psi\, d\hbar + \int_\Phi b\nabla v \nabla\psi\, d\hbar + \int_\Phi \lambda\beta e^{-\lambda\tau}\, \nabla u \nabla \psi\, d\hbar \\ \displaystyle = \int_\Phi \beta e^{-\lambda\tau} \nabla f_1 \nabla\psi\, d\hbar - \int_\Phi \tau\beta e^{-\lambda\tau} \nabla F_2 \nabla\psi\, d\hbar + \int_\Phi \left(f_4 + (q+\lambda) f_2\right)\psi\, d\hbar, \end{array}$$

such that

$$F_1(\hbar) = \int_0^1 f_6(\hbar,\theta) e^{\lambda\tau\theta} d\theta \quad \text{and} \quad F_2(\hbar) = \int_0^1 f_5(\hbar,\theta) e^{\lambda\tau\theta} d\theta.$$

Consequently, problem (3.9) is equivalent to the problem

$$\varpi\left((u,v),(\varphi,\psi)\right) = \pi\left(\varphi,\psi\right), \tag{3.11}$$

where the bilinear form

$$\varpi : \left(H_0^1(\Phi) \times H_0^1(\Phi)\right)^2 \to \mathbb{R}$$

and the linear form

$$\pi : H_0^1(\Phi) \times H_0^1(\Phi) \to \mathbb{R}$$

are defined by

$$\begin{array}{rl} \varpi\left((u,v),(\varphi,\psi)\right) = & \displaystyle\int_\Phi \lambda^2 u\varphi\, d\hbar + \int_\Phi \lambda^2 v\psi\, d\hbar + \int_\Phi \lambda p u\varphi\, d\hbar + \int_\Phi \lambda q v\psi\, d\hbar \\ \displaystyle + \int_\Phi a\nabla u\nabla\varphi\, d\hbar + \int_\Phi b\nabla v \nabla\psi\, d\hbar + \int_\Phi \lambda\alpha e^{-\lambda\tau}\, \nabla v\nabla\varphi\, d\hbar + \int_\Phi \lambda\beta e^{-\lambda\tau}\, \nabla u \nabla\psi\, d\hbar & \end{array}$$

and

$$\begin{array}{rl} \pi\left(\varphi,\psi\right) = & \displaystyle\int_\Phi \alpha e^{-\lambda\tau}\nabla f_2 \nabla\varphi\, d\hbar + \int_\Phi \beta e^{-\lambda\tau} \nabla f_1\nabla\psi\, d\hbar - \int_\Phi \tau\alpha e^{-\lambda\tau}\nabla F_1 \nabla\varphi\, d\hbar \\ \displaystyle - \int_\Phi \tau\beta e^{-\lambda\tau}\nabla F_2\nabla\psi\, d\hbar + \int_\Phi \left(f_3 + (p+\lambda) f_1\right)\varphi\, d\hbar + \int_\Phi \left(f_4 + (q+\lambda) f_2\right)\psi\, d\hbar. & \end{array}$$

One can readily confirm that ϖ exhibits continuity and coercivity, while π is continuous. By invoking the Lax-Milgram theorem, we establish that for every pair $(\varphi,\psi) \in H_0^1(\Phi) \times H_0^1(\Phi)$, the system (3.11) possesses a unique solution $(u,v) \in H_0^1(\Phi) \times H_0^1(\Phi)$. Utilizing standard elliptic regularity results, it follows from (3.9) that $(u,v) \in H^2(\Phi) \times H^2(\Phi)$.
Therefore, the operator $\lambda I - \mathcal{A}$ exhibits surjectivity for every $\lambda > 0$. As a result, the outcome of Theorem 2.1 is established by applying the Hille-Yosida theorem.

4. Asymptotic Behavior

Henceforth, we use c to represent various positive constants, which may differ across instances. In (2.9), we multiply the first equation by $\aleph^{\mu}u$ and the second by $\aleph^{\mu}v$, where $\mu > 0$, yielding

$$\begin{aligned}
0 &= \int_S^T \aleph^{\mu} \int_{\Phi} u \left(u_{\mathsf{tt}} - a\Delta u + p(\hbar)\, u_{\mathsf{t}} - div\left(\alpha(\hbar)\nabla z(\hbar,1,\mathsf{t})\right)\right) d\hbar\, d\mathsf{t} \\
&= \left[\aleph^{\mu}\int_{\Phi} uu_{\mathsf{t}}\, d\hbar\right]_S^T - \int_S^T \mu\aleph'\aleph^{\mu-1}\int_{\Phi} uu_{\mathsf{t}}\, d\hbar\, d\mathsf{t} \\
&\quad -2\int_S^T \aleph^{\mu}\int_{\Phi} u_{\mathsf{t}}^2\, d\hbar\, d\mathsf{t} + \int_S^T \aleph^{\mu}\int_{\Phi}\left(u_{\mathsf{t}}^2 + a|\nabla u|^2\right) d\hbar\, d\mathsf{t} \\
&\quad + \int_S^T \aleph^{\mu}\int_{\Phi} p(\hbar) uu_{\mathsf{t}}\, d\hbar\, d\mathsf{t} + \int_S^T \aleph^{\mu}\int_{\Phi}\alpha(\hbar)\nabla u \nabla z(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t},
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
0 &= \int_S^T \aleph^{\mu} \int_{\Phi} v \left(v_{\mathsf{tt}} - b\Delta v + q(\hbar)\, v_{\mathsf{t}} - div\left(\beta(\hbar)\nabla y(\hbar,1,\mathsf{t})\right)\right) d\hbar\, d\mathsf{t} \\
&= \left[\aleph^{\mu}\int_{\Phi} vv_{\mathsf{t}}\, d\hbar\right]_S^T - \int_S^T \mu\aleph'\aleph^{\mu-1}\int_{\Phi} vv_{\mathsf{t}}\, d\hbar\, d\mathsf{t} \\
&\quad -2\int_S^T \aleph^{\mu}\int_{\Phi} v_{\mathsf{t}}^2\, d\hbar\, d\mathsf{t} + \int_S^T \aleph^{\mu}\int_{\Phi}\left(v_{\mathsf{t}}^2 + b|\nabla v|^2\right) d\hbar\, d\mathsf{t} \\
&\quad + \int_S^T \aleph^{\mu}\int_{\Phi} q(\hbar) vv_{\mathsf{t}}\, d\hbar\, d\mathsf{t} + \int_S^T \aleph^{\mu}\int_{\Phi}\beta(\hbar)\nabla v \nabla y(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t},
\end{aligned} \tag{4.2}$$

Similarly, we multiply the third equation of (2.9) by $\aleph^{\mu}e^{-2\kappa\tau}y(\hbar,\kappa,\mathsf{t})$ and fifth equation by $\aleph^{\mu}e^{-2\kappa\tau}z(\hbar,\kappa,\mathsf{t})$, we get

$$\begin{aligned}
0 &= \int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau} y\left(\tau y_{\mathsf{t}} + y_{\kappa}\right) d\kappa\, d\hbar\, d\mathsf{t} \\
&= \left[\frac{\tau}{2}\aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\right]_S^T - \frac{\tau}{2}\int_S^T \mu\aleph'\aleph^{\mu-1}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\, d\mathsf{t} \\
&\quad + \frac{1}{2}\int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 \frac{\partial}{\partial\kappa}\left(e^{-2\kappa\tau}y^2\right) d\kappa\, d\hbar\, d\mathsf{t} + \tau\int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\, d\mathsf{t} \\
&= \left[\frac{\tau}{2}\aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\right]_S^T - \frac{\tau}{2}\int_S^T \mu\aleph'\aleph^{\mu-1}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\, d\mathsf{t} \\
&\quad + \frac{1}{2}\int_S^T \aleph^{\mu}\int_{\Phi}\left(e^{-2\tau}y^2(\hbar,1,\mathsf{t}) - y^2(\hbar,0,\mathsf{t})\right) d\hbar\, d\mathsf{t} + \tau\int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}y^2\, d\kappa\, d\hbar\, d\mathsf{t},
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
0 &= \int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau} z\left(\tau z_{\mathsf{t}} + z_{\kappa}\right) d\kappa\, d\hbar\, d\mathsf{t} \\
&= \left[\frac{\tau}{2}\aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}z^2\, d\kappa\, d\hbar\right]_S^T - \frac{\tau}{2}\int_S^T \mu\aleph'\aleph^{\mu-1}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}z^2\, d\kappa\, d\hbar\, d\mathsf{t} \\
&\quad + \frac{1}{2}\int_S^T \aleph^{\mu}\int_{\Phi}\left(e^{-2\tau}z^2(\hbar,1,\mathsf{t}) - z^2(\hbar,0,\mathsf{t})\right) d\hbar\, d\mathsf{t} + \tau\int_S^T \aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}z^2\, d\kappa\, d\hbar\, d\mathsf{t}.
\end{aligned} \tag{4.4}$$

Taking their sum, we obtain

$$\begin{aligned}
&\Lambda \int_S^T \aleph^{\mu+1}\, d\mathsf{t} \\
&\leq \int_S^T \aleph^{\mu} \int_\Phi \left(u_\mathsf{t}^2 + v_\mathsf{t}^2 + a|\nabla u|^2 + b|\nabla v|^2 + \tau \int_0^1 e^{-2\kappa\tau} \left(y^2+z^2\right) d\kappa \right) d\hbar\, d\mathsf{t} \\
&= -\left[\aleph^{\mu} \int_\Phi u u_\mathsf{t}\, d\hbar \right]_S^T + \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi u u_\mathsf{t}\, d\hbar\, d\mathsf{t} + 2\int_S^T \aleph^{\mu} \int_\Phi u_\mathsf{t}^2\, d\hbar\, d\mathsf{t} \\
&- \left[\aleph^{\mu} \int_\Phi v v_\mathsf{t}\, d\hbar \right]_S^T + \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi v v_\mathsf{t}\, d\hbar\, d\mathsf{t} + 2\int_S^T \aleph^{\mu} \int_\Phi v_\mathsf{t}^2\, d\hbar\, d\mathsf{t} \\
&- \int_S^T \aleph^{\mu} \int_\Phi p(\hbar) u u_\mathsf{t}\, d\hbar\, d\mathsf{t} - \int_S^T \aleph^{\mu} \int_\Phi \alpha(\hbar) \nabla u \nabla z(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t} \\
&- \int_S^T \aleph^{\mu} \int_\Phi q(\hbar) v v_\mathsf{t}\, d\hbar\, d\mathsf{t} - \int_S^T \aleph^{\mu} \int_\Phi \beta(\hbar) \nabla v \nabla y(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t} \\
&- \left[\frac{\tau}{2} \aleph^{\mu} \int_\Phi \int_0^1 e^{-2\kappa\tau} y^2\, d\kappa\, d\hbar \right]_S^T + \frac{\tau}{2} \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi \int_0^1 e^{-2\kappa\tau} y^2\, d\kappa\, d\hbar\, d\mathsf{t} \\
&- \left[\frac{\tau}{2} \aleph^{\mu} \int_\Phi \int_0^1 e^{-2\kappa\tau} z^2\, d\kappa\, d\hbar \right]_S^T + \frac{\tau}{2} \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi \int_0^1 e^{-2\kappa\tau} z^2\, d\kappa\, d\hbar\, d\mathsf{t}
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{2} \int_S^T \aleph^{\mu} \int_\Phi \left(y^2(\hbar,0,\mathsf{t}) - e^{-2\tau} y^2(\hbar,1,\mathsf{t}) \right) d\hbar\, d\mathsf{t} \\
&+\frac{1}{2} \int_S^T \aleph^{\mu} \int_\Phi \left(z^2(\hbar,0,\mathsf{t}) - e^{-2\tau} z^2(\hbar,1,\mathsf{t}) \right) d\hbar\, d\mathsf{t},
\end{aligned} \tag{4.5}$$

where $\Lambda = 2\min\{1, e^{-2\tau}\}$.
Using the Cauchy-Schwarz and Young inequalities along with (2.2), we obtain

$$\begin{aligned}
&- \int_S^T \aleph^{\mu} \int_\Phi \alpha(\hbar) \nabla u \nabla z(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t} \\
&\leq \frac{e^{2\tau}}{2} \int_S^T \aleph^{\mu} \int_\Phi \alpha(\hbar) |\nabla u|^2\, d\hbar\, d\mathsf{t} + \frac{e^{-2\tau}}{2} \int_S^T \aleph^{\mu} \int_\Phi \alpha(\hbar) |\nabla z(\hbar,1,\mathsf{t})|^2\, d\hbar\, d\mathsf{t} \\
&\leq \frac{e^{2\tau}}{2a} \left(2 - \frac{1}{q_0}\right) \int_S^T \aleph^{\mu} \int_\Phi a|\nabla u|^2\, d\hbar\, d\mathsf{t} + \frac{e^{-2\tau}}{2} \int_S^T \aleph^{\mu} \int_\Phi |y(\hbar,1,\mathsf{t})|^2\, d\hbar\, d\mathsf{t},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&- \int_S^T \aleph^{\mu} \int_\Phi \beta(\hbar) \nabla v \nabla y(\hbar,1,\mathsf{t})\, d\hbar\, d\mathsf{t} \\
&\leq \frac{e^{2\tau}}{2b} \left(2 - \frac{1}{p_0}\right) \int_S^T \aleph^{\mu} \int_\Phi b|\nabla v|^2\, d\hbar\, d\mathsf{t} + \frac{e^{-2\tau}}{2} \int_S^T \aleph^{\mu} \int_\Phi |z(\hbar,1,\mathsf{t})|^2\, d\hbar\, d\mathsf{t}.
\end{aligned} \tag{4.7}$$

Then, we obtian

$$\begin{aligned}
&\Lambda \int_S^T \aleph^{\mu+1}\, d\mathsf{t} \;\leq\; -\left[\aleph^{\mu} \int_\Phi u u_\mathsf{t}\, d\hbar \right]_S^T - \left[\mathcal{E}^{\mu} \int_\Phi v v_\mathsf{t}\, d\hbar \right]_S^T + \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi u u_\mathsf{t}\, d\hbar\, d\mathsf{t} \\
&+ \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi v v_\mathsf{t}\, d\hbar\, d\mathsf{t} - \int_S^T \aleph^{\mu} \int_\Phi p(\hbar) u u_\mathsf{t}\, d\hbar\, d\mathsf{t} - \int_S^T \aleph^{\mu} \int_\Phi q(\hbar) v v_\mathsf{t}\, d\hbar\, d\mathsf{t} \\
&+\frac{5}{2} \int_S^T \aleph^{\mu} \int_\Phi \left(|u_\mathsf{t}|^2 + |v_\mathsf{t}|^2 \right) d\hbar\, d\mathsf{t} \\
&+\frac{e^{2\tau}}{2a} \left(2 - \frac{1}{q_0}\right) \int_S^T \aleph^{\mu} \int_\Phi a|\nabla u|^2\, d\hbar\, d\mathsf{t} + \frac{e^{2\tau}}{2b} \left(2 - \frac{1}{p_0}\right) \int_S^T \aleph^{\mu} \int_\Phi b|\nabla v|^2\, d\hbar\, d\mathsf{t} \\
&+ \left[\frac{\tau}{2} \aleph^{\mu} \int_\Phi \int_0^1 e^{-2\kappa\tau} \left(y^2+z^2\right) d\kappa\, d\hbar \right]_T^S + \frac{\tau}{2} \int_S^T \mu \aleph' \aleph^{\mu-1} \int_\Phi \int_0^1 e^{-2\kappa\tau} \left(y^2+z^2\right) d\kappa\, d\hbar\, d\mathsf{t}.
\end{aligned} \tag{4.8}$$

Using the Cauchy-Schwarz, Young's and Poincaré's inequalities and the definition of $\aleph$, we get

$$\begin{aligned} \left|\aleph^{\mu}\int_{\Phi} u u_{\mathsf{t}}\,d\hbar\right| &\leq \aleph^{\mu}\|u\|_2\|u_{\mathsf{t}}\|_2 \\ &\leq \frac{1}{2}\aleph^{\mu}\int_{\Phi}|u|^2\,d\hbar+\frac{1}{2}\aleph^{\mu}\int_{\Phi}|u_{\mathsf{t}}|^2\,d\hbar \\ &\leq \frac{c_*}{2}\aleph^{\mu}\int_{\Phi}|\nabla u|^2\,d\hbar+\frac{1}{2}\aleph^{\mu}\int_{\Phi}|u_{\mathsf{t}}|^2\,d\hbar \\ &\leq \frac{c}{2}\aleph^{\mu}\int_{\Phi}\left(a|\nabla u|^2+|u_{\mathsf{t}}|^2\right)\,d\hbar. \end{aligned}$$

Then, we conclure

$$\left|\aleph^{\mu}\int_{\Phi} u u_{\mathsf{t}}\,d\hbar\right| \leq c\,\aleph^{\mu+1}(\mathsf{t}). \tag{4.9}$$

Similarly, we have

$$\left|\aleph^{\mu}\int_{\Phi} v v_{\mathsf{t}}\,d\hbar\right| \leq \frac{c}{2}\aleph^{\mu}\int_{\Phi}\left(b|\nabla v|^2+|v_{\mathsf{t}}|^2\right)\,d\hbar \leq c\,\aleph^{\mu+1}(\mathsf{t}). \tag{4.10}$$

By recalling (2.12), we obtian

$$-\left[\aleph^{\mu}\int_{\Phi} u u_{\mathsf{t}}\,d\hbar\right]_S^T-\left[\aleph^{\mu}\int_{\Phi} v v_{\mathsf{t}}\,d\hbar\right]_S^T \leq c\,\aleph^{\mu+1}(S), \tag{4.11}$$

$$\int_S^T\left|\mu\aleph'\aleph^{\mu-1}\int_{\Phi} u u_{\mathsf{t}}\,d\hbar\right|\,d\mathsf{t}\leq c\int_S^T\aleph^{\mu}|\aleph'|\,d\mathsf{t}\leq c\int_S^T\aleph^{\mu}(-\aleph')\,d\mathsf{t}\leq c\,\aleph^{\mu+1}(S), \tag{4.12}$$

$$\int_S^T\left|\mu\aleph'\aleph^{\mu-1}\int_{\Phi} v v_{\mathsf{t}}\,d\hbar\right|\,d\mathsf{t}\leq c\,\aleph^{\mu+1}(S), \tag{4.13}$$

$$\begin{aligned} -\int_S^T\aleph^{\mu}\int_{\Phi}p(\hbar)u u_{\mathsf{t}}\,d\hbar\,d\mathsf{t} &\leq \|p\|_{\infty}\int_S^T\aleph^{\mu}\int_{\Phi}|u u_{\mathsf{t}}|\,d\hbar\,d\mathsf{t} \\ &\leq c\int_S^T\aleph^{\mu+1}(\mathsf{t})\,d\mathsf{t} \\ &\leq c\,\aleph^{\mu+1}(S)\int_S^T d\mathsf{t} \\ &\leq c\,\aleph^{\mu+1}(S), \end{aligned} \tag{4.14}$$

and

$$-\int_S^T\aleph^{\mu}\int_{\Phi}q(\hbar)v v_{\mathsf{t}}\,d\hbar\,d\mathsf{t} \leq \|q\|_{\infty}\int_S^T\aleph^{\mu}\int_{\Phi}|v v_{\mathsf{t}}|\,d\hbar\,d\mathsf{t} \leq c\,\aleph^{\mu+1}(S), \tag{4.15}$$

Recalling the definition of $\aleph$, we have

$$\begin{aligned} &\frac{5}{2}\int_S^T\aleph^{\mu}\int_{\Phi}\left(|u_{\mathsf{t}}|^2+|v_{\mathsf{t}}|^2\right)\,d\hbar\,d\mathsf{t}+\frac{e^{2\tau}}{2a}\left(2-\frac{1}{q_0}\right)\int_S^T\aleph^{\mu}\int_{\Phi}a|\nabla u|^2\,d\hbar\,d\mathsf{t} \\ &+\frac{e^{2\tau}}{2b}\left(2-\frac{1}{p_0}\right)\int_S^T\aleph^{\mu}\int_{\Phi}b|\nabla v|^2\,d\hbar\,d\mathsf{t} \\ &\leq c\int_S^T\aleph^{\mu+1}(\mathsf{t})\,d\mathsf{t} \leq c\,\aleph^{\mu+1}(S)\int_S^T d\mathsf{t} \leq c\,\aleph^{\mu+1}(S), \end{aligned} \tag{4.16}$$

$$\left[\frac{\tau}{2}\aleph^{\mu}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}\left(y^2+z^2\right)\,d\kappa\,d\hbar\right]_T^S \leq c\,\aleph^{\mu+1}(S)+c\,\aleph^{\mu+1}(T) \leq c\,\aleph^{\mu+1}(S), \tag{4.17}$$

and

$$\frac{\tau}{2}\int_S^T\mu\aleph'\aleph^{\mu-1}\int_{\Phi}\int_0^1 e^{-2\kappa\tau}\left(y^2+z^2\right)\,d\kappa\,d\hbar\,d\mathsf{t}\leq c\int_S^T|\aleph'|\aleph^{\mu}\,d\mathsf{t}\leq c\int_S^T(-\aleph')\aleph^{\mu}\,d\mathsf{t}\leq c\,\aleph^{\mu+1}(S). \tag{4.18}$$

Combining these estimates, we conclude from (4.8) that

$$\int_S^T \aleph^{\mu+1} d\mathsf{t} \;\leq\; c\,\aleph^{\mu+1}(S) \;\leq\; c\,\aleph(S). \tag{4.19}$$

Thus, we conclude based on Lemma 2.1.

$$\aleph(\mathsf{t}) \;\leq\; c\,\aleph(0)\,e^{-\gamma \mathsf{t}}, \qquad \forall\, \mathsf{t} \geq 0. \tag{4.20}$$

This concludes the demonstration of Theorem 2.1.

To create the Acknowledgments

Acknowledgments

We thank the editor and the referee for their valuable and constructive comments.

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