Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.79863

Unrestricted Mersenne and Mersenne-Lucas Hybrid Octonions and Sedenions

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ABSTRACT: In this paper, we introduce unrestricted Mersenne and Mersenne-Lucas Hybrid Octonions and Sedenions sequences and establish recurrence relations, generating functions, and Binet formulas for the preceding sequences. Also verified the above sequences through some widely acknowledged identities and furthered a few relationships among them

Key Words: Mersenne sequence, Mersenne-Lucas sequence, hybrid numbers, octonions, sedenions.

Contents

1. Introduction

In hypercomplex number, Octonions have eight dimensions, which is twice the number of dimensions of quaternions, of which they are an extension. They are neither commutative nor associative. The Sedenions form a 16-dimensional non-commutative and non-associative algebra over the real numbers which are obtained by applying the Cayley-Dickson construction to the Octonions.

Marin Mersenne, a French mathematician, initially introduced a number of the type $M_n = 2^n - 1$, where n is an integer, in 1644. Various investigations have been carried out on the Mersenne sequences. Mersenne-Lucas sequences are defined as $ML_n = 2^n + 1, n \geqslant 2$, with $ML_0 = 2, ML_1 = 3$. In [1,2], the Mersenne Lucas sequences, including its generating functions and Binet formulas were discussed.

The hybrid number [3,4], which comprises real, complex, dual, and hyperbolic numbers, was introduced by Ozdemir in 2018. It is of the form $\mathcal{H} = z_0 + z_1 i + z_2 \varepsilon + z_3 h$, where $z_0, z_1, z_2, z_3 \in \mathbb{R}$ and i, ε, h are operators such that $i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = i + \varepsilon$.

Many studies have been conducted on various Octonions [5,6] and Sedenions [7,8,9] sequences. We present unconstrained Mersenne and Mersenne-Lucas Hybrid Octonions and Sedenions sequences in this study, as well as recurrence relations, generating functions, and Binet formulas for the preceding sequences. The above sequences are additionally validated using several widely recognized identities, and a few correlations between them were also shown.

2. Unconstrained Mersenne and Mersenne-Lucas Hybrid Octonions

Definition 2.1 *For any integers p, q, υ and non-negative integer n, unrestricted Mersenne and Mersenne-Lucas hybrid Octonions are defined by*

(i) $\widetilde{M_n}^{(p,q,\upsilon)} = \widetilde{M_n} + i\widetilde{M_{n+p}} + \varepsilon\widetilde{M_{n+q}} + h\widetilde{M_{n+\upsilon}}$

(ii) $\widetilde{ML_n}^{(p,q,\upsilon)} = \widetilde{ML_n} + i\widetilde{ML_{n+p}} + \varepsilon\widetilde{ML_{n+q}} + h\widetilde{ML_{n+\upsilon}}$

where i, ε, h are hybrid units.

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2010 *Mathematics Subject Classification:* 11B99, 11R52.
Submitted November 02, 2025. Published December 20, 2025

Theorem 2.1 *Unrestricted Mersenne and Mersenne-Lucas Hybrid Octonions satisfy the recurrence relations:*

i) $\widetilde{M_n}^{(p,q,\upsilon)} = 3\widetilde{M_{n-1}}^{(p,q,\upsilon)} - 2\widetilde{M_{n-2}}^{(p,q,\upsilon)}$

ii) $\widetilde{ML_n}^{(p,q,\upsilon)} = 3\widetilde{ML_{n-1}}^{(p,q,\upsilon)} - 2\widetilde{ML_{n-2}}^{(p,q,\upsilon)}$

Proof:
i) For $n = 2$, we have

$$\begin{aligned}
\widetilde{M_2}^{(p,q,\upsilon)} &= 3\widetilde{M_1}^{(p,q,\upsilon)} - 2\widetilde{M_0}^{(p,q,\upsilon)} \\
&= 3(\widetilde{M_1} + i\widetilde{M_{1+p}} + \varepsilon\widetilde{M_{1+q}} + h\widetilde{M_{1+\upsilon}}) - 2(\widetilde{M_0} + i\widetilde{M_{0+p}} + \varepsilon\widetilde{M_{0+q}} + h\widetilde{M_{0+\upsilon}}) \\
&= 3[(M_1e_0 + M_2e_1 + M_3e_2 + M_4e_3 + M_5e_4 + M_6e_5 + M_7e_6 + M_8e_7) \\
&\quad +i(M_{1+p}e_0 + M_{2+p}e_1 + M_{3+p}e_2 + M_{4+p}e_3 + M_{5+p}e_4 + M_{6+p}e_5 \\
&\quad +M_{7+p}e_6 + M_{8+p}e_7) + \varepsilon(M_{1+q}e_0 + M_{2+q}e_1 + M_{3+q}e_2 + M_{4+q}e_3 \\
&\quad +M_{5+q}e_4 + M_{6+q}e_5 + M_{7+q}e_6 + M_{8+q}e_7) + h(M_{1+\upsilon}e_0 + M_{2+\upsilon}e_1 \\
&\quad +M_{3+\upsilon}e_2 + M_{4+\upsilon}e_3 + M_{5+\upsilon}e_4 + M_{6+\upsilon}e_5 + M_{7+\upsilon}e_6 + M_{8+\upsilon}e_7)] \\
&\quad -2[(M_0e_0 + M_1e_1 + M_2e_2 + M_3e_3 + M_4e_4 + M_5e_5 + M_6e_6 + M_7e_7) \\
&\quad +i(M_{0+p}e_0 + M_{1+p}e_1 + M_{2+p}e_2 + M_{3+p}e_3 + M_{4+p}e_4 + M_{5+p}e_5 \\
&\quad +M_{6+p}e_6 + M_{7+p}e_7) + \varepsilon(M_{0+q}e_0 + M_{1+q}e_1 + M_{2+q}e_2 + M_{3+q}e_3 \\
&\quad +M_{4+q}e_4 + M_{5+q}e_5 + M_{6+q}e_6 + M_{7+q}e_7) + h(M_{0+\upsilon}e_0 + M_{1+\upsilon}e_1 \\
&\quad +M_{2+\upsilon}e_2 + M_{3+\upsilon}e_3 + M_{4+\upsilon}e_4 + M_{5+\upsilon}e_5 + M_{6+\upsilon}e_6 + M_{7+\upsilon}e_7)] \\
&= e_0(3M_1 - 2M_0) + e_1(3M_2 - 2M_1) + e_2(3M_3 - 2M_2) + e_3(3M_4 - 2M_3) \\
&\quad +e_4(3M_5 - 2M_4) + e_5(3M_6 - 2M_5) + e_6(3M_7 - 2M_6) + e_7(3M_8 - 2M_7) \\
&\quad +i(e_0(3M_{1+p} - 2M_{0+p}) + e_1(3M_{2+p} - 2M_{1+p}) + e_2(3M_{3+p} - 2M_{2+p}) \\
&\quad +e_3(3M_{4+p} - 2M_{3+p}) + e_4(3M_{5+p} - 2M_{4+p}) + e_5(3M_{6+p} - 2M_{5+p}) \\
&\quad +e_6(3M_{7+p} - 2M_{6+p}) + e_7(3M_{8+p} - 2M_{7+p})) + \varepsilon(e_0(3M_{1+q} - 2M_{0+q}) \\
&\quad +e_1(3M_{2+q} - 2M_{1+q}) + e_2(3M_{3+q} - 2M_{2+q}) + e_3(3M_{4+q} - 2M_{3+q}) \\
&\quad +e_4(3M_{5+q} - 2M_{4+q}) + e_5(3M_{6+q} - 2M_{5+q}) + e_6(3M_{7+q} - 2M_{6+q}) \\
&\quad +e_7(3M_{8+q} - 2M_{7+q})) + h(e_0(3M_{1+\upsilon} - 2M_{0+\upsilon}) + e_1(3M_{2+\upsilon} - 2M_{1+\upsilon}) \\
&\quad +e_2(3M_{3+\upsilon} - 2M_{2+\upsilon}) + e_3(3M_{4+\upsilon} - 2M_{3+\upsilon}) + e_4(3M_{5+\upsilon} - 2M_{4+\upsilon}) \\
&\quad +e_5(3M_{6+\upsilon} - 2M_{5+\upsilon}) + e_6(3M_{7+\upsilon} - 2M_{6+\upsilon}) + e_7(3M_{8+\upsilon} - 2M_{7+\upsilon})) \\
&= e_0M_2 + e_1M_3 + e_2M_4 + e_3M_5 + e_4M_6 + e_5M_7 + e_6M_8 + e_7M_9 \\
&\quad +i(e_0M_{2+p} + e_1M_{3+p} + e_2M_{4+p} + e_3M_{5+p} + e_4M_{6+p} + e_5M_{7+p} \\
&\quad +e_6M_{8+p} + e_7M_{9+p}) + \varepsilon(e_0M_{2+q} + e_1M_{3+q} + e_2M_{4+q} + e_3M_{5+q} \\
&\quad +e_4M_{6+q} + e_5M_{7+q} + e_6M_{8+q} + e_7M_{9+q}) + h(e_0M_{2+\upsilon} + e_1M_{3+\upsilon} \\
&\quad +e_2M_{4+\upsilon} + e_3M_{5+\upsilon} + e_4M_{6+\upsilon} + e_5M_{7+\upsilon} + e_6M_{8+\upsilon} + e_7M_{9+\upsilon}) \\
&= \widetilde{M_2} + i\widetilde{M_{2+p}} + \varepsilon\widetilde{M_{2+q}} + h\widetilde{M_{2+\upsilon}}
\end{aligned}$$

By using the definition 2.1 (i), we have

$$\begin{aligned}
\widetilde{M_n}^{(p,q,\upsilon)} &= \widetilde{M_n} + i\widetilde{M_{n+p}} + \varepsilon\widetilde{M_{n+q}} + h\widetilde{M_{n+\upsilon}} \\
&= M_ne_0 + M_{n+1}e_1 + M_{n+2}e_2 + M_{n+3}e_3 + M_{n+4}e_4 + M_{n+5}e_5 + M_{n+6}e_6 \\
&\quad + M_{n+7}e_7 + i(M_{n+p}e_0 + M_{n+p+1}e_1 + M_{n+p+2}e_2 + M_{n+p+3}e_3 + M_{n+p+4}e_4 \\
&\quad + M_{n+p+5}e_5 + M_{n+p+6}e_6 + M_{n+p+7}e_7) + \varepsilon(M_{n+q}e_0 + M_{n+q+1}e_1 \\
&\quad + M_{n+q+2}e_2 + M_{n+q+3}e_3 + M_{n+q+4}e_4 + M_{n+q+5}e_5 + M_{n+q+6}e_6 \\
&\quad + M_{n+q+7}e_7) + h(M_{n+\upsilon}e_0 + M_{n+\upsilon+1}e_1 + M_{n+\upsilon+2}e_2 + M_{n+\upsilon+3}e_3 \\
&\quad + M_{n+\upsilon+4}e_4 + M_{n+\upsilon+5}e_5 + M_{n+\upsilon+6}e_6 + M_{n+\upsilon+7}e_7) \\
&= (3M_{n-1} - 2M_{n-2})e_0 + (3M_n - 2M_{n-1})e_1 + (3M_{n+1} - 2M_n)e_2 \\
&\quad + (3M_{n+2} - 2M_{n+1})e_3 + (3M_{n+3} - 2M_{n+2})e_4 + (3M_{n+4} - 2M_{n+3})e_5 \\
&\quad + (3M_{n+5} - 2M_{n+4})e_6 + (3M_{n+6} - 2M_{n+5})e_7 + i((3M_{n+p-1} - 2M_{n+p-2})e_0 \\
&\quad + (3M_{n+p} - 2M_{n+p-1})e_1 + (3M_{n+p+1} - 2M_{n+p})e_2 \\
&\quad + (3M_{n+p+2} - 2M_{n+p+1})e_3 + (3M_{n+p+3} - 2M_{n+p+2})e_4 \\
&\quad + (3M_{n+p+4} - 2M_{n+p+3})e_5 + (3M_{n+p+5} - 2M_{n+p+4})e_6 \\
&\quad + (3M_{n+p+6} - 2M_{n+p+5})e_7) + \varepsilon((3M_{n+q-1} - 2M_{n+q-2})e_0
\end{aligned}$$

$$
\begin{aligned}
&+(3M_{n+q}-2M_{n+q-1})e_1+(3M_{n+q+1}-2M_{n+q})e_2+(3M_{n+q+2}-2M_{n+q+1})e_3\\
&+(3M_{n+q+3}-2M_{n+q+2})e_4+(3M_{n+q+4}-2M_{n+q+3})e_5\\
&+(3M_{n+q+5}-2M_{n+q+4})e_6+(3M_{n+q+6}-2M_{n+q+5})e_7)\\
&+h((3M_{n+\upsilon-1}-2M_{n+\upsilon-2})e_0+(3M_{n+\upsilon}-2M_{n+\upsilon-1})e_1\\
&+(3M_{n+\upsilon+1}-2M_{n+\upsilon})e_2+(3M_{n+\upsilon+2}-2M_{n+\upsilon+1})e_3\\
&+(3M_{n+\upsilon+3}-2M_{n+\upsilon+2})e_4+(3M_{n+\upsilon+4}-2M_{n+\upsilon+3})e_5\\
&+(3M_{n+\upsilon+5}-2M_{n+\upsilon+4})e_6+(3M_{n+\upsilon+6}-2M_{n+\upsilon+5})e_7)\\
=&3(M_{n-1}e_0+M_ne_1+M_{n+1}e_2+M_{n+2}e_3+M_{n+3}e_4+M_{n+4}e_5+M_{n+5}e_6\\
&+M_{n+6}e_7+i(M_{n+p-1}e_0+M_{n+p}e_1+M_{n+p+1}e_2+M_{n+p+2}e_3+M_{n+p+3}e_4\\
&+M_{n+p+4}e_5+M_{n+p+5}e_6+M_{n+p+6}e_7)+\varepsilon(M_{n+q-1}e_0+M_{n+q}e_1+M_{n+q+1}e_2\\
&+M_{n+q+2}e_3+M_{n+q+3}e_4+M_{n+q+4}e_5+M_{n+q+5}e_6+M_{n+q+6}e_7)\\
&+h(M_{n+\upsilon-1}e_0+M_{n+\upsilon}e_1+M_{n+\upsilon+1}e_2+M_{n+\upsilon+2}e_3+M_{n+\upsilon+3}e_4+M_{n+\upsilon+4}e_5\\
&+M_{n+\upsilon+5}e_6+M_{n+\upsilon+6}e_7))-2(M_{n-2}e_0+M_{n-1}e_1+M_ne_2+M_{n+1}e_3\\
&+M_{n+2}e_4+M_{n+3}e_5+M_{n+4}e_6+M_{n+5}e_7+i(M_{n+p-2}e_0+M_{n+p-1}e_1\\
&+M_{n+p}e_2+M_{n+p+1}e_3+M_{n+p+2}e_4+M_{n+p+3}e_5+M_{n+p+4}e_6+M_{n+p+5}e_7)\\
&+\varepsilon(M_{n+q-2}e_0+M_{n+q-1}e_1+M_{n+q}e_2+M_{n+q+1}e_3+M_{n+q+2}e_4+M_{n+q+3}e_5\\
&+M_{n+q+4}e_6+M_{n+q+5}e_7)+h(M_{n+\upsilon-2}e_0+M_{n+\upsilon-1}e_1+M_{n+\upsilon}e_2+M_{n+\upsilon+1}e_3\\
&+M_{n+\upsilon+2}e_4+M_{n+\upsilon+3}e_5+M_{n+\upsilon+4}e_6+M_{n+\upsilon+5}e_7))\\
=&3(\breve{M}_{n-1}+i\breve{M}_{n+p-1}+\varepsilon\breve{M}_{n+q-1}+h\breve{M}_{n+\upsilon-1})-2(\breve{M}_{n-2}+i\breve{M}_{n+p-2}\\
&+\varepsilon\breve{M}_{n+q-2}+h\breve{M}_{n+\upsilon-2})\\
=&3\breve{M}_{n-1}^{(p,q,\upsilon)}-2\breve{M}_{n-2}^{(p,q,\upsilon)}
\end{aligned}
$$

ii) For $n=2$, we have

$$
\begin{aligned}
\breve{ML}_2^{(p,q,\upsilon)} &= 3\breve{ML}_1^{(p,q,\upsilon)}-2\breve{ML}_0^{(p,q,\upsilon)}\\
&= 3(\breve{ML}_1+i\breve{ML}_{1+p}+\varepsilon\breve{ML}_{1+q}+h\breve{ML}_{1+\upsilon})\\
&\quad -2(\breve{ML}_0+i\breve{ML}_{0+p}+\varepsilon\breve{ML}_{0+q}+h\breve{ML}_{0+\upsilon})\\
&= \breve{ML}_2+i\breve{ML}_{2+p}+\varepsilon\breve{ML}_{2+q}+h\breve{ML}_{2+\upsilon}
\end{aligned}
$$

By using the definition 2.1 (ii), we have

$$
\begin{aligned}
\breve{ML}_n^{(p,q,\upsilon)} &= \breve{ML}_n+i\breve{ML}_{n+p}+\varepsilon\breve{ML}_{n+q}+h\breve{ML}_{n+\upsilon}\\
&= ML_ne_0+ML_{n+1}e_1+ML_{n+2}e_2+ML_{n+3}e_3+ML_{n+4}e_4+ML_{n+5}e_5\\
&\quad +ML_{n+6}e_6+ML_{n+7}e_7+i(ML_{n+p}e_0+ML_{n+p+1}e_1+ML_{n+p+2}e_2\\
&\quad +ML_{n+p+3}e_3+ML_{n+p+4}e_4+ML_{n+p+5}e_5+ML_{n+p+6}e_6\\
&\quad +ML_{n+p+7}e_7)+\varepsilon(ML_{n+q}e_0+ML_{n+q+1}e_1+ML_{n+q+2}e_2\\
&\quad +ML_{n+q+3}e_3+ML_{n+q+4}e_4+ML_{n+q+5}e_5+ML_{n+q+6}e_6\\
&\quad +ML_{n+q+7}e_7)+h(ML_{n+\upsilon}e_0+ML_{n+\upsilon+1}e_1+ML_{n+\upsilon+2}e_2\\
&\quad +ML_{n+\upsilon+3}e_3+ML_{n+\upsilon+4}e_4+ML_{n+\upsilon+5}e_5+ML_{n+\upsilon+6}e_6\\
&\quad +ML_{n+\upsilon+7}e_7)\\
&= 3(\breve{ML}_{n-1}+i\breve{ML}_{n+p-1}+\varepsilon\breve{ML}_{n+q-1}+h\breve{ML}_{n+\upsilon-1})\\
&\quad -2(\breve{ML}_{n-2}+i\breve{ML}_{n+p-2}+\varepsilon\breve{ML}_{n+q-2}+h\breve{ML}_{n+\upsilon-2})\\
\breve{ML}_n^{(p,q,\upsilon)} &= 3\breve{ML}_{n-1}^{(p,q,\upsilon)}-2\breve{ML}_{n-2}^{(p,q,\upsilon)}
\end{aligned}
$$

□

Theorem 2.2 *The Binet formula for* $\breve{M}_n^{(p,q,\upsilon)}$ *and* $\breve{ML}_n^{(p,q,\upsilon)}$ *are given by*

i) $\breve{M}_n^{(p,q,\upsilon)}=2^n\breve{\mathbb{a}}\mathbb{a}^*-\breve{\mathbb{b}}\mathbb{b}^*$

ii) $\breve{ML}_n^{(p,q,\upsilon)}=2^n\breve{\mathbb{a}}\mathbb{a}^*+\breve{\mathbb{b}}\mathbb{b}^*$

where $\mathbb{a}^* = 1 + 2^p i + 2^q \varepsilon + 2^v h$, $\mathbb{b}^* = 1 + i + \varepsilon + h$ *and*

$$\breve{\mathbb{a}} = \sum_{s=0}^{7} 2^s e_s, \quad \breve{\mathbb{b}} = \sum_{s=0}^{7} e_s$$

Proof:

$$\begin{aligned}
\text{i) } \breve{M_n}^{(p,q,v)} &= \breve{M_n} + i\breve{M_{n+p}} + \varepsilon\breve{M_{n+q}} + h\breve{M_{n+v}} \\
&= 2^n\breve{\mathbb{a}} - \breve{\mathbb{b}} + \left(2^{n+p}\breve{\mathbb{a}} - \breve{\mathbb{b}}\right) i + \left(2^{n+q}\breve{\mathbb{a}} - \breve{\mathbb{b}}\right)\varepsilon + \left(2^{n+v}\breve{\mathbb{a}} - \breve{\mathbb{b}}\right) h \\
&= 2^n\breve{\mathbb{a}}\left(1 + 2^p i + 2^q\varepsilon + 2^v h\right) - \breve{\mathbb{b}}\left(1 + i + \varepsilon + h\right) \\
\breve{M_n}^{(p,q,v)} &= 2^n\breve{\mathbb{a}}\mathbb{a}^* - \breve{\mathbb{b}}\mathbb{b}^*
\end{aligned}$$

where $\mathbb{a}^* = 1 + 2^p i + 2^q \varepsilon + 2^v h$ and $\mathbb{b}^* = 1 + i + \varepsilon + h$

$$\begin{aligned}
\text{ii) } \breve{ML_n}^{(p,q,v)} &= \breve{ML_n} + \breve{ML_{n+p}}i + \breve{ML_{n+q}}\varepsilon + \breve{ML_{n+v}}h \\
&= 2^n\breve{\mathbb{a}} + \breve{\mathbb{b}} + \left(2^{n+p}\breve{\mathbb{a}} + \breve{\mathbb{b}}\right) i + \left(2^{n+q}\breve{\mathbb{a}} + \breve{\mathbb{b}}\right)\varepsilon + \left(2^{n+v}\breve{\mathbb{a}} + \breve{\mathbb{b}}\right) h \\
&= 2^n\breve{\mathbb{a}}\left(1 + 2^p i + 2^q\varepsilon + 2^v h\right) + \breve{\mathbb{b}}\left(1 + i + \varepsilon + h\right) \\
\breve{ML_n}^{(p,\ q,v)} &= 2^n\breve{\mathbb{a}}\mathbb{a}^* + \breve{\mathbb{b}}\mathbb{b}^*
\end{aligned}$$

where $\mathbb{a}^* = 1 + 2^p i + 2^q \varepsilon + 2^v h$ and $\mathbb{b}^* = 1 + i + \varepsilon + h$. □

Theorem 2.3 *The generating functions for* $\breve{M_n}^{(p,q,v)}$ *and* $\breve{ML_n}^{(p,q,v)}$ *are given by*

i) $\breve{f}(x) = \frac{\breve{M}_0^{(p,q,v)} + \left(\breve{M}_1^{(p,q,v)} - 3\breve{M}_0^{(p,q,v)}\right)x}{1-3x+2x^2}$

ii) $\breve{g}(x) = \frac{\breve{ML}_0^{(p,q,v)} + \left(\breve{ML}_1^{(p,q,v)} - 3\breve{ML}_0^{(p,q,v)}\right)x}{1-3x+2x^2}$

Proof:
i) Let us define

$$\begin{aligned}
\breve{f}(x) &= \sum_{n=0}^{\infty} \breve{M}_n^{(p,q,v)} x^n \\
\breve{f}(x) &= \breve{M}_0^{(p,q,v)} + x\breve{M}_1^{(p,q,v)} + \sum_{n=2}^{\infty} \breve{M}_n^{(p,q,v)} x^n
\end{aligned}$$

Multiply this equation by $-3x$ and $2x^2$, we obtain

$$\begin{aligned}
-3\breve{f}(x)\,x &= -3x\breve{M}_0^{(p,q,v)} - 3\sum_{n=2}^{\infty} \breve{M}_{n-1}^{(p,q,v)} x^n \\
2\breve{f}(x)\,x^2 &= 2\sum_{n=2}^{\infty} \breve{M}_{n-2}^{(p,q,v)} x^n
\end{aligned}$$

Adding the above three equations, we get

$$\begin{aligned}
\left(1 - 3x + 2x^2\right)\breve{f}(x) &= \breve{M}_0^{(p,q,v)} + x\left(\breve{M}_1^{(p,q,v)} - 3\breve{M}_0^{(p,q,v)}\right) \\
&\quad + \sum_{n=2}^{\infty}\left[\breve{M}_n^{(p,q,v)} - 3\breve{M}_{n-1}^{(p,q,v)} + 2\breve{M}_{n-2}^{(p,q,v)}\right] x^n \\
\breve{f}(x) &= \frac{\breve{M}_0^{(p,q,v)} + \left(\breve{M}_1^{(p,q,v)} - 3\breve{M}_0^{(p,q,v)}\right)x}{1 - 3x + 2x^2}.
\end{aligned}$$

ii) In a similar way, we obtain $\breve{g}(x) = \frac{\widetilde{ML}_0^{(p,q,\upsilon)} + \left(\widetilde{ML}_1^{(p,q,\upsilon)} - 3\widetilde{ML}_0^{(p,q,\upsilon)}\right)x}{1-3x+2x^2}$. □

Theorem 2.4

i) The exponential generating function for $\widetilde{M_n}^{(p,q,\upsilon)}$ is given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\widetilde{M}_n^{(p,q,\upsilon)} l^k}{k!} &= \sum_{k=0}^{\infty} (2^k \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{b}}\mathrm{b}^*) \frac{l^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(2l)^k}{k!} \breve{\mathbb{a}}\mathrm{a}^* - \sum_{k=0}^{\infty} \frac{l^k}{k!} \breve{\mathbb{b}}\mathrm{b}^* \\ \sum_{k=0}^{\infty} \frac{\widetilde{M}_n^{(p,q,\upsilon)} l^k}{k!} &= \breve{\mathbb{a}}\mathrm{a}^* e^{2l} - \breve{\mathbb{b}}\mathrm{b}^* e^l \end{aligned}$$

ii) The exponential generating function for $\widetilde{ML_n}^{(p,q,\upsilon)}$ is given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\widetilde{ML}_n^{(p,q,\upsilon)} l^k}{k!} &= \sum_{k=0}^{\infty} (2^k \breve{\mathbb{a}}\mathrm{a}^* + \breve{\mathbb{b}}\mathrm{b}^*) \frac{l^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(2l)^k}{k!} \breve{\mathbb{a}}\mathrm{a}^* + \sum_{k=0}^{\infty} \frac{l^k}{k!} \breve{\mathbb{b}}\mathrm{b}^* \\ \sum_{k=0}^{\infty} \frac{\widetilde{ML}_n^{(p,q,\upsilon)} l^k}{k!} &= \breve{\mathbb{a}}\mathrm{a}^* e^{2l} + \breve{\mathbb{b}}\mathrm{b}^* e^l \end{aligned}$$

Theorem 2.5 (Vajda Identity) *For any integers m, n, k, p, q, υ, we have*

i) $\widetilde{M}_{m+n}^{(p,q,\upsilon)} \widetilde{M}_{m+k}^{(p,q,\upsilon)} - \widetilde{M}_{m}^{(p,q,\upsilon)} \widetilde{M}_{m+n+k}^{(p,q,\upsilon)} = 2^m M_n \left(2^k \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^*\right)$

ii) $\widetilde{ML}_{m+n}^{(p,q,\upsilon)} \widetilde{ML}_{m+k}^{(p,q,\upsilon)} - \widetilde{ML}_{m}^{(p,q,\upsilon)} \widetilde{ML}_{m+n+k}^{(p,q,\upsilon)} = 2^m M_n \left(\breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^* - 2^k \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^*\right)$

Proof:

i) $\widetilde{M}_{m+n}^{(p,q,\upsilon)} \widetilde{M}_{m+k}^{(p,q,\upsilon)} - \widetilde{M}_{m}^{(p,q,\upsilon)} \widetilde{M}_{m+n+k}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^{m+n} \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{b}}\mathrm{b}^*\right) \left(2^{m+k} \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{b}}\mathrm{b}^*\right) \\ &\quad -(2^m \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{b}}\mathrm{b}^*)(2^{m+n+k} \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{b}}\mathrm{b}^*) \\ &= 2^{m+k} \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^* (2^n - 1) - 2^m \breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^* (2^n - 1) \\ &= 2^m M_n \left(2^k \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^* - \breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^*\right) \end{aligned}$$

ii) $\widetilde{ML}_{m+n}^{(p,q,\upsilon)} \widetilde{ML}_{m+k}^{(p,q,\upsilon)} - \widetilde{ML}_{m}^{(p,q,\upsilon)} \widetilde{ML}_{m+n+k}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^{m+n} \breve{\mathbb{a}}\mathrm{a}^* + \breve{\mathbb{b}}\mathrm{b}^*\right) \left(2^{m+k} \breve{\mathbb{a}}\mathrm{a}^* + \breve{\mathbb{b}}\mathrm{b}^*\right) \\ &\quad -(2^m \breve{\mathbb{a}}\mathrm{a}^* + \breve{\mathbb{b}}\mathrm{b}^*)(2^{m+n+k} \breve{\mathbb{a}}\mathrm{a}^* + \breve{\mathbb{b}}\mathrm{b}^*) \\ &= 2^{m+k} \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^* (1 - 2^n) + 2^m \breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^* (2^n - 1) \\ &= 2^m M_n \left(\breve{\mathbb{a}}\mathrm{a}^* \breve{\mathbb{b}}\mathrm{b}^* - 2^k \breve{\mathbb{b}}\mathrm{b}^* \breve{\mathbb{a}}\mathrm{a}^*\right) \end{aligned}$$

If we substitute $k \to -n$ in the Vajda identities, we obtain the Catalan's identities. $\square$

Theorem 2.6 (Catalan's identity) *For any integers m, n, p, q, υ, we have*

i) $\breve{M}_{m+n}^{(p,q,\upsilon)} \breve{M}_{m-n}^{(p,q,\upsilon)} - \left[\breve{M}_{m}^{(p,q,\upsilon)}\right]^2 = 2^{m-n} M_n \left(\breve{b}b^* \breve{a}a^* - 2^n \breve{a}a^* \breve{b}b^*\right)$

ii) $\breve{ML}_{m+n}^{(p,q,\upsilon)} \breve{ML}_{m-n}^{(p,q,\upsilon)} - \left[\breve{ML}_{m}^{(p,q,\upsilon)}\right]^2 = 2^{m-n} M_n \left(2^n \breve{a}a^* \breve{b}b^* - \breve{b}b^* \breve{a}a^*\right)$

If we substitute $n \to 1$ in Catalan's identities, we get Cassini's identities.

Theorem 2.7 (Cassini's identity) *For any integers m, p, q, υ, we have*

i) $\breve{M}_{m+1}^{(p,q,\upsilon)} \breve{M}_{m-1}^{(p,q,\upsilon)} - \left[\breve{M}_{m}^{(p,q,\upsilon)}\right]^2 = 2^{m-1} \left(\breve{b}b^* \breve{a}a^* - 2\breve{a}a^* \breve{b}b^*\right)$

ii) $\breve{ML}_{m+1}^{(p,q,\upsilon)} \breve{ML}_{m-1}^{(p,q,\upsilon)} - \left[\breve{ML}_{m}^{(p,q,\upsilon)}\right]^2 = 2^{m-1} \left(2\breve{a}a^* \breve{b}b^* - \breve{b}b^* \breve{a}a^*\right)$

Theorem 2.8 (d'Ocagne's identity) *For any integers m, n, p, q, υ, we have*

i) $\breve{M}_{m}^{(p,q,\upsilon)} \breve{M}_{n+1}^{(p,q,\upsilon)} - \breve{M}_{m+1}^{(p,q,\upsilon)} \breve{M}_{n}^{(p,q,\upsilon)} = 2^m \breve{a}a^* \breve{b}b^* - 2^n \breve{b}b^* \breve{a}a^*$

ii) $\breve{ML}_{m}^{(p,q,\upsilon)} \breve{ML}_{n+1}^{(p,q,\upsilon)} - \breve{ML}_{m+1}^{(p,q,\upsilon)} \breve{ML}_{n}^{(p,q,\upsilon)} = 2^n \breve{b}b^* \breve{a}a^* - 2^m \breve{a}a^* \breve{b}b^*$

Proof:

i) $\breve{M}_{m}^{(p,q,\upsilon)} \breve{M}_{n+1}^{(p,q,\upsilon)} - \breve{M}_{m+1}^{(p,q,\upsilon)} \breve{M}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^m \breve{a}a^* - \breve{b}b^*\right)\left(2^{n+1} \breve{a}a^* - \breve{b}b^*\right) \\ &\quad -(2^{m+1} \breve{a}a^* - \breve{b}b^*)(2^n \breve{a}a^* - \breve{b}b^*) \\ &= 2^n \breve{b}b^* \breve{a}a^* (1-2) - 2^m \breve{a}a^* \breve{b}b^* (1-2) \\ &= 2^m \breve{a}a^* \breve{b}b^* - 2^n \breve{b}b^* \breve{a}a^* \end{aligned}$$

ii) $\breve{ML}_{m}^{(p,q,\upsilon)} \breve{ML}_{n+1}^{(p,q,\upsilon)} - \breve{ML}_{m+1}^{(p,q,\upsilon)} \breve{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^m \breve{a}a^* + \breve{b}b^*\right)\left(2^{n+1} \breve{a}a^* + \breve{b}b^*\right) \\ &\quad -(2^{m+1} \breve{a}a^* + \breve{b}b^*)(2^n \breve{a}a^* + \breve{b}b^*) \\ &= 2^n \breve{b}b^* \breve{a}a^* (2-1) - 2^m \breve{a}a^* \breve{b}b^* (2-1) \\ &= 2^n \breve{b}b^* \breve{a}a^* - 2^m \breve{a}a^* \breve{b}b^* \end{aligned}$$

$\square$

Theorem 2.9 (Honsberger Identity) *For any integers $m,\ n,\ p, q, \upsilon$, we have*

i) $\breve{M}_{m-1}^{(p,q,\upsilon)} \breve{M}_{n}^{(p,q,\upsilon)} + \breve{M}_{m}^{(p,q,\upsilon)} \breve{M}_{n+1}^{(p,q,\upsilon)}$

$$= 2^{m+n-1} (\breve{a}a^*)^2 ML_2 - 2^n \breve{b}b^* \breve{a}a^* ML_1 - 2^{m-1} \breve{a}a^* \breve{b}b^* ML_1 + 2\left(\breve{b}b^*\right)^2$$

ii) $\breve{ML}_{m-1}^{(p,q,\upsilon)} \breve{ML}_{n}^{(p,q,\upsilon)} + \breve{ML}_{m}^{(p,q,\upsilon)} \breve{ML}_{n+1}^{(p,q,\upsilon)}$

$$= 2^{m+n-1} (\breve{a}a^*)^2 ML_2 + 2^n \breve{b}b^* \breve{a}a^* ML_1 + 2^{m-1} \breve{a}a^* \breve{b}b^* ML_1 + 2\left(\breve{b}b^*\right)^2$$

Proof:

i) $\breve{M}_{m-1}^{(p,q,\upsilon)}\breve{M}_{n}^{(p,q,\upsilon)}+\breve{M}_{m}^{(p,q,\upsilon)}\breve{M}_{n+1}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{m-1}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*\right)\left(2^{n}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*\right)+(2^{m}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*)(2^{n+1}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*)\\
&= 2^{m+n-1}(\breve{\mathbb{a}}^*)^2\left(2^2+1\right)-2^{n}\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*\left(2+1\right)\\
&\quad -2^{m-1}\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*\left(2+1\right)+2\left(\breve{\mathbb{b}}^*\right)^2\\
&= 2^{m+n-1}(\breve{\mathbb{a}}^*)^2ML_2-2^{n}\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*ML_1-2^{m-1}\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*ML_1+2\left(\breve{\mathbb{b}}^*\right)^2
\end{aligned}$$

ii) $\breve{ML}_{m-1}^{(p,q,\upsilon)}\breve{ML}_{n}^{(p,q,\upsilon)}+\breve{ML}_{m}^{(p,q,\upsilon)}\breve{ML}_{n+1}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{m-1}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*\right)\left(2^{n}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*\right)+(2^{m}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*)(2^{n+1}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*)\\
&= 2^{m+n-1}(\breve{\mathbb{a}}^*)^2\left(2^2+1\right)+2^{n}\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*\left(2+1\right)\\
&\quad +2^{m-1}\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*\left(2+1\right)+2\left(\breve{\mathbb{b}}^*\right)^2\\
&= 2^{m+n-1}(\breve{\mathbb{a}}^*)^2ML_2+2^{n}\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*ML_1+2^{m-1}\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*ML_1+2\left(\breve{\mathbb{b}}^*\right)^2
\end{aligned}$$

□

Theorem 2.10 *For any integers n,p,q,υ, we have*

i) $\breve{M}_{n}^{(p,q,\upsilon)}\breve{ML}_{n}^{(p,q,\upsilon)}=2^{2n}(\breve{\mathbb{a}}^*)^2+2^n\left(\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*-\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*\right)-\left(\breve{\mathbb{b}}^*\right)^2$

ii) $\breve{M}_{n}^{(p,q,\upsilon)}+\breve{ML}_{n}^{(p,q,\upsilon)}=2^{n+1}\breve{\mathbb{a}}^*$

iii) $\breve{M}_{n}^{(p,q,\upsilon)}-\breve{ML}_{n}^{(p,q,\upsilon)}=-2\breve{\mathbb{b}}^*$

Proof:

i) $\breve{M}_{n}^{(p,q,\upsilon)}\breve{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{n}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*\right)\left(2^{n}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*\right)\\
&= 2^{2n}(\breve{\mathbb{a}}^*)^2+2^n\left(\breve{\mathbb{a}}^*\breve{\mathbb{b}}^*-\breve{\mathbb{b}}^*\breve{\mathbb{a}}^*\right)-\left(\breve{\mathbb{b}}^*\right)^2
\end{aligned}$$

ii) $\breve{M}_{n}^{(p,q,\upsilon)}+\breve{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{n}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*\right)+\left(2^{n}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*\right)\\
&= 2^{n+1}\breve{\mathbb{a}}^*
\end{aligned}$$

iii) $\breve{M}_{n}^{(p,q,\upsilon)}-\breve{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{n}\breve{\mathbb{a}}^*-\breve{\mathbb{b}}^*\right)-\left(2^{n}\breve{\mathbb{a}}^*+\breve{\mathbb{b}}^*\right)\\
&= -2\breve{\mathbb{b}}^*
\end{aligned}$$

□

Theorem 2.11 *For any integers n, p, q, υ, we have*

i) $\widetilde{M}_{n+1}^{(p,q,\upsilon)} + \widetilde{M}_{n}^{(p,q,\upsilon)} = 3\left(2^n\right)\breve{a}a^* - 2\breve{b}b^*$

ii) $\widetilde{ML}_{n+1}^{(p,q,\upsilon)} + \widetilde{ML}_{n}^{(p,q,\upsilon)} = 3\left(2^n\right)\breve{a}a^* + 2\breve{b}b^*$

Proof:

i) $\widetilde{M}_{n+1}^{(p,q,\upsilon)} + \widetilde{M}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^{n+1}\breve{a}a^* - \breve{b}b^*\right) + \left(2^{n}\breve{a}a^* - \breve{b}b^*\right) \\ &= 3\left(2^n\right)\breve{a}a^* - 2\breve{b}b^* \end{aligned}$$

ii) $\widetilde{ML}_{n+1}^{(p,q,\upsilon)} + \widetilde{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^{n+1}\breve{a}a^* + \breve{b}b^*\right) + \left(2^{n}\breve{a}a^* + \breve{b}b^*\right) \\ &= 3\left(2^n\right)\breve{a}a^* + 2\breve{b}b^* \end{aligned}$$

□

Theorem 2.12 *For any integers n, p, q, υ, we have*

i) $\left[\widetilde{M}_{n}^{(p,q,\upsilon)}\right]^2 + \left[\widetilde{M}_{n+1}^{(p,q,\upsilon)}\right]^2 + \left[\widetilde{M}_{n+2}^{(p,q,\upsilon)}\right]^2$

$$= 21\left(2^{2n}\right)\left(\breve{a}a^*\right)^2 + 3\left(\breve{b}b^*\right)^2 - 7(2^n)(\breve{b}b^*\breve{a}a^* + \breve{a}a^*\breve{b}b^*)$$

ii) $\left[\widetilde{ML}_{n+1}^{(p,q,\upsilon)}\right]^2 - \left[\widetilde{ML}_{n}^{(p,q,\upsilon)}\right]^2 = 2^{2n}(\breve{a}a^*)^2 M_2 + 2^n(\breve{a}a^*\breve{b}b^* + \breve{b}b^*\breve{a}a^*)$

Proof:

i)

$$\begin{aligned} \left[\widetilde{M}_{n}^{(p,q,\upsilon)}\right]^2 &= 2^{2n}(\breve{a}a^*)^2 - 2^n\breve{b}b^*\breve{a}a^* - 2^n\breve{a}a^*\breve{b}b^* + \left(\breve{b}b^*\right)^2 \\ \left[\widetilde{M}_{n+1}^{(p,q,\upsilon)}\right]^2 &= 2^{2n+2}(\breve{a}a^*)^2 - 2^{n+1}\breve{b}b^*\breve{a}a^* - 2^{n+1}\breve{a}a^*\breve{b}b^* + \left(\breve{b}b^*\right)^2 \\ \left[\widetilde{M}_{n+2}^{(p,q,\upsilon)}\right]^2 &= 2^{2n+4}(\breve{a}a^*)^2 - 2^{n+2}\breve{b}b^*\breve{a}a^* - 2^{n+2}\breve{a}a^*\breve{b}b^* + \left(\breve{b}b^*\right)^2 \end{aligned}$$

$\left[\widetilde{M}_{n}^{(p,q,\upsilon)}\right]^2 + \left[\widetilde{M}_{n+1}^{(p,q,\upsilon)}\right]^2 + \left[\widetilde{M}_{n+2}^{(p,q,\upsilon)}\right]^2$

$$\begin{aligned} &= 2^{2n}(\breve{a}a^*)^2(1 + 2^2 + 2^4) - 2^n\breve{b}b^*\breve{a}a^*(1 + 2 + 2^2) \\ &\quad -2^n\breve{a}a^*\breve{b}b^*(1 + 2 + 2^2) + 3\left(\breve{b}b^*\right)^2 \\ &= 21\left(2^{2n}\right)\left(\breve{a}a^*\right)^2 + 3\left(\breve{b}b^*\right)^2 - 7(2^n)(\breve{b}b^*\breve{a}a^* + \breve{a}a^*\breve{b}b^*) \end{aligned}$$

ii)

$$\left[\widetilde{ML}_n^{(p,q,\upsilon)}\right]^2 = 2^{2n}(\breve{a}a^*)^2 + 2^n\breve{b}b^*\breve{a}a^* + 2^n\breve{a}a^*\breve{b}b^* + \left(\breve{b}b^*\right)^2$$

$$\left[\widetilde{ML}_{n+1}^{(p,q,\upsilon)}\right]^2 = 2^{2n+2}(\breve{a}a^*)^2 + 2^{n+1}\breve{b}b^*\breve{a}a^* + 2^{n+1}\breve{a}a^*\breve{b}b^* + \left(\breve{b}b^*\right)^2$$

$$\begin{aligned}\left[\widetilde{ML}_{n+1}^{(p,q,\upsilon)}\right]^2 - \left[\widetilde{ML}_n^{(p,q,\upsilon)}\right]^2 &\\ &= 2^{2n}(\breve{a}a^*)^2(2^2-1) + 2^n\breve{b}b^*\breve{a}a^*(2-1) + 2^n\breve{a}a^*\breve{b}b^*(2-1) \\ &= 2^{2n}(\breve{a}a^*)^2 M_2 + 2^n(\breve{a}a^*\breve{b}b^* + \breve{b}b^*\breve{a}a^*)\end{aligned}$$

□

3. Unconstrained Mersenne and Mersenne-Lucas Hybrid Sedenions

Definition 3.1 *For any integers p, q, υ and non-negative integer n, unrestricted Mersenne and Mersenne-Lucas Hybrid Sedenions are defined by*

i) $\widehat{M_n}^{(p,q,\upsilon)} = \widehat{M_n} + i\hat{M}_{n+p} + \varepsilon\hat{M}_{n+q} + h\hat{M}_{n+\upsilon}$

ii) $\widehat{ML_n}^{(p,q,\upsilon)} = \widehat{ML_n} + i\widehat{ML}_{n+p} + \varepsilon\widehat{ML}_{n+q} + h\widehat{ML}_{n+\upsilon}$

where $i,\ \varepsilon,\ h$ are hybrid units.

Theorem 3.1 *Unrestricted Mersenne and Mersenne-Lucas Hybrid Sedenions satisfy the recurrence relations:*

i) $\widehat{M_n}^{(p,q,\upsilon)} = 3\hat{M}_{n-1}^{(p,q,\upsilon)} - 2\hat{M}_{n-2}^{(p,q,\upsilon)}$

ii) $\widehat{ML_n}^{(p,q,\upsilon)} = 3\widehat{ML}_{n-1}^{(p,q,\upsilon)} - 2\widehat{ML}_{n-2}^{(p,q,\upsilon)}$

Proof:
i) For $n = 2$, we have

$$\begin{aligned}\widehat{M_2}^{(p,q,\upsilon)} &= 3\widehat{M_1}^{(p,q,\upsilon)} - 2\widehat{M_0}^{(p,q,\upsilon)} \\ &= 3\left(\widehat{M_1} + i\widehat{M_{1+p}} + \varepsilon\widehat{M_{1+q}} + h\widehat{M_{1+\upsilon}}\right) \\ &\quad -2\left(\widehat{M_0} + i\widehat{M_{0+p}} + \varepsilon\widehat{M_{0+q}} + h\widehat{M_{0+\upsilon}}\right) \\ &= \widehat{M_2} + i\widehat{M_{2+p}} + \varepsilon\widehat{M_{2+q}} + h\widehat{M_{2+\upsilon}}\end{aligned}$$

By using the definition 3.1 (i), we have

$$\begin{aligned}\widehat{M_n}^{(p,q,\upsilon)} &= \widehat{M_n} + i\widehat{M_{n+p}} + \varepsilon\widehat{M_{n+q}} + h\widehat{M_{n+\upsilon}} \\ &= 3\widehat{M_{n-1}} - 2\widehat{M_{n-2}} + i\left(3\widehat{M_{n+p-1}} - 2\widehat{M_{n+p-2}}\right) \\ &\quad +\varepsilon(3\widehat{M_{n+q-1}} - 2\widehat{M_{n+q-2}}) + h\left(3\widehat{M_{n+\upsilon-1}} - 2\widehat{M_{n+\upsilon-2}}\right) \\ &= 3\left(\widehat{M_{n-1}} + i\widehat{M_{n+p-1}} + \varepsilon\widehat{M_{n+q-1}} + h\widehat{M_{n+\upsilon-1}}\right) \\ &\quad -2\left(\widehat{M_{n-2}} + i\widehat{M_{n+p-2}} + \varepsilon\widehat{M_{n+q-2}} + h\widehat{M_{n+\upsilon-2}}\right) \\ \widehat{M_n}^{(p,q,\upsilon)} &= 3\widehat{M_{n-1}}^{(p,q,\upsilon)} - 2\widehat{M_{n-2}}^{(p,q,\upsilon)}\end{aligned}$$

ii) For $n = 2$, we have

$$\begin{aligned}
\widehat{ML_2}^{(p,q,\upsilon)} &= 3\widehat{ML_1}^{(p,q,\upsilon)} - 2\widehat{ML_0}^{(p,q,\upsilon)} \\
&= 3\left(\widehat{ML_1} + i\widehat{ML_{1+p}} + \varepsilon\widehat{ML_{1+q}} + h\widehat{ML_{1+\upsilon}}\right) \\
&\quad -2\left(\widehat{ML_0} + i\widehat{ML_{0+p}} + \varepsilon\widehat{ML_{0+q}} + h\widehat{ML_{0+\upsilon}}\right) \\
&= \widehat{ML_2} + i\widehat{ML_{2+p}} + \varepsilon\widehat{ML_{2+q}} + h\widehat{ML_{2+\upsilon}}
\end{aligned}$$

By using the definition 3.1 (ii), we have

$$\begin{aligned}
\widehat{ML_n}^{(p,q,\upsilon)} &= \widehat{ML_n} + i\widehat{ML_{n+p}} + \varepsilon\widehat{ML_{n+q}} + h\widehat{ML_{n+\upsilon}} \\
&= 3\widehat{ML_{n-1}} - 2\widehat{ML_{n-2}} + i\left(3\widehat{ML_{n+p-1}} - 2\widehat{ML_{n+p-2}}\right) \\
&\quad +\varepsilon(3\widehat{ML_{n+q-1}} - 2\widehat{ML_{n+q-2}}) + h\left(3\widehat{ML_{n+\upsilon-1}} - 2\widehat{ML_{n+\upsilon-2}}\right) \\
&= 3\left(\widehat{ML_{n-1}} + i\widehat{ML_{n+p-1}} + \varepsilon\widehat{ML_{n+q-1}} + h\widehat{ML_{n+\upsilon-1}}\right) \\
&\quad -2\left(\widehat{ML_{n-2}} + i\widehat{ML_{n+p-2}} + \varepsilon\widehat{ML_{n+q-2}} + h\widehat{ML_{n+\upsilon-2}}\right) \\
\widehat{ML_n}^{(p,q,\upsilon)} &= 3\widehat{ML_{n-1}}^{(p,q,\upsilon)} - 2\widehat{ML_{n-2}}^{(p,q,\upsilon)}
\end{aligned}$$

□

Theorem 3.2 *The Binet formula for* $\widehat{M_n}^{(p,q,\upsilon)}$ *and* $\widehat{ML_n}^{(p,q,\upsilon)}$ *are given by*

i) $\widehat{M_n}^{(p,q,\upsilon)} = 2^n\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*$

ii) $\widehat{ML_n}^{(p,q,\upsilon)} = 2^n\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^*$

where $\mathbb{a}^* = 1 + 2^p i + 2^q\varepsilon + 2^\upsilon h$, $\mathbb{b}^* = 1 + i + \varepsilon + h$ *and*

$$\hat{\mathbb{a}} = \sum_{s=0}^{15} 2^s e_s, \quad \hat{\mathbb{b}} = \sum_{s=0}^{15} e_s$$

Proof:

i)

$$\begin{aligned}
\widehat{M_n}^{(p,q,\upsilon)} &= \widehat{M_n} + i\widehat{M_{n+p}} + \varepsilon\widehat{M_{n+q}} + h\widehat{M_{n+\upsilon}} \\
&= 2^n\hat{\mathbb{a}} - \hat{\mathbb{b}} + \left(2^{n+p}\hat{\mathbb{a}} - \hat{\mathbb{b}}\right)i + \left(2^{n+q}\hat{\mathbb{a}} - \hat{\mathbb{b}}\right)\varepsilon + \left(2^{n+\upsilon}\hat{\mathbb{a}} - \hat{\mathbb{b}}\right)h \\
&= 2^n\hat{\mathbb{a}}\left(1 + 2^p i + 2^q\varepsilon + 2^\upsilon h\right) - \hat{\mathbb{b}}\left(1 + i + \varepsilon + h\right) \\
\widehat{M_n}^{(p,q,\upsilon)} &= 2^n\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*
\end{aligned}$$

where $\mathbb{a}^* = 1 + 2^p i + 2^q\varepsilon + 2^\upsilon h$ and $\mathbb{b}^* = 1 + i + \varepsilon + h$

ii)

$$\begin{aligned}
\widehat{ML_n}^{(p,q,\upsilon)} &= \widehat{ML_n} + \widehat{ML_{n+p}}i + \widehat{ML_{n+q}}\varepsilon + \widehat{ML_{n+\upsilon}}h \\
&= 2^n\hat{\mathbb{a}} + \hat{\mathbb{b}} + \left(2^{n+p}\hat{\mathbb{a}} + \hat{\mathbb{b}}\right)i + \left(2^{n+q}\hat{\mathbb{a}} + \hat{\mathbb{b}}\right)\varepsilon + \left(2^{n+\upsilon}\hat{\mathbb{a}} + \hat{\mathbb{b}}\right)h \\
&= 2^n\hat{\mathbb{a}}\left(1 + 2^p i + 2^q\varepsilon + 2^\upsilon h\right) + \hat{\mathbb{b}}\left(1 + i + \varepsilon + h\right) \\
\widehat{ML_n}^{(p,q,\upsilon)} &= 2^n\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^*
\end{aligned}$$

where $\mathbb{a}^* = 1 + 2^p i + 2^q\varepsilon + 2^\upsilon h$ and $\mathbb{b}^* = 1 + i + \varepsilon + h$.

$\square$

Theorem 3.3 *The generating functions for* $\widehat{M_n}^{(p,q,\upsilon)}$ *and* $\widehat{ML_n}^{(p,q,\upsilon)}$ *are given by*

i) $\hat{f}(x) = \frac{\hat{M}_0^{(p,q,\upsilon)}+\left(\hat{M}_1^{(p,q,\upsilon)}-3\hat{M}_0^{(p,q,\upsilon)}\right)x}{1-3x+2x^2}$

ii) $\hat{g}(x) = \frac{\widehat{ML}_0^{(p,q,\upsilon)}+\left(\widehat{ML}_1^{(p,q,\upsilon)}-3\widehat{ML}_0^{(p,q,\upsilon)}\right)x}{1-3x+2x^2}$

Proof:
i) Let us define

$$\begin{aligned}\hat{f}(x) &= \sum_{n=0}^{\infty}\hat{M}_n^{(p,q,\upsilon)}x^n\\ \hat{f}(x) &= \hat{M}_0^{(p,q,\upsilon)}+x\hat{M}_1^{(p,q,\upsilon)}+\sum_{n=2}^{\infty}\hat{M}_n^{(p,q,\upsilon)}x^n\end{aligned}$$

Multiplying this equation by $-3x$ and $2x^2$, we obtain

$$\begin{aligned}-3\hat{f}(x)x &= -3x\hat{M}_0^{(p,q,\upsilon)}-3\sum_{n=2}^{\infty}\hat{M}_{n-1}^{(p,q,\upsilon)}x^n\\ 2\hat{f}(x)x^2 &= 2\sum_{n=2}^{\infty}\hat{M}_{n-2}^{(p,q,\upsilon)}x^n\end{aligned}$$

Adding the above three equations, we get

$$\begin{aligned}\left(1-3x+2x^2\right)\hat{f}(x) &= \hat{M}_0^{(p,q,\upsilon)}+x\left(\hat{M}_1^{(p,q,\upsilon)}-3\hat{M}_0^{(p,q,\upsilon)}\right)\\ &\quad+\sum_{n=2}^{\infty}\left[\hat{M}_n^{(p,q,\upsilon)}-3\hat{M}_{n-1}^{(p,q,\upsilon)}+2\hat{M}_{n-2}^{(p,q,\upsilon)}\right]x^n\\ \hat{f}(x) &= \frac{\hat{M}_0^{(p,q,\upsilon)}+\left(\hat{M}_1^{(p,q,\upsilon)}-3\hat{M}_0^{(p,q,\upsilon)}\right)x}{1-3x+2x^2}.\end{aligned}$$

ii) In a similar way, we obtain $\hat{g}(x) = \frac{\widehat{ML}_0^{(p,q,\upsilon)}+\left(\widehat{ML}_1^{(p,q,\upsilon)}-3\widehat{ML}_0^{(p,q,\upsilon)}\right)x}{1-3x+2x^2}$. $\square$

Theorem 3.4

i) The exponential generating function for $\widehat{M_n}^{(p,q,\upsilon)}$ *is given by*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\hat{M}_n^{(p,q,\upsilon)}l^k}{k!} &= \sum_{k=0}^{\infty}(2^k\hat{\mathbb{a}}\mathbb{a}^*-\hat{\mathbb{b}}\mathbb{b}^*)\frac{l^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{(2l)^k}{k!}\hat{\mathbb{a}}\mathbb{a}^*-\sum_{k=0}^{\infty}\frac{l^k}{k!}\hat{\mathbb{b}}\mathbb{b}^*\\ \sum_{k=0}^{\infty}\frac{\hat{M}_n^{(p,q,\upsilon)}l^k}{k!} &= \hat{\mathbb{a}}\mathbb{a}^*e^{2l}-\hat{\mathbb{b}}\mathbb{b}^*e^l\end{aligned}$$

ii) The exponential generating function for $\widehat{ML}_n^{(p,q,\upsilon\ )}$ *is given by*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\widehat{ML}_n^{(p,q,\upsilon)}l^k}{k!} &= \sum_{k=0}^{\infty}(2^k\hat{a}a^*+\hat{\mathbb{b}}\mathbb{b}^*)\frac{l^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{(2l)^k}{k!}\hat{a}a^*+\sum_{k=0}^{\infty}\frac{l^k}{k!}\hat{\mathbb{b}}\mathbb{b}^*\\ \sum_{k=0}^{\infty}\frac{\widehat{ML}_n^{(p,q,\upsilon)}l^k}{k!} &= \hat{a}a^*e^{2l}+\hat{\mathbb{b}}\mathbb{b}^*e^l\end{aligned}$$

Theorem 3.5 (Vajda Identity) *For any integers* m, n, k, p, q, υ, *we have*

i) $\hat{M}_{m+n}^{(p,q,\upsilon)}\hat{M}_{m+k}^{(p,q,\upsilon)}-\hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{m+n+k}^{(p,q,\upsilon)}=2^mM_n\left(2^k\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*-\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*\right)$

ii) $\widehat{ML}_{m+n}^{(p,q,\upsilon)}\widehat{ML}_{m+k}^{(p,q,\upsilon)}-\widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{m+n+k}^{(p,q,\upsilon)}=2^mM_n\left(\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*-2^k\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*\right)$

Proof:

i) $\hat{M}_{m+n}^{(p,q,\upsilon)}\hat{M}_{m+k}^{(p,q,\upsilon)}-\hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{m+n+k}^{(p,q,\upsilon)}$

$$\begin{aligned}&= \left(2^{m+n}\hat{a}a^*-\hat{\mathbb{b}}\mathbb{b}^*\right)\left(2^{m+k}\hat{a}a^*-\hat{\mathbb{b}}\mathbb{b}^*\right)\\ &\quad -(2^m\hat{a}a^*-\hat{\mathbb{b}}\mathbb{b}^*)(2^{m+n+k}\hat{a}a^*-\hat{\mathbb{b}}\mathbb{b}^*)\\ &= 2^{m+k}\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*\left(2^n-1\right)-2^m\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*(2^n-1)\\ &= 2^mM_n\left(2^k\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*-\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*\right)\end{aligned}$$

ii) $\widehat{ML}_{m+n}^{(p,q,\upsilon)}\widehat{ML}_{m+k}^{(p,q,\upsilon)}-\widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{m+n+k}^{(p,q,\upsilon)}$

$$\begin{aligned}&= \left(2^{m+n}\hat{a}a^*+\hat{\mathbb{b}}\mathbb{b}^*\right)\left(2^{m+k}\hat{a}a^*+\hat{\mathbb{b}}\mathbb{b}^*\right)\\ &\quad -(2^m\hat{a}a^*+\hat{\mathbb{b}}\mathbb{b}^*)(2^{m+n+k}\hat{a}a^*+\hat{\mathbb{b}}\mathbb{b}^*\\ &= 2^{m+k}\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*\left(1-2^n\right)+2^m\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*(2^n-1)\\ &= 2^mM_n\left(\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*-2^k\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*\right)\end{aligned}$$

If we substitute $k\to -n$ in the Vajda identities, we obtain the Catalan's identities. □

Theorem 3.6 (Catalan's identity) *For any integers* m, n, p, q, υ, *we have*

i) $\hat{M}_{m+n}^{(p,q,\upsilon)}\hat{M}_{m-n}^{(p,q,\upsilon)}-\left[\hat{M}_{m}^{(p,q,\upsilon)}\right]^2=2^{m-n}M_n\left(\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*-2^n\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*\right)$

ii) $\widehat{ML}_{m+n}^{(p,q,\upsilon)}\widehat{ML}_{m-n}^{(p,q,\upsilon)}-\left[\widehat{ML}_{m}^{(p,q,\upsilon)}\right]^2=2^{m-n}M_n\left(2^n\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*-\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*\right)$

If we substitute $n\to 1$ *in Catalan's identities, we get Cassini's identities.*

Theorem 3.7 (Cassini's identity) *For any integers* m, p, q, υ, *we have*

i) $\hat{M}_{m+1}^{(p,q,\upsilon)}\hat{M}_{m-1}^{(p,q,\upsilon)}-\left[\hat{M}_{m}^{(p,q,\upsilon)}\right]^2=2^{m-1}\left(\hat{\mathbb{b}}\mathbb{b}^*\hat{a}a^*-2\hat{a}a^*\hat{\mathbb{b}}\mathbb{b}^*\right)$

ii) $\widehat{ML}_{m+1}^{(p,q,\upsilon)}\widehat{ML}_{m-1}^{(p,q,\upsilon)} - \left[\widehat{ML}_{m}^{(p,q,\upsilon)}\right]^2 = 2^{m-1}\left(2\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* - \hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*\right)$

Theorem 3.8 (d'Ocagne's identity) *For any integers m, n, p, q, υ, we have*

i) $\hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{n+1}^{(p,q,\upsilon)} - \hat{M}_{m+1}^{(p,q,\upsilon)}\hat{M}_{n}^{(p,q,\upsilon)} = 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* - 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*$

ii) $\widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{n+1}^{(p,q,\upsilon)} - \widehat{ML}_{m+1}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)} = 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* - 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*$

Proof:

i) $\hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{n+1}^{(p,q,\upsilon)} - \hat{M}_{m+1}^{(p,q,\upsilon)}\hat{M}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^m\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*\right)\left(2^{n+1}\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*\right) \\ &\quad -(2^{m+1}\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*)(2^n\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*) \\ &= 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*(1-2) - 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*(1-2) \\ &= 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* - 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* \end{aligned}$$

ii) $\widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{n+1}^{(p,q,\upsilon)} - \widehat{ML}_{m+1}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^m\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^*\right)\left(2^{n+1}\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^*\right) \\ &\quad -(2^{m+1}\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^*)(2^n\hat{\mathbb{a}}\mathbb{a}^* + \hat{\mathbb{b}}\mathbb{b}^* \\ &= 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*(2-1) - 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*(2-1) \\ &= 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* - 2^m\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* \end{aligned}$$

□

Theorem 3.9 (Honsberger Identity) *For any integers m, n, p, q, υ, we have*

i) $\hat{M}_{m-1}^{(p,q,\upsilon)}\hat{M}_{n}^{(p,q,\upsilon)} + \hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{n+1}^{(p,q,\upsilon)}$

$$= 2^{m+n-1}(\hat{\mathbb{a}}\mathbb{a}^*)^2 ML_2 - 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* ML_1 - 2^{m-1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* ML_1 + 2\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2$$

ii) $\widehat{ML}_{m-1}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)} + \widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{n+1}^{(p,q,\upsilon)}$

$$= 2^{m+n-1}(\hat{\mathbb{a}}\mathbb{a}^*)^2 ML_2 + 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* ML_1 + 2^{m-1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* ML_1 + 2\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2$$

Proof:

i) $\hat{M}_{m-1}^{(p,q,\upsilon)}\hat{M}_{n}^{(p,q,\upsilon)} + \hat{M}_{m}^{(p,q,\upsilon)}\hat{M}_{n+1}^{(p,q,\upsilon)}$

$$\begin{aligned} &= \left(2^{m-1}\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*\right)\left(2^n\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*\right) + (2^m\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^*)(2^{n+1}\hat{\mathbb{a}}\mathbb{a}^* - \hat{\mathbb{b}}\mathbb{b}^* \\ &= 2^{m+n-1}(\hat{\mathbb{a}}\mathbb{a}^*)^2\left(2^2+1\right) - 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*(2+1) - 2^{m-1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*(2+1) + 2\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2 \\ &= 2^{m+n-1}(\hat{\mathbb{a}}\mathbb{a}^*)^2 ML_2 - 2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^* ML_1 - 2^{m-1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^* ML_1 + 2\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2 \end{aligned}$$

ii) $\widehat{ML}_{m-1}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)}+\widehat{ML}_{m}^{(p,q,\upsilon)}\widehat{ML}_{n+1}^{(p,q,\upsilon)}$

$$\begin{aligned}
&= \left(2^{m-1}\hat{a}a^* + \hat{\mathbb{b}}b^*\right)\left(2^{n}\hat{a}a^* + \hat{\mathbb{b}}b^*\right) + (2^{m}\hat{a}a^* + \hat{\mathbb{b}}b^*)(2^{n+1}\hat{a}a^* + \hat{\mathbb{b}}b^*\\
&= 2^{m+n-1}(\hat{a}a^*)^2\left(2^2+1\right) + 2^{n}\hat{\mathbb{b}}b^*\hat{a}a^*\left(2+1\right) + 2^{m-1}\hat{a}a^*\hat{\mathbb{b}}b^*\left(2+1\right) + 2\left(\hat{\mathbb{b}}b^*\right)^2\\
&= 2^{m+n-1}(\hat{a}a^*)^2 ML_2 + 2^{n}\hat{\mathbb{b}}b^*\hat{a}a^* ML_1 + 2^{m-1}\hat{a}a^*\hat{\mathbb{b}}b^* ML_1 + 2\left(\hat{\mathbb{b}}b^*\right)^2
\end{aligned}$$

□

Theorem 3.10 *For any integers n, p, q, υ, we have*

i) $\hat{M}_{n}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)} = 2^{2n}(\hat{a}a^*)^2 + 2^{n}\left(\hat{a}a^*\hat{\mathbb{b}}b^* - \hat{\mathbb{b}}b^*\hat{a}a^*\right) - \left(\hat{\mathbb{b}}b^*\right)^2$

ii) $\hat{M}_{n}^{(p,q,\upsilon)} + \widehat{ML}_{n}^{(p,q,\upsilon)} = 2^{n+1}\hat{a}a^*$

iii) $\hat{M}_{n}^{(p,q,\upsilon)} - \widehat{ML}_{n}^{(p,q,\upsilon)} = -2\hat{\mathbb{b}}b^*$

Proof:

i)

$$\begin{aligned}
\hat{M}_{n}^{(p,q,\upsilon)}\widehat{ML}_{n}^{(p,q,\upsilon)} &= \left(2^{n}\hat{a}a^* - \hat{\mathbb{b}}b^*\right)\left(2^{n}\hat{a}a^* + \hat{\mathbb{b}}b^*\right)\\
&= 2^{2n}(\hat{a}a^*)^2 + 2^{n}\left(\hat{a}a^*\hat{\mathbb{b}}b^* - \hat{\mathbb{b}}b^*\hat{a}a^*\right) - \left(\hat{\mathbb{b}}b^*\right)^2
\end{aligned}$$

ii)

$$\begin{aligned}
\hat{M}_{n}^{(p,q,\upsilon)} + \widehat{ML}_{n}^{(p,q,\upsilon)} &= \left(2^{n}\hat{a}a^* - \hat{\mathbb{b}}b^*\right) + \left(2^{n}\hat{a}a^* + \hat{\mathbb{b}}b^*\right)\\
&= 2^{n+1}\hat{a}a^*
\end{aligned}$$

iii)

$$\begin{aligned}
\hat{M}_{n}^{(p,q,\upsilon)} - \widehat{ML}_{n}^{(p,q,\upsilon)} &= \left(2^{n}\hat{a}a^* - \hat{\mathbb{b}}b^*\right) - \left(2^{n}\hat{a}a^* + \hat{\mathbb{b}}b^*\right)\\
&= -2\hat{\mathbb{b}}b^*
\end{aligned}$$

□

Theorem 3.11 *For any integers n, p, q, υ, we have*

i) $\hat{M}_{n+1}^{(p,q,\upsilon)} + \hat{M}_{n}^{(p,q,\upsilon)} = 3\left(2^{n}\right)\hat{a}a^* - 2\hat{\mathbb{b}}b^*$

ii) $\widehat{ML}_{n+1}^{(p,q,\upsilon)} + \widehat{ML}_{n}^{(p,q,\upsilon)} = 3\left(2^{n}\right)\hat{a}a^* + 2\hat{\mathbb{b}}b^*$

Proof:

i)

$$\begin{aligned}
\hat{M}_{n+1}^{(p,q,\upsilon)} + \hat{M}_{n}^{(p,q,\upsilon)} &= \left(2^{n+1}\hat{a}a^* - \hat{\mathbb{b}}b^*\right) + \left(2^{n}\hat{a}a^* - \hat{\mathbb{b}}b^*\right)\\
&= 3\left(2^{n}\right)\hat{a}a^* - 2\hat{\mathbb{b}}b^*
\end{aligned}$$

ii)

$$\begin{aligned}\widehat{ML}_{n+1}^{(p,q,\upsilon)}+\widehat{ML}_{n}^{(p,q,\upsilon)} &= \left(2^{n+1}\hat{\mathbb{a}}\mathbb{a}^*+\hat{\mathbb{b}}\mathbb{b}^*\right)+\left(2^{n}\hat{\mathbb{a}}\mathbb{a}^*+\hat{\mathbb{b}}\mathbb{b}^*\right)\\ &= 3\left(2^{n}\right)\hat{\mathbb{a}}\mathbb{a}^*+2\hat{\mathbb{b}}\mathbb{b}^*\end{aligned}$$

□

Theorem 3.12 *For any integers n,p,q,υ, we have*

i) $\left[\hat{M}_{n}^{(p,q,\upsilon)}\right]^2+\left[\hat{M}_{n+1}^{(p,q,\upsilon)}\right]^2+\left[\hat{M}_{n+2}^{(p,q,\upsilon)}\right]^2$

$$=21\left(2^{2n}\right)\left(\hat{\mathbb{a}}\mathbb{a}^*\right)^2+3\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2-7(2^n)(\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*+\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*)$$

ii) $\left[\widehat{ML}_{n+1}^{(p,q,\upsilon)}\right]^2-\left[\widehat{ML}_{n}^{(p,q,\upsilon)}\right]^2=2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2M_2+2^n(\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*)$

Proof:

i)

$$\begin{aligned}\left[\hat{M}_{n}^{(p,q,\upsilon)}\right]^2 &= 2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2-2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*-2^n\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\\ \left[\hat{M}_{n+1}^{(p,q,\upsilon)}\right]^2 &= 2^{2n+2}(\hat{\mathbb{a}}\mathbb{a}^*)^2-2^{n+1}\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*-2^{n+1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\\ \left[\hat{M}_{n+2}^{(p,q,\upsilon)}\right]^2 &= 2^{2n+4}(\hat{\mathbb{a}}\mathbb{a}^*)^2-2^{n+2}\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*-2^{n+2}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\end{aligned}$$

$$\begin{aligned}&\left[\hat{M}_{n}^{(p,q,\upsilon)}\right]^2+\left[\hat{M}_{n+1}^{(p,q,\upsilon)}\right]^2+\left[\hat{M}_{n+2}^{(p,q,\upsilon)}\right]^2\\ &= 2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2(1+2^2+2^4)-2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*(1+2+2^2)\\ &\quad -2^n\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*(1+2+2^2)+3\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\\ &= 21\left(2^{2n}\right)\left(\hat{\mathbb{a}}\mathbb{a}^*\right)^2+3\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2-7(2^n)(\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*+\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*)\end{aligned}$$

ii)

$$\begin{aligned}\left[\widehat{ML}_{n}^{(p,q,\upsilon)}\right]^2 &= 2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2+2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*+2^n\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\\ \left[\widehat{ML}_{n+1}^{(p,q,\upsilon)}\right]^2 &= 2^{2n+2}(\hat{\mathbb{a}}\mathbb{a}^*)^2+2^{n+1}\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*+2^{n+1}\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\left(\hat{\mathbb{b}}\mathbb{b}^*\right)^2\end{aligned}$$

$$\begin{aligned}&\left[\widehat{ML}_{n+1}^{(p,q,\upsilon)}\right]^2-\left[\widehat{ML}_{n}^{(p,q,\upsilon)}\right]^2\\ &= 2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2(2^2-1)+2^n\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*(2-1)+2^n\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*(2-1)\\ &= 2^{2n}(\hat{\mathbb{a}}\mathbb{a}^*)^2M_2+2^n(\hat{\mathbb{a}}\mathbb{a}^*\hat{\mathbb{b}}\mathbb{b}^*+\hat{\mathbb{b}}\mathbb{b}^*\hat{\mathbb{a}}\mathbb{a}^*)\end{aligned}$$

□

4. Conclusion

An infinite number of terms of the unrestricted Mersenne and Mersenne-Lucas hybrid Octonions and Sedenions sequences are found. Also verified the above sequences through some well known identities. One can generate various number patterns whose characteristics satisfying unrestricted hybrid sequences.

References

1. Malini Devi, B. and Devibala, S., *A View on the Mersenne and Mersenne-Lucas Numbers*, GIS Science Journal 8(8), 784-789, (2021).
2. Malini Devi, B. and Devibala, S., *Some Sums Formulae for Products of terms of Mersenne and Mersenne-Lucas Numbers*, Adalya Journal 10(8), 74-79, (2021).
3. Ozdemir, M., *Introduction to Hybrid Numbers*, Adv. Appl. Clifford Algebras 28(11), (2018).
4. Kuruza, F. and Dagdevirenb, A., *Pell and Pell-Lucas hybrid quaternions*, Filomat 37(25), 8425-8434, (2023).
5. Kumari, M., Prasad, K. and Mahato, H., *On k-Mersenne and k-Mersenne-Lucas Octonions*, arXiv preprint arXiv:2207.12243, (2022).
6. Irmak, N. and Acıkel, A., *More Identities for Fibonacci and Lucas Octonions*, Mathematical Sciences and Applications E-Notes 8(2), 48-53, (2020).
7. Malini Devi, B. and Devibala, S., *On Mersenne And Mersenne-Lucas Sedenions*, Advances and Applications in Mathematical Sciences 21(1), 383-392, (2021).
8. Ozimamoglu, H., *On Leonardo Sedenions*, Afrika Matematika 34(26), (2023).
9. Catarino, P. and Srivastava, H. M., *k-Pell, K-Pell-Lucas and modified k-Pell Sedenions*, Asian-European Journal of Mathematics 12(2), (2019).

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