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Blow Up and Growth of Solutions to a Viscoelastic Parabolic Type Kirchhoff Equation with Variable-Exponents in the Damping and Source Terms

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ABSTRACT: In this paper, we consider a Kirchhoff-type viscoelastic equation with nonlinear damping and source terms involving variable exponents. We establish results on the blow-up and exponential growth of solutions corresponding to negative initial energy.

Key Words: Stability, blow up, degenerate damping, exponential growth, Kirchhoff-type equation, viscoelastic equation, variable exponents.

Contents

1. Introduction

In this work, we consider the following system of a parabolic-type Kirchhoff equation with variable exponents:

$$\begin{cases} u_t - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g_1(t-s)\Delta u\,(s)\,ds + |u|^{q(x)-2}\,u_t = f_1\,(x,u,v)\,, \ (x,t)\in\Omega\times(0,\infty), \\ v_t - M\left(\|\nabla v\|^2\right)\Delta v + \int_0^t g_2(t-s)\Delta v\,(s)\,ds + |v|^{q(x)-2}\,v_t = f_2\,(x,u,v)\,, \ (x,t)\in\Omega\times(0,\infty), \\ u(x,t)=v(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), \quad x\in\Omega. \end{cases} \tag{1.1}$$

where Ω be a bounded and regular domain of $\mathbb{R}^n$, $n\geq 1$, with a smooth boundary $\partial\Omega$. The relaxation functions g_i: $\mathbb{R}^+\to\mathbb{R}^+$ $(i=1,2)$ satisfy assumptions to be specified later. The Kirchhoff function is defined as

$$M(s) = 1+s^\gamma, \quad \gamma>0.$$

The source terms $f_1\,(.,.,.)\,,\ f_2\,(.,.,.) : \Omega\times\mathbb{R}^2\to L^1((0,T),H_0^1(\Omega))$ are given by

$$\begin{aligned} f_1(x,u,v) &= a\,|u+v|^{2(p(x)+1)}\,(u+v) + bu\,|u|^{p(x)}\,|v|^{p(x)+2}\,, \\ f_2(x,u,v) &= a\,|u+v|^{2(p(x)+1)}\,(u+v) + bv\,|v|^{p(x)}\,|u|^{p(x)+2}\,. \end{aligned} \tag{1.2}$$

$u,v\in L^1((0,T),H_0^1(\Omega))$.
where a, $b>0$ are constants and $p(.)$, $q(.):\Omega\to[1,\infty)$ are continuous variable exponent functions on $\overline{\Omega}$. Assume that the exponent $p(\cdot)$ satisfies the log-Hölderer continuity condition

$$|p\,(x)-p\,(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)}, \text{ for all } x,y\in\Omega \quad \text{with} \quad |x-y|<\delta. \tag{1.3}$$

where $A>0$ and $0<\delta<1$.

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From the definitions of $f_1(u,v)$ and $f_2(u,v)$, we have:

$$uf_1(x,u,v)+vf_2(x,u,v)=2(p(x)+2)F(x,u,v), \quad \forall(u,v)\in\mathbb{R}^2, \tag{1.4}$$

where the potential function $F(u,v)$ is given by:

$$F(x,u,v)=\frac{1}{2(p(x)+2)}\left[a|u+v|^{2(p(x)+2)}+2b|uv|^{p(x)+2}\right],$$

and

$$\frac{\partial F}{\partial u}(x,u,v)=f_1(x,u,v), \frac{\partial F}{\partial v}(x,u,v)=f_2(x,u,v).$$

We assume the following bounds for the variable exponents

$$\begin{cases} 2\le q^-\le q(x)\le q^+\le q^*, \\ 2\le p^-\le p(x)\le p^+\le p^*, \end{cases} \tag{1.5}$$

where

$$\begin{cases} q^-=\text{ess}\inf_{x\in\Omega} q(x), \quad q^+=\text{ess}\sup_{x\in\Omega} m(x), \\ p^-=\text{ess}\inf_{x\in\Omega} p(x), \quad p_2=\text{ess}\sup_{x\in\Omega} p(x), \end{cases}$$

and we assume

$$\begin{cases} 2<q^*,p^*<\infty \text{ if } n\le 2, \\ 2<q^*,p^*<\frac{2n}{n-2} \text{ if } n>2. \end{cases}$$

Then the embedding $H_0^1(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Before discussing our problem, we present some earlier work related to wave equations in the case of constant-exponent nonlinearity. For instance, Erhan Pişkina, Fatma Ekinci [20] studied a system of viscoelastic parabolic type Kirchhoff equation with multiple nonlinearities This article deals with the following initial value problem:

$$\begin{cases} u_t-M\left(\|\nabla u\|^2\right)\Delta u+\int_0^t \omega_1(t-s)\Delta u(s)\,ds+|u|^{q-2}u_t=f_1(u,v), \; x\in\Omega, \; t>0, \\ v_t-M\left(\|\nabla v\|^2\right)\Delta v+\int_0^t \omega_2(t-s)\Delta v(s)\,ds+|v|^{q-2}v_t=f_2(u,v), \; x\in\Omega, \; t>0, \\ u(x,t)=v(x,t)=0, \quad x\in\partial\Omega, \; t\ge 0, \\ u(x,0)=u_0(x), \; v(x,0)=v_0(x), \quad x\in\Omega \end{cases} \tag{1.6}$$

The system consists of two viscoelastic parabolic-type Kirchhoff equations with memory terms. $M\left(\|\nabla u\|^2\right)$ and $M\left(\|\nabla v\|^2\right)$ represent Kirchhoff-type coefficients depending on the gradient norms, $\omega_1$and ω_2 are memory kernels.The terms $|u|^{q-2}u_t$ and $|v|^{q-2}v_t$, represent nonlinear damping. $f_1(u,v)$ and $f_2(u,v)$ are nonlinear source terms depending on both u and v.

The system is subjected to homogeneous Dirichlet boundary conditions and given initial data in the domain Ω.

Partial differential equations with nonlinear terms and variable exponents have received significant attention in recent years, as they provide more realistic models for complex physical phenomena. For example, Wu et al [29] established the blow-up of solutions with positive initial energy for the equation

$$u_t-Du=u^{p(x)}. \tag{1.7}$$

These results were later extended by other authors (see [2,28]). Qu et al [23] investigated the fourth-order equation

$$u_t+D^2u=u^{p(x)}. \tag{1.8}$$

Analyzing the asymptotic behavior of its solutions. In the absence of the fourth-order term, the equation takes the form

$$u_t-M\left(\|\nabla u\|^2\right)Du+|u|^{m(x)-2}u_t=|u|^{r(x)-2}u. \tag{1.9}$$

which was studied by Khaldi et al [8], where they addressed the global existence and stability of solutions. More broadly, numerous recent studies have addressed local and global existence, blow-up, and stability results for problems involving variable exponents (see [1,25]). These works motivate our current study, in which we aim to deepen the understanding of such systems by analyzing a new viscoelastic Kirchhoff-type model with variable coefficients and complex nonlinearities.

Finally, in [17], Pişkin, Erhan, and Gülistan Butakın studied the following parabolic-type Kirchhoff equation with variable exponents

$$\begin{cases} \left(1+|u|^{q(x)-2}\right) u_t + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u = |u|^{q(x)-2}u, \text{ in } (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, \quad \text{on } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad \text{in } x \in \Omega. \end{cases} \tag{1.10}$$

This paper is organized as follows: In Section 2, we recall the definition of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$, as well as some of their properties. We also give some hypotheses and some necessary preliminaries. In Section 3, we prove the blow up of solution with negative initial energy. In Section 4, we prove the exponential growth of solution with negative initial energy.

2. Preliminaries and assumptions

Let $q : \Omega \to [1, \infty)$ be a continuous function. We define the Lebesgue space with a variable exponent by:

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable in } \Omega : \rho_{p(\cdot)}(\lambda u) < +\infty, \text{ for some } \lambda > 0\right\}$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg-type norm

$$\|u\|_{p(.)} = \|u\|_{L^{p(\cdot)}(\Omega)} = inf\left\{\lambda > 0 : \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space.
We also defne the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}.$$

For the relaxation function g we assume the following.

Lemma 2.1 *(Hölder's inequality) [9] Let Ω be a bounded domain in $\mathbb{R}^n$*
(i) The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv dx\right| \leq 2\|u\|_{p(x)}\|v\|_{q(x)}.$$

(ii) If $p, q \in C^+(\overline{\Omega})$ with $q(x) \leq p(x)$ for any $x \in \overline{\Omega}$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

Lemma 2.2 *(Young's inequality) [9] Let p, q and s be measurable functions defined on Ω such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)}, \quad for \quad a.e. \ x \in \Omega.$$

Then for all $a, b \geq 0$, we have

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \leq \frac{(a)^{p(\cdot)}}{p(\cdot)} + \frac{(b)^{q(\cdot)}}{q(\cdot)}.$$

For $p = q = 2$, $a > 0$, $b > 0$, we have

$$ab \leq \delta a^2 + \frac{1}{4\delta}b^2.$$

We assume the following hypotheses:
(H_1) The relaxation functions $g_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$ are a differentiable and decreasing functions such that

$$g_i(0) > 0, \quad g_i'(s) < 0, \quad 1 - \int_0^\infty g_i(s)ds = l_i > 0, \quad i = 1, 2. \tag{2.1}$$

(H_2) Assume there exist two positive non-increasing differentiable functions ξ_1 and ξ_2 such that

$$g_i'(t) < -\xi_i(t)g_i(t) \quad t \geq 0, \qquad \int_0^\infty \xi_i(s)ds = \infty, \ i = 1, 2.$$

By using the direct calculations, we have

$$\begin{aligned}\int_0^t g(t-s)\left(\nabla u_t\left(t\right), \nabla u(s)\right) ds &= -\frac{1}{2}g(t)\left\|u\left(t\right)\right\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) \\ &\quad - \frac{1}{2}\frac{d}{dt}\left[(g \circ \nabla u)(t) - \left(\int_0^t g(s)ds\right)\left\|\nabla u\left(t\right)\right\|_2^2\right],\end{aligned}$$

where

$$(g \circ u)(t) = \int_0^t g(t-s)\left\|u(t) - u(s)\right\|_2^2 ds.$$

Lemma 2.3 *(Poincaré's inequality) [9] Let Ω be a bounded domain of $\mathbb{R}^n$ and $q(.)$ satisfies (1.3), and $1 \leq q^- \leq q(x) \leq q^+ < +\infty$, Then*

$$\left\|u\right\|_{q(\cdot)} \leq C\left\|\nabla u\right\|_{q(\cdot)}, \quad \textit{for all} \quad u \in W_0^{1,q(\cdot)}(\Omega).$$

where the positive constant C depending on q^-, q^+ and Ω only. In particular, the space $W_0^{1,q(\cdot)}(\Omega)$ has an equivalent norm given by $\left\|u\right\|_{W_0^{1,q(\cdot)}(\Omega)} = \left\|\nabla u\right\|_{q(\cdot)}$.

Lemma 2.4 *[11] There exist two positive constants c_0 and c_1 such that, for all $x \in \overline{\Omega}$, $(u, v) \in \mathbb{R}^2$.*

$$\frac{c_0}{2(p(x)+2)}\left(|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)}\right) \leq F\left(x, u, v\right) \leq \frac{c_1}{2(p(x)+2)}\left(|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)}\right).$$

Corollary 2.1 *[11] For all $x \in \overline{\Omega}$ and $(u, v) \in \mathbb{R}^2$, we have*

$$c_0\left(\rho_{p(x)+1}\left(u\right) + \rho_{p(x)+1}\left(v\right)\right) \leq \int_\Omega F\left(x, u, v\right) dx \leq c_1\left(\rho_{p(x)+1}\left(u\right) + \rho_{p(x)+1}\left(v\right)\right). \tag{2.2}$$

Lemma 2.5 *[9] Let p be a measurable function on Ω. Then*

$$\left\|u\right\|_{p(.)} \leq 1 \textit{ if and only if } \rho_{p(.)}\left(u\right) \leq 1.$$

If $1 < p^- \leq p(x) \leq p^+ < +\infty$, hold then

$$min\left\{\left\|u\right\|_{p(.)}^{p^-}, \left\|u\right\|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}\left(u\right) \leq max\left\{\left\|u\right\|_{p(.)}^{p^-}, \left\|u\right\|_{p(.)}^{p^+}\right\}.$$

for any $u \in L^{p(.)}(\Omega)$.

3. Blow-up result

In this section we state and prove our main blow-up result. For this purpose, we define the energy functional of (1.1) as

$$\begin{aligned}E(t) &= \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\left\|\nabla u\right\|^2 + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\left\|\nabla v\right\|^2 \\ &+ \frac{1}{2\left(\gamma+1\right)}\left(\left\|\nabla u\right\|^{2(\gamma+1)} + \left\|\nabla v\right\|^{2(\gamma+1)}\right) + \frac{1}{2}\left(\left(g_1 \circ \nabla u\right)(t) + \left(g_2 \circ \nabla v\right)(t)\right) - \int_\Omega F\left(x, u, v\right) dx.\end{aligned} \tag{3.1}$$

where u, $v \in W_0^{1,2(\gamma+1)}(\Omega)$.

Lemma 3.1 *$E(t)$ energy functional is nonincreasing function.*

Proof: Multiplying the frst equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, since $g_i(s) \leq 0$, we get

$$\begin{aligned} E'(t) &= -\|u_t\|_2^2 - \|v_t\|_2^2 - \frac{1}{2}\left[g_1(t)\int_\Omega |\nabla u(t,x)|^2 dx + g_2(t)\int_\Omega |\nabla v(t,x)|^2 dx\right] \\ &\quad + \frac{1}{2}\left[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)\right] - \int_\Omega |u_t|^2 |u|^{q(x)-2} dx - \int_\Omega |v_t|^2 |v|^{q(x)-2} dx \\ &\leq 0. \end{aligned} \tag{3.2}$$

□

Now, the main result of this work is given in the following theorem.

Theorem 3.1 *[9] Let (1.5)−(2.1) hold, u_0, $v_0 \in W_0^{1,2(\gamma+1)}(\Omega)$ and (u, v) is a local solution of the system (1.1). Assume further that*

$$\int_0^t g_i(s)ds \geq \frac{\gamma}{\gamma + \frac{1}{4}}, \quad i = 1, 2,$$

and

$$E(0) < 0, \quad \max\left\{q^-, q^+\right\} \leq p^- + 1. \tag{3.3}$$

Then there exists a fnite time T^* such that the solution of problem (1.1)blows up in a fnite time, and

$$T^* \leq \frac{1-\alpha}{\Lambda \alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

Lemma 3.2 *[11] Assume that (1.5) holds. Then, we have the following inequalities*

$$\left[\rho_{p(.)}(u) + \rho_{p(.)}(v)\right]^{\frac{s}{2(p^-+2)}} \leq C\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \rho_{p(.)}(u) + \rho_{p(.)}(v)\right), \tag{3.4}$$

$$\rho_{p(.)}(u) + \rho_{p(.)}(v) \leq C\left(\|u\|_{2(p^-+2)}^{2(p^-+2)} + \|v\|_{2(p^-+2)}^{2(p^-+2)}\right), \tag{3.5}$$

$$\|u\|_{2(p^-+2)}^s \leq C\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(p^-+2)}^{2(p^-+2)} + \|v_t\|_{2(p^-+2)}^{2(p^-+2)}\right), \tag{3.6}$$

$$\|v\|_{2(p^-+2)}^s \leq C\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(p^-+2)}^{2(p^-+2)} + \|v_t\|_{2(p^-+2)}^{2(p^-+2)}\right), \tag{3.7}$$

$$\int_\Omega |u|^{q(x)+1} dx \leq c_3\left[\left(\rho_{p(.)}(u) + \rho_{p(.)}(v)\right)^{\frac{q^-+1}{2(p^-+2)}} + \left(\rho_{p(.)}(u) + \rho_{p(.)}(v)\right)^{\frac{q^++1}{2(p^-+2)}}\right], \tag{3.8}$$

$$\|u\|_{p^-+1}^{p^-+1} + \|v\|_{p^-+1}^{p^-+1} \leq c_3\left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v)\right), \tag{3.9}$$

for any $u, v \in H_0^1(\Omega)$ and $2 \leq s \leq 2(p^- + 2)$. Where $C > 1$, $c_3 > 0$, $c_4 > 0$ are constants.

Proof of Theorem 1. Let us set

$$H(t) = -E(t). \tag{3.10}$$

By using (3.10), we have

$$\begin{aligned} H'(t) &= -E'(t) \\ &\geq \int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \geq 0. \end{aligned} \tag{3.11}$$

Then $E(0)<0$ and (3.11) gives $H(t)\geq H(0)>0$. By the defnition $H(t)$ and (2.2), we get

$$\begin{aligned} H(t) &\leq \int_{\Omega} F(x,u,v)\,dx \\ &\leq c_1\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right). \end{aligned} \tag{3.12}$$

We defne

$$L(t):=H^{1-\alpha}(t)+\frac{\varepsilon}{2}\int_{\Omega} u^2dx++\frac{\varepsilon}{2}\int_{\Omega} v^2dx,\quad t\geq 0. \tag{3.13}$$

where $\varepsilon>0$ to be fixed later and

$$0<\alpha\leq \min\left\{\frac{p^- - q^+ +1}{(p^-+1)(q^+-1)},\frac{p^- -1}{2(p^-+1)}\right\}. \tag{3.14}$$

By derivating $L(t)$ and using Eq (1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H'(t)H^{-\alpha}(t)+\varepsilon\int_{\Omega} u_t u dx+\varepsilon\int_{\Omega} v_t v dx \\ &= (1-\alpha)H'(t)H^{-\alpha}(t)-\varepsilon\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &\quad -\varepsilon\left(\left(1-\int_0^t g_1(s)ds\right)\|\nabla u\|^2+\left(1-\int_0^t g_2(s)ds\right)\|\nabla v\|^2\right) \\ &\quad -\varepsilon\int_{\Omega}\nabla u(t)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))dsdx-\varepsilon\int_{\Omega}\nabla v(t)\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx \\ &\quad +2\varepsilon\int_{\Omega}(p(x)+2)F(x,u,v)\,dx \\ &\quad -\varepsilon\int_{\Omega} uu_t|u|^{q(x)-2}dx-\varepsilon\int_{\Omega} vv_t|v|^{q(x)-2}dx. \end{aligned} \tag{3.15}$$

In order to estimate the last term in (3.15), we make use of the following Young's inequality, we have

$$XY\leq\frac{1}{4\delta}X^2+\delta Y^2.$$

for $\delta>0$, with $X=u_t|u|^{\frac{q(x)-2}{2}}$ and $Y=u|u|^{\frac{q(x)-2}{2}}$, we find

$$\begin{aligned} \int_{\Omega} u_t u|u|^{q(x)-2}dx &= \int_{\Omega} u_t|u|^{\frac{q(x)-2}{2}}u|u|^{\frac{q(x)-2}{2}}dx \\ &\leq \frac{1}{4\delta}\int_{\Omega} u_t^2|u|^{q(x)-2}dx+\delta\int_{\Omega}|u|^{q(x)}dx. \end{aligned} \tag{3.16}$$

Similarly, we have

$$\int_{\Omega} v_t v|v|^{q(x)-2}dx\leq\frac{1}{4\delta}\int_{\Omega} v_t^2|v|^{q(x)-2}dx+\delta\int_{\Omega}|v|^{q(x)}dx. \tag{3.17}$$

Cachy-Schwarz ans Young inequalities, we have

$$\begin{aligned} &\int_0^t g_1(t-s)\int_{\Omega}\nabla u(t,x)(\nabla u(t,x)-\nabla u(s,x))\,dxds \\ \leq &\int_0^t g_1(t-s)\|\nabla u(t,x)\|_2\|\nabla u(t,x)-\nabla u(s,x)\|_2\,ds \\ \leq &\frac{1}{4\delta}\left(\int_0^t g_1(s)ds\right)\|\nabla u\|_2^2+\delta(g_1\circ\nabla u)(t),\quad\forall\delta>0. \end{aligned} \tag{3.18}$$

Similarly,

$$\begin{aligned} &\int_0^t g_2(t-s)\int_{\Omega}\nabla v(t,x).(\nabla v(t,x)-\nabla v(s,x))\,dxds \\ \leq &\frac{1}{4\delta}\left(\int_0^t g_2(s)ds\right)\|\nabla v\|_2^2+\delta(g_2\circ\nabla v)(t),\quad\forall\delta>0. \end{aligned} \tag{3.19}$$

Combining (3.16) , (3.17) , (3.18) and (3.19), give

$$
\begin{aligned}
L'(t) \geq\ & (1-\alpha) H'(t)H^{-\alpha}(t) - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\
& -\varepsilon \left(1 - \left(1 - \frac{1}{4\delta} \right) \int_0^t g_1(s)ds \right) \|\nabla u\|^2 - \varepsilon \left(1 - \left(1 - \frac{1}{4\delta} \right) \int_0^t g_2(s)ds \right) \|\nabla v\|^2 \\
& -\delta\varepsilon \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\
& -\delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) - \frac{\varepsilon}{4\delta} \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right) \\
\geq\ & (1-\alpha) H'(t)H^{-\alpha}(t) - \varepsilon \left(\|\nabla u\|_{2(\gamma+1)}^{2(\gamma+1)} + \|\nabla v\|_{2(\gamma+1)}^{2(\gamma+1)} \right) \\
& -\varepsilon \left(\frac{1 - \left(1 - \frac{1}{4\delta} \right) \int_0^t g_1(s)ds}{1 - \int_0^t g_1(s)ds} \right) l_1 \|\nabla u\|^2 - \varepsilon \left(\frac{1 - \left(1 - \frac{1}{4\delta} \right) \int_0^t g_2(s)ds}{1 - \int_0^t g_2(s)ds} \right) l_2 \|\nabla v\|^2 \\
& -\delta\varepsilon \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\
& -\delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) - \frac{\varepsilon}{4\delta} \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right).
\end{aligned} \tag{3.20}
$$

We denote $\eta_i = \frac{1-\left(1-\frac{1}{4\delta}\right)\int_0^t g_i(s)ds}{1-\int_0^t g_i(s)ds}, \ \ i = 1, 2.$

By using Lemma 3.8 (2.1) and estimate (3.20), we get

$$
\begin{aligned}
L'(t) \geq\ & (1-\alpha) H'(t)H^{-\alpha}(t) - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\
& -\varepsilon \left(\eta_1 l_1 \|\nabla u\|^2 + \eta_2 l_2 \right) \|\nabla v\|^2 \\
& -\delta\varepsilon \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\
& -\delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) - \frac{\varepsilon}{4\delta} \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right) \\
\geq\ & (1-\alpha) H'(t)H^{-\alpha}(t) - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\
& -\varepsilon \min_{i=1,2} (\eta_i) \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right) \\
& -\delta\varepsilon \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\
& -2\delta\varepsilon c_4 \left(\left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right)^{\frac{q^+}{p^-+1}} + \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right)^{\frac{q^-}{p^-+1}} \right) \\
& -\frac{\varepsilon}{4\delta} \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right).
\end{aligned} \tag{3.21}
$$

Then, from the defnition $H(t)$ and l_i, $(i = 1, 2)$, we get

$$
\begin{aligned}
& -\left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right) \\
\geq\ & 2H(t) + \frac{1}{(\gamma+1)} \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) - 2\int_\Omega F(x,u,v)\, dx \\
\geq\ & 2H(t) + \frac{1}{(\gamma+1)} \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \\
& -2c_1 \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right).
\end{aligned} \tag{3.22}
$$

Using the last inequality, for $z \geq 0$, $0 < \beta \leq 1$ and $d > 0$, we have

$$
z^\beta \leq z + 1 \leq \left(1 + \frac{1}{d} \right) (z + d). \tag{3.23}
$$

Applying (3.23) with $z=\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)$, $d=H(0)$, $\beta=\frac{q^+}{p^-+1}\leq 1$ and then with $\beta=\frac{q^-}{p^-+1}\leq 1$ respectively, we get

$$\begin{aligned}\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right)^{\frac{q^+}{p^-+1}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)+H(0)\right) \\ &\leq c_5\left(\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right)+H(t)\right).\end{aligned}\tag{3.24}$$

where $c_5=1+\frac{1}{H(0)}$.

Similar calculations give, for some $c_5>0$

$$\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right)^{\frac{q^-}{p^-+1}}\leq c_5\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)+H(t)\right).\tag{3.25}$$

A combination of (3.5) – (3.24) – (3.25) implies that, for some $c_6>0$

$$\begin{aligned}\int_\Omega |u|^{q(x)}dx+\int_\Omega |v|^{q(x)}dx &\leq 2c_4\left(\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right)^{\frac{q^+}{p^-+1}}+\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right)^{\frac{q^-}{p^-+1}}\right) \\ &\leq c_6\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)+H(t)\right),\end{aligned}\tag{3.26}$$

where $c_6=4c_4.c_5$.

Consequently,

$$\begin{aligned}L'(t) \geq\ & (1-\alpha)H'(t)H^{-\alpha}(t)-\varepsilon\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &+\varepsilon\min_{i=1,2}(\eta_i)\left(-2H(t)+\frac{1}{(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right) \\ &-2\varepsilon\min_{i=1,2}(\eta_i)\int_\Omega F(x,u,v)\,dx \\ &-\delta\varepsilon\left((g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right)+\left(2\varepsilon c_0\left(p^-+2\right)-\delta\varepsilon c_6\right)\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right) \\ &-\frac{\varepsilon}{4\delta}\left(\int_\Omega u_t^2|u|^{q(x)-2}dx+\int_\Omega v_t^2|v|^{q(x)-2}dx\right) \\ \geq\ & (1-\alpha)H'(t)H^{-\alpha}(t)+2\varepsilon\left(\min_{i=1,2}(\eta_i)-\delta c_4c_6\right)H(t) \\ &+\varepsilon\left(\frac{\min_{i=1,2}(\eta_i)}{(\gamma+1)}-1\right)\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &+\varepsilon\left(\min_{i=1,2}(\eta_i)-\delta\right)\left((g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right) \\ &+\varepsilon\left(2c_0\left(p^-+2\right)-2\min_{i=1,2}(\eta_i)c_1-\delta c_6\right)\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right) \\ &-\frac{\varepsilon}{4\delta}\left(\int_\Omega u_t^2|u|^{q(x)-2}dx+\int_\Omega v_t^2|v|^{q(x)-2}dx\right).\end{aligned}\tag{3.27}$$

So by choosing δ such that $\frac{1}{4\delta}=\kappa H^{-\alpha}(t)$, where κ is a large constant to be specified later, $\varepsilon\left(\frac{\min_{i=1,2}(\eta_i)}{(\gamma+1)}-1\right)\geq 0$, $\varepsilon\left(\min_{i=1,2}(\eta_i)-\delta\right)\geq 0$

$$c_7=\varepsilon\left(2c_0\left(p^-+2\right)-2\min_{i=1,2}(\eta_i)-\delta\varepsilon c_6\right)>0.$$

substituting in(3.27), we obtain

$$\begin{aligned}L'(t) &\geq (1-\alpha-\varepsilon\kappa)H'(t)H^{-\alpha}(t)+2\varepsilon\min_{i=1,2}(\eta_i)H(t)+c_7\left(\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right) \\ &\geq c_8\left(H(t)+\rho_{p(x)+1}(u)+\rho_{p(x)+1}(v)\right),\end{aligned}\tag{3.28}$$

where $c_8 = \min\left(2\varepsilon \min_{i=1,2}(\eta_i),\ c_7\right)$ and we pick ε small enough such that $1-\alpha-\varepsilon\kappa \geq 0$.
Next, using the algebraic inequality

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right).$$

and on the other hand, thanks to the Hölder's inequality and the embedding $L^{(p^-+1)}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain
Finally, by noting that

$$\begin{aligned}
L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \frac{\varepsilon}{2}\int_\Omega u^2 dx + \frac{\varepsilon}{2}\int_\Omega v^2 dx\right)^{\frac{1}{1-\alpha}} \qquad (3.29)\\
&\leq 2^{\frac{\alpha}{1-\alpha}}\left[H(t) + \left(\frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2\right)^{\frac{1}{1-\alpha}}\right]\\
&\leq C\left[H(t) + \|u\|_{p^-+1}^{\frac{2}{1-\alpha}} + \|v\|_{p^-+1}^{\frac{2}{1-\alpha}}\right]\\
&= C\left[H(t) + \left(\|u\|_{p^-+1}^{p^-+1}\right)^{\frac{2}{1-\alpha}} + \left(\|v\|_{p^-+1}^{p^-+1}\right)^{\frac{2}{1-\alpha}}\right]\\
&\leq C\left[\left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v)\right)^{\frac{2}{1-\alpha}}\right]\\
&\leq C\left[H(t) + \rho_{p(x)+1}(u) + \rho_{p(x)+1}(v)\right].
\end{aligned}$$

we get, for some

$$L^{\frac{1}{1-\alpha}}(t) \leq C\left[H(t) + \rho_{p(x)+1}(u) + \rho_{p(x)+1}(v)\right]. \qquad (3.30)$$

Thus,(3.28) and (3.30) arrive at

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \text{ for } t \geq 0. \qquad (3.31)$$

where $\Lambda = \frac{c_8}{C}$ is a positive constant.
A direct integration over $(0,t)$ of (3.31) then yields

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda\frac{\alpha t}{1-\alpha}}, \text{ for } t \geq 0.$$

Therefore, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

which implies that $L(t) \to +\infty$, as $t \to T^*$, where $T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\frac{\alpha}{1-\alpha}}(0)}$. Consequently, the solution of problem (1.1) blows-up in a finite time.
This completes the proof.

4. Exponential growt

In this section, we aim to indicate that the energy grow up as an exponential function as time as goes to infinity.

Theorem 4.1 *Suppose that* (1.5) – (2.1) *hold.* (u,v) *be a any solution to* (1.1)*. Suppose further that* $E(0) < 0$ *and*

$$\int_0^t g_i(s)ds \geq \frac{\gamma}{\gamma+\frac{1}{2}}, \quad i = 1,2.$$

Then, the solution to (1.1) *grows exponentially.*

Proof: We define

$$G(t) := H(t) + \frac{\varepsilon}{2}\int_{\Omega} u^2 dx + + \frac{\varepsilon}{2}\int_{\Omega} v^2 dx, \quad t \geq 0. \tag{4.1}$$

where $H(t) = -E(t)$ and choose $0 < \varepsilon \leq 1$ in this interval to obtain small perturbation of $E(t)$ and we will indicate that $G(t)$ grows exponentialy, namely $G(t)$ satisfies a differential inequality of the form

$$G'(t) \geq \xi \Gamma G(t), \quad \text{for} \quad t \geq 0.$$

By derivating (4.1) and using Eq (1.1), we have

$$\begin{aligned}
G'(t) &= H'(t) + \varepsilon\int_{\Omega} u_t u dx + \varepsilon\int_{\Omega} v_t v dx \qquad (4.2)\\
&= \|u_t\|_2^2 + \|v_t\|_2^2 + \frac{1}{2}\left[g_1(t)\|\nabla u\|^2 + g_2(t)\|\nabla v\|^2\right]\\
&\quad -\frac{1}{2}\left[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)\right] + \int_{\Omega} |u_t|^2 |u|^{q(x)-2} dx + \int_{\Omega} |v_t|^2 |v|^{q(x)-2} dx\\
&\quad -\varepsilon\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) - \varepsilon\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\\
&\quad +\varepsilon\int_{\Omega} \nabla u(t)\int_0^t g_1(t-s)\nabla u(s)\, ds dx + \varepsilon\int_{\Omega} \nabla v(t)\int_0^t g_2(t-s)\nabla v(s)\, ds dx\\
&\quad +\varepsilon\int_{\Omega} \left(u f_1(x,u,v) + v f_2(x,u,v)\right) dx\\
&\quad -\varepsilon\int_{\Omega} u u_t |u|^{q(x)-2} dx - \varepsilon\int_{\Omega} v v_t |v|^{q(x)-2} dx\\
&\geq \|u_t\|_2^2 + \|v_t\|_2^2 + \frac{1}{2}\left[g_1(t)\|\nabla u\|^2 + g_2(t)\|\nabla v\|^2\right]\\
&\quad -\varepsilon\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\\
&\quad -\frac{1}{2}\left[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)\right]. + \int_{\Omega} |u_t|^2 |u|^{q(x)-2} dx + \int_{\Omega} |v_t|^2 |v|^{q(x)-2} dx\\
&\quad -\varepsilon\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) + 2\varepsilon\left(p^- + 2\right)\int_{\Omega} F(x,u,v)\, dx\\
&\quad +\varepsilon\int_{\Omega} \nabla u(t)\int_0^t g_1(t-s)\nabla u(s)\, ds dx + \varepsilon\int_{\Omega} \nabla v(t)\int_0^t g_2(t-s)\nabla v(s)\, ds dx\\
&\quad -\varepsilon\int_{\Omega} u u_t |u|^{q(x)-2} dx - \varepsilon\int_{\Omega} v v_t |v|^{q(x)-2} dx.
\end{aligned}$$

Terms in (4.2) is estimated as follows:

$$\int_{\Omega} u_t u |u|^{q(x)-2} dx \leq \frac{1}{4\delta}\int_{\Omega} u_t^2 |u|^{q(x)-2} dx + \delta\int_{\Omega} |u|^{q(x)} dx. \tag{4.3}$$

Similarly, we have

$$\int_{\Omega} v_t v |v|^{q(x)-2} dx \leq \frac{1}{4\delta}\int_{\Omega} v_t^2 |v|^{q(x)-2} dx + \delta\int_{\Omega} |v|^{q(x)} dx. \tag{4.4}$$

Therefore, combining with and Cachy-Schwarz ans Young inequalities, we have

$$\begin{aligned}
&\int_{\Omega} \nabla u(t)\int_0^t g_1(t-s)\nabla u(s)\, ds dx \qquad (4.5)\\
&= \int_0^t g_1(s) ds \|\nabla u\|^2 - \int_0^t g_1(t-s)\int_{\Omega} \nabla u(t,x)\left(\nabla u(t,x) - \nabla u(s,x)\right) dx ds\\
&\geq \frac{1}{2}\int_0^t g_1(s) ds \|\nabla u\|^2 - \frac{1}{2}(g_1 \circ \nabla u)(t).
\end{aligned}$$

Similarly,

$$\int_{\Omega} \nabla v(t) \int_0^t g_2(t-s) \nabla v(s)\, ds dx \geq \frac{1}{2} \int_0^t g_2(s) ds \left\| \nabla v \right\|^2 - \frac{1}{2} (g_2 \circ \nabla v)(t). \tag{4.6}$$

Combining (4.3) , (4.4) ,(4.5)and (4.6), give

$$\begin{aligned} G'(t) \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\ & - \varepsilon \left(1 - \frac{1}{2} \int_0^t g_1(s) ds \right) \|\nabla u\|^2 - \varepsilon \left(1 - \frac{1}{2} \int_0^t g_2(s) ds \right) \|\nabla v\|^2 \\ & - \frac{\varepsilon}{2} \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\ & - \delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) + \left(1 - \frac{\varepsilon}{4\delta} \right) \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right) \\ \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\ & - \varepsilon \left(\frac{1 - \frac{1}{2}\int_0^t g_1(s) ds}{1 - \int_0^t g_1(s) ds} \right) l_1 \|\nabla u\|_2^2 - \varepsilon \left(\frac{1 - \frac{1}{2}\int_0^t g_2(s) ds}{1 - \int_0^t g_2(s) ds} \right) l_2 \|\nabla v\|_2^2 \\ & - \frac{\varepsilon}{2} \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\ & - \delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) + \left(1 - \frac{\varepsilon}{4\delta} \right) \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right). \end{aligned} \tag{4.7}$$

We denote $w_i = \frac{1 - \frac{1}{2}\int_0^t g_i(s) ds}{1 - \int_0^t g_i(s) ds}$, $i = 1, 2$.

$$\begin{aligned} G'(t) \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\ & - \varepsilon \left(w_1 l_1 \|\nabla u\|^2 + w_2 l_2 \right) \|\nabla v\|^2 \\ & - \frac{\varepsilon}{2} \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\ & - \delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) + \left(1 - \frac{\varepsilon}{4\delta} \right) \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right) \\ \geq\ & \|u_t\|^2 + \|v_t\|^2 - \varepsilon \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\ & - \varepsilon \min_{i=1,2} (w_i) \left(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2 \right) \\ & - \frac{\varepsilon}{2} \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) + 2\varepsilon c_0 \left(p^- + 2 \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\ & - \delta\varepsilon \left(\int_\Omega |u|^{q(x)} dx + \int_\Omega |v|^{q(x)} dx \right) + \left(1 - \frac{\varepsilon}{4\delta} \right) \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right). \end{aligned} \tag{4.8}$$

Then, from the defnition $H(t)$, consequently,

$$\begin{aligned} G'(t) \geq\ & \|u_t\|^2 + \|v_t\|^2 + 2\varepsilon \min_{i=1,2} (w_i) H(t) \\ & + \varepsilon \left(\frac{\min_{i=1,2}(w_i)}{\gamma+1} - 1 \right) \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\ & + \varepsilon \left(\min_{i=1,2} (w_i) - 1 \right) \left((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right) \\ & + \varepsilon \left(2c_0 \left(p^- + 2 \right) - 2c_4\delta - 2c_1 \min_{i=1,2} (w_i) \right) \left(\rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right) \\ & + \left(1 - \frac{\varepsilon}{4\delta} \right) \left(\int_\Omega u_t^2 |u|^{q(x)-2} dx + \int_\Omega v_t^2 |v|^{q(x)-2} dx \right). \end{aligned} \tag{4.9}$$

Taking δ small enough such that $k_1 = 2c_0(p^- + 2) - 2c_4\delta - 2c_1 \min_{i=1,2}(w_i) > 0$ and $k_2 = \min_{i=1,2}(w_i) - \frac{1}{2} > 0$ and $k_3 = \frac{\min_{i=1,2}(w_i)}{\gamma+1} - 1 > 0$ then taking ε small enough such that $1 - \frac{\varepsilon}{4\delta} > 0$, we have

$$G'(t) \geq C' \left[H(t) + \|u_t\|^2 + \|v_t\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) + \rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right]. \tag{4.10}$$

Thus, the functional $G(t)$ is strictly positive and increasing for all $t \geq 0$.
Conversely, from $G(t)$ function we obtain

$$\begin{aligned} G(t) &= H(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2 \\ &\leq C'' \left[H(t) + \|\nabla u\|^2 + \|\nabla v\|^2 \right] \\ &\leq C'' \left[H(t) + \|u_t\|^2 + \|v_t\|^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) + \rho_{p(x)+1}(u) + \rho_{p(x)+1}(v) \right]. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11) we arrive at

$$G'(t) \geq \lambda G(t), \text{ for } t \geq 0. \tag{4.12}$$

where $\lambda = \frac{C'}{C''} > 0$ is a constant.
Integration of (4.12) over $(0, t)$ gives us

$$G(t) \geq G(0) \exp(\lambda t), \text{ for } t \geq 0. \tag{4.13}$$

his completes the proof. □

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