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Common Coupled Fixed Point Theorems on k-Fuzzy Metric Spaces and Applications to Functional Equations in Dynamic Programming

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ABSTRACT: This paper explores a coupled coincidence point theorem on k-fuzzy metric spaces involving an inequality that includes several mappings. Specifically, theorems that prove common fixed points have been derived using w-compatibility. Moreover, we discuss the consequences of our main theorem and provide a couple of examples to validate our result. Finally, we apply our result in proving the existence of solutions to functional equations that frequently appear in Dynamic Programming.

Keywords: Common coupled fixed point, k-fuzzy metric space, w-compatible, dynamical programming, functional equations.

Contents

1. Introduction

The notion of fuzzy set theory was proposed by Lotfi A. Zadeh [36] in 1965 as a broader version of classical set theory. They give an approach for expressing and reasoning with vague or inaccurate concepts. In classical set theory, a member either belongs to a set or does not, according to a clear binary condition. However, in many situations in reality, the separations between categories are not well defined, and elements may possess partial membership to several sets at the same time, such as human perception or qualitative data analysis. Fuzzy set enable us to reflect this intrinsic fuzziness by giving degrees of membership to elements that range from 0 to 1. To utilize this idea in topology as well as analysis, many different researchers have substantially developed the fuzzy set theory and its applications. In [13], George and Veeramani revised and formalized the notion of a Cauchy sequence in a fuzzy metric space (in short, $\mathbb{FMS}$) presented by Kromosil and Michalek [24] and established the Hausdorff topology, which has very rigorous applications in quantum particle theory(see, [28] and references therein). Remarkably, the advent of $\mathbb{FMS}$ has produced applications in many domains, including image processing, artificial intelligence and pattern recognition, where uncertainty and imperfection are common. George and Veeramani's [13] work paved the way for future study and advancement in $\mathbb{FMS}$ theory and applications, and it also made contributions to the larger field of fuzzy logic and its applications.

The distance between two points in an $\mathbb{FMS}$ is determined by the degree of closeness concerning a parameter $t > 0$, so it has evoked much interest to see the variations in the distance function by changing the different parameters (k-parameter). Recently, the concept of k-fuzzy metric space (in short, $k-\mathbb{FMS}$) was introduced by Dhananjay Gopal et al. [15], in which they extended the work of George and Veeramani [13] into $k-\mathbb{FMS}$. The concept of coupled fixed point (in short, $\mathbb{CFP}$) broadens the idea of fixed points from single mappings to pairs of mappings. In 1987, Guo and Lakshmikantham [17] initiated the idea of the $\mathbb{CFP}$ theory, which was further expanded by Bhaskar and Lakshmikantham [8], who demonstrated the

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$\mathbb{CFP}$ results in metric spaces with partial ordering. In recent times, the study of the metric theory of $\mathbb{CFP}$s has taken on a fuzzy aspect. Several coupled fixed point consequences are being observed with various contractions. In [32], Sedghi et al. came up with a $\mathbb{CFP}$ theorem for $\mathbb{FMS}$ under contractive settings. Afterward, Hu [18] observed some results on ϕ-contractions in $\mathbb{FMS}$ with compatible mappings. Utilizing Hadžić type t-norm Binayak et al. [10] provided a coupled coincidence point (in short, $\mathbb{CCP}$) results in $\mathbb{FMS}$ involving different contractive approaches.

Fortunately, in fixed point theory, a property known as CLR_g enables mappings that do not possess continuity. Jain et al. [23] generalized the idea of CLR_g property for coupled mappings in $\mathbb{FMS}$ and applied their results to extend the work of Hu [18]. Roldán-López-de-Hierro and Sintunavarat [31] discovered and generalized certain results from [29], using some control functions. In [21], Jain et al. went further from the concept of property CLR_g to property CLR_{ST} for $\phi-$ contraction $\mathbb{CFP}$ problems. In 2020, Jain et al. [22] presented a new Φ contraction principle for w-compatible mappings and proved a $\mathbb{CFP}$ theorem with the support of the CLR property, beneficially enhancing recent research in [18], [29], [19]. In addition, they gave an open problem that requires more analysis. Refer to [37], [27], [14], [16], [33] for further information on fixed point theory and its usage in $\mathbb{FMS}$s. Recently, a new ϕ-contraction has been found to obtain $\mathbb{CCP}$s in partially ordered $\mathbb{FMS}$ by Jain and Atangana [20], which extends certain pre-existing findings and presents an illustrative example for elucidating the presence of ϕ-contraction using 3-dimensional analysis.

Indeed, $\mathbb{FMS}$s find applications in various fields, including Dynamic Programming, where they provide a versatile framework for modeling uncertainty and imprecision. The basic structure underlying Dynamical Programming's functional equation is on

$$L(p) = \max_q [U(p, q, L(G(p, q)))], \tag{1.1}$$

where p is the state vector and q is the decision vector, G denotes the transformation of process, and $L(p)$ denotes the function that yields the optimal return from the initial state p. This functional equation lends itself to various approaches, including examining the nature of processes that lead to (1.1), delving into the specific form of (1.1), or concerning the precise form of (1.1). Dynamical Programming adopts a distinct approach to problem formulation. Rather than focusing solely on a single problem of fixed duration, it addresses a family of problems spanning from zero duration to the original problem's duration.
The theory of Dynamic Programming originates from the realm of multistage decision-making problems [4], [5], [6], [34], [35]. In 1984, certain types of continuous multistage decision processes were proven by Bhakta and Mitra [7] to yield existence theorems for functional equations. In 1986, Baskaran et al. [3] obtained a solution for the same category of functional equations under weaker conditions compared to those described in [7]. Soon after, Chang and Ma [9] investigated the existence of solutions to a set of functional equations encountered in Dynamic Programming, employing iterative approximation techniques involving $\mathbb{CFP}$s for mixed monotone set-valued mappings.

In [11], Binayak et al. employ an inequality that incorporates both a coupled multivalued mapping and a single-valued mapping to derive a theorem on $\mathbb{CCP}$s, which amalgamates various concepts commonly found in the study of fixed point theory. Inspired by this research, in this paper, we broaden this approach by encompassing $k-\mathbb{FMS}$, making pertinent substitutions. So, we validated that our result can be proved in generalized $\mathbb{FMS}$ ($k-\mathbb{FMS}$), and in numerous scenarios, we restricted our domain and co-domain to align with our inequality. In our inequality, we used one of the functions ζ, which indeed holds considerable importance in demonstrating our theorem. Ultimately, applying our main theorem, we ascertain the existence of solutions for the set of functional equations. Moreover, it is observed that in [7] and [3], unique solutions to the specified functional equations were derived by the authors through their fixed point results, while in [9] and [20], the authors have established the existence of solutions for the system of functional equations via the application of coupled fixed point theorems. In contrast, our contribution lies in demonstrating the same system of functional equations can be resolved utilizing a common coupled fixed point approach.

2. Preliminaries

Definition 2.1 [26] A binary operation $\star : [0,1]^2 \to [0,1]$ is called a continuous t-norm if the following properties are satisfied:

(i) $\star$ is associative and commutative,

(ii) $\star(p,1) = p$ for all $p \in [0,1]$,

(iii) $\star(p,q) \leq \star(r,s)$, whenever $p \leq r$ and $q \leq s$, for all $p,q,r,s \in [0,1]$,

(iv) $\star$ is continuous.

Without $\star$ being continuous, we refer to the above definition as t-norm, and henceforth, we adopt the notation $\star(p,q) = p \star q$.
Three usual examples of t-norms are $p \star q = \min\{p,q\}$(minimum t-norm), $p \star q = pq$(product t-norm) and $p \star q = \max\{p+q-1, 0\}$(Łukasiewicz t-norm).

Remark 2.1 [13] *For each t-norm $\star : [0,1]^2 \to [0,1]$, the following conditions hold:*

(1) For any $p_1, p_2 \in (0,1)$ with $p_1 > p_2$, we can find $p_3 \in (0,1)$ such that $p_1 \star p_3 \geq p_2$.

(2) For any $p_4 \in (0,1)$, we can find $p_5 \in (0,1)$ such that $p_5 \star p_5 \geq p_4$.

The following definition of $\mathbb{FMS}$ was introduced by Kramosil and Michálek [24], aiming to establish the fuzzy Banach contraction principle.

Definition 2.2 [24] An ordered triple $(X, M, \star)$ is called fuzzy metric space if X is any nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all $p,q,r \in X$ and $l,t > 0$:

(i) $M(p,q,0) = 0$;

(ii) $M(p,q,t) = 1$ iff $p = q$;

(iii) $M(p,q,t) = M(q,p,t)$;

(iv) $M(p,r,t+l) \geq M(p,q,t) \star M(q,r,l)$;

(v) $M(p,q,.) : (0,\infty) \to [0,1]$ is left continuous.

In the subsequent definition, George and Veeramani [13] refined the above definition of $\mathbb{FMS}$ to generate the Hausdorff topology within $\mathbb{FMS}$.

Definition 2.3 [13] An ordered triple $(X, M, \star)$ is called fuzzy metric space if X is any nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all $p,q,r \in X$ and $l,t > 0$:

(i) $M(p,q,t) > 0$;

(ii) $M(p,q,t) = 1$ iff $p = q$;

(iii) $M(p,q,t) = M(q,p,t)$;

(iv) $M(p,r,t+l) \geq M(p,q,t) \star M(q,r,l)$;

(v) $M(p,q,.) : (0,\infty) \to [0,1]$ is continuous.

In the next definition, Gopal et al. [15] introduced the $k-\mathbb{FMS}$, a novel way of measuring the fuzzy distance, which contains multiple parameters($t_1, t_2, \ldots, t_k > 0$).

Definition 2.4 [15] An ordered triple $(X, M, \star)$ is called k-fuzzy metric space if X is any nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)^k$ satisfying the following conditions for all $p,q,r \in X$ and $t,l,t_1,t_2,\ldots,t_k > 0$:

(i) $M(p,q,t_1,t_2,\ldots,t_k) > 0$;

(ii) $M(p,q,t_1,t_2,\ldots,t_k) = 1$ iff $p = q$;

(iii) $M(p,q,t_1,t_2,\ldots,t_k) = M(q,p,t_1,t_2,\ldots,t_k)$;

(iv) for any $u \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned} M(p,r,t_1,t_2,\ldots,t_u,t+l,t_{u+1},\ldots,t_k) \quad &\geq \quad M(p,q,t_1,t_2,\ldots,t_u,t,t_{u+1},\ldots,t_k) \\ &\quad \star M(q,r,t_1,t_2,\ldots,t_u,l,t_{u+1},\ldots,t_k); \end{aligned}$$

(v) $M(p,q,.) : (0,\infty)^k \to [0,1]$ is continuous.

Throughout, for a given $k-\mathbb{FMS}$ $(X,M,\star)$, for notational simplification, we use $M(p,q,t_1^k)$ instead of $M(p,q,t_1,t_2,\dots,t_k), p,q \in X$ and $t_1,t_2,\dots,t_k > 0$.

Definition 2.5 [15] Let $(X,M,\star)$ be a $k-\mathbb{FMS}$, then a sequence $\{p_n\}$ in X is said to be

(i) convergent to a point p_0 in X if $\lim_{n\to\infty} M(p_n,p_0,t_1^k) = 1$, for all $t_1,t_2,\dots,t_k > 0$;

(ii) a Cauchy if for every $0 < \epsilon < 1$ and for all $t_1,t_2,\dots,t_k > 0$ there exists a positive integer N such that for all $n,m \geq N, M(p_n,p_m,t_1^k) > 1-\epsilon$.

A $k-\mathbb{FMS}$ is complete if every Cauchy sequence is convergent.
Inspired by the notion of mixed monotone function [30] on $\mathbb{R}^2$, Bhaskar et. al [8] introduced the concept of the coupled fixed point.

Definition 2.6 [8] Let X be a nonempty set and consider the mapping $U : X \times X \to X$, then an element (p,q) is said to be a coupled fixed point of the mapping U if

$$U(p,q) = p, U(q,p) = q.$$

Definition 2.7 [25] Let X be a nonempty set and consider the mappings $U : X \times X \to X$ and $L : X \to X$. Then an element (p,q) is said to be a

(i) coupled coincidence point of the mappings U and L if

$$U(p,q) = L(p), U(q,p) = L(q);$$

(ii) common coupled fixed point of the mappings U and L if

$$U(p,q) = L(p) = p, U(q,p) = L(q) = q.$$

Definition 2.8 [12] Let X be a nonempty set and consider the mappings $U : X \times X \to X$ and $L : X \to X$. Then an element $p \in X$ is said to be a common fixed point of the mappings U and L if

$$U(p,p) = L(p) = p.$$

Definition 2.9 [1] The mappings $U : X \times X \to X$ and $L : X \to X$, are said to be w-compatible if

$$L(U(p,q)) = U(L(p),L(q)), \text{ whenever } (p,q) \in C(U,L),$$

where $C(U,L)$ is the collection of all coupled coincidence point of U and L.

Definition 2.10 [2] Consider the mappings $U : X \times X \to X$ and $L : X \to X$. Then we say the pair (U,L) is weakly commutative at $(p,q) \in X \times X$ if

$$L(U(p)) = U(L(p),L(q)) \text{ and } L(U(q)) = U(L(q),L(p)).$$

Proposition 2.1 [15] *Let $(X,M,\star)$ be a $k-\mathbb{FMS}$, $l,t_1,t_2,\dots,t_k > 0$. Suppose for some $u \in \{1,2,\dots,k\}$ we have $t_u < l$, then*

$$M(p,q,t_1^k) \leq M(p,q,t_1,t_2,\dots,t_{u-1},l,t_{u+1},\dots,t_k), \text{ for all } p,q \in X.$$

3. Main Results

Now, we define certain ideas that serve as rudimentary tools for achieving our main result. Let $(X, M, \star)$ be a $k-\mathbb{FMS}$, and for any $\Omega, \Theta \subseteq X$ we define

$$F(\Omega,\Theta) = \min\{\max_{p\in\Omega} \Phi(p,\Theta), \max_{q\in\Theta} \Phi(q,\Omega)\},$$

where

$$\Phi(p,\Theta) = \sup\{M(p,q,t_1^k)\}, q \in \Theta, t_1, t_2, \ldots, t_k > 0,$$

and

$$\Phi(q,\Omega) = \sup\{M(p,q,t_1^k)\}, p \in \Omega, t_1, t_2, \ldots, t_k > 0.$$

Consider the mappings $U : X \times X \to X$ and $L : X \to X$, and for any $p, q, a, b \in X$, we assume

$$S(p,q,a,b) = \max\{M(Lp, La, t_1^k), M(Lq, Lb, t_1^k)\},$$

and

$$\begin{aligned} T(p,q,a,b) \quad = \quad & \zeta\Big\{M(Lp, La, t_1^k), M(Lq, Lb, t_1^k), \frac{[1+\Phi(Lp,U(p,q))]\Phi(Lp,U(p,q))}{1+M(Lp,La,t_1^k)}, \\ & \frac{[1+\Phi(La,Ua,b))]\Phi(La,U(p,q))}{1+M(Lp,La,t_1^k)}, \frac{[1+\Phi(Lq,U(q,p))]\Phi(Lq,U(q,p))}{1+M(Lq,Lb,t_1^k)}, \\ & \frac{[1+\Phi(Lb,U(b,a))]\Phi(Lb,U(q,p))}{1+M(Lq,Lb,t_1^k)}\Big\}, \end{aligned}$$

where $\zeta : [0,1]^2 \times [0,2]^4 \to [0,1]$ is continuous in each coordinate such that

$$\zeta(p_1, p_2, p_3, p_4, p_5, p_6) = 1,$$

whenever $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$.
Now, we are ready to prove the main result of this section.

Theorem 3.1 *Let $(X, M, \star)$ be a complete $k-\mathbb{FMS}$, $a \star b \geq ab$ for all $a, b \in [0,1]$, $U : X \times X \to X$ and $L : X \to X$ be two mappings. Assume that there exists a continuous function $\omega : [0,1] \to [0,1]$ such that*

$$F(U(p,q), U(a,b)) \geq \omega(S(p,q,a,b))T(p,q,a,b), \tag{3.1}$$

for all $p, q, a, b \in X$. $U(X \times X) \subseteq L(X)$ and U commutes with L. Then U and L have a coupled coincidence point.
Furthermore, the mappings U and L have a common coupled fixed point if the pair (U, L) is w-compatible, and there exists $(p, q) \in C(U, L)$, $a, b \in X$ such that $\lim\limits_{n\to\infty} L^n p = a$ and $\lim\limits_{n\to\infty} L^n q = b$, and also L is continuous at a and b.

Proof: Let p_0 and q_0 be arbitrary points of X. Since $U(X \times X) \subseteq L(X)$, we can choose $p_1, q_1 \in X$ such that $Lp_1 = U(p_0, q_0)$ and $Lq_1 = U(q_0, p_0)$. Again from $U(X \times X) \subseteq L(X)$, we can choose $p_2, q_2 \in X$ such that $Lp_2 = U(p_1, q_1)$ and $Lq_2 = U(q_1, p_1)$. Continuing in this manner, we obtain two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$$Lp_{n+1} = U(p_n, q_n) \text{ and } Lq_{n+1} = U(q_n, p_n), \text{ for all } n \geq 0.$$

Since,

$$\begin{aligned} \sup(M(Lp_n, Lp_{n+1}, t_1^k)) \quad &\geq \quad F(U(p_{n-1}, q_{n-1}), U(p_n, q_n)) \\ &\geq \quad \omega(S(p_{n-1}, q_{n-1}, p_n, q_n))T(p_{n-1}, q_{n-1}, p_n, q_n). \end{aligned} \tag{3.2}$$

By the definition of $T(p, q, a, b)$, we have

$$\begin{aligned}
T(p_{n-1}, q_{n-1}, p_n, q_n) = \zeta\Big\{ & M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k), \\
& \frac{[1 + \Phi(Lp_{n-1}, U(p_{n-1}, q_{n-1}))]\Phi(Lp_{n-1}, U(p_{n-1}, q_{n-1}))}{1 + M(Lp_{n-1}, Lp_n, t_1^k)}, \\
& \frac{[1 + \Phi(Lp_n, U(p_n, q_n))]\Phi(Lp_n, U(p_{n-1}, q_{n-1}))}{1 + M(Lp_{n-1}, Lp_n, t_1^k)}, \\
& \frac{[1 + \Phi(Lq_{n-1}, U(q_{n-1}, p_{n-1}))]\Phi(Lq_{n-1}, U(q_{n-1}, p_{n-1}))}{1 + M(Lq_{n-1}, Lq_n, t_1^k)}, \\
& \frac{[1 + \Phi(Lq_n, U(q_n, p_n))]\Phi(Lq_n, U(q_{n-1}, p_{n-1}))}{1 + M(Lq_{n-1}, Lq_n, t_1^k)}\Big\}
\end{aligned}$$

$$\begin{aligned}
&\geq \zeta\Big\{ M(Lp_{n-1}, Lp_n, t_1^k),\ M(Lq_{n-1}, Lq_n, t_1^k), \\
&\quad \frac{[1 + M(Lp_{n-1}, Lp_n, t_1^k)]M(Lp_{n-1}, Lp_n, t_1^k)}{1 + M(Lp_{n-1}, Lp_n, t_1^k)}, \\
&\quad \frac{[1 + M(Lp_n, Lp_{n+1}, t_1^k)]M(Lp_n, Lp_n, t_1^k)}{1 + M(Lp_{n-1}, Lp_n, t_1^k)}, \\
&\quad \frac{[1 + M(Lq_{n-1}, Lq_n, t_1^k)]M(Lq_{n-1}, Lq_n, t_1^k)}{1 + M(Lq_{n-1}, Lq_n, t_1^k)}, \\
&\quad \frac{[1 + M(Lq_n, L(q_{n+1}, t_1^k))]M(Lq_n, Lq_n, t_1^k)}{1 + M(Lq_{n-1}, Lq_n, t_1^k)}\Big\} \\
&\geq \zeta\Big\{ \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big), \\
&\quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big), \\
&\quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big), \\
&\quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big), \\
&\quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big), \\
&\quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big)\Big\} \\
&\geq \quad \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big).
\end{aligned}$$

Now, (3.2) can be written as

$$\begin{aligned}
\sup\big(M(Lp_n, Lp_{n+1}, t_1^k)\big) \geq \omega\big(S(p_{n-1}, q_{n-1}, p_n, q_n)\big) \\
\min\big(M(Lp_{n-1}, Lp_n, t_1^k),\ M(Lq_{n-1}, Lq_n, t_1^k)\big)
\end{aligned}$$

which implies,

$$\sup(M(Lp_n, Lp_{n+1}, t_1^k)) \geq \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big).$$

Similarly, we can prove that

$$\sup(M(Lq_n, Lq_{n+1}, t_1^k)) \geq \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big). \tag{3.3}$$

Now, by the definition of t-norm, we get

$$\begin{aligned} M(Lp_n, Lp_{n+1}, t_1^k) \star M(Lq_n, Lq_{n+1}, t_1^k) \\ &\geq \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big) \\ &\quad \star \min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big) \end{aligned} \tag{3.4}$$

Suppose, $\min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big) = M(Lp_{n-1}, Lp_n, t_1^k)$.

Then (3.4) implies that

$$M(Lp_n, Lp_{n+1}, t_1^k) \star M(Lq_n, Lq_{n+1}, t_1^k) \geq [M(Lp_{n-1}, Lp_n, t_1^k)]^2. \tag{3.5}$$

Now for every $m > n$ and $0 < l < 1$, we have

$$(1-l)(1+l+\cdots+l^{m-n-1}) = 1 - l^{m-n} < 1,$$

and so for every $t > 0$, we have

$$t(1-l)(1+l+\cdots+l^{m-n-1}) = t(1 - l^{m-n}) < t.$$

Thus,

$$\begin{aligned} M(Lp_n, Lp_m, t^k) \ \star\ M(Lq_n, Lq_m, t^k) \\ &\geq M(Lp_n, Lp_m, (t(1-l)(1+l+\cdots+l^{m-n-1}))^k) \\ &\quad \star M(Lq_n, Lq_m, (t(1-l)(1+l+\cdots+l^{m-n-1})^k) \\ &\geq M(Lp_n, Lp_{n+1}, (t(1-l))^k) \star M(Lp_{n+1}, Lp_{n+2}, (tl(1-l))^k) \\ &\quad \star \cdots \star M(Lp_{m-1}, Lp_m, (tl^{m-n-1}(1-l))^k) \\ &\quad \star M(Lq_n, Lq_{n+1}, (t(1-l))^k) \star M(Lq_{n+1}, Lq_{n+2}, (tl(1-l))^k) \\ &\quad \star \cdots \star M(Lq_{m-1}, Lq_m, (tl^{m-n-1}(1-l))^k). \end{aligned}$$

Using (3.5), we have

$$\begin{aligned} M(Lp_n, Lp_m, t^k) \star M(Lq_n, Lq_m, t^k) \geq\ & [M(Lp_{n-1}, Lp_n, (t(1-l))^k)]^2 \\ & \star [M(Lp_n, Lp_{n+1}, (tl(1-l))^k)]^2 \\ & \star \cdots \star \\ & [M(Lp_{m-2}, Lp_{m-1}, (tl^{m-n-1}(1-l))^k)]^2. \end{aligned} \tag{3.6}$$

Now, by the property of $k-\mathbb{FMS}$, we have

$$\begin{aligned} M(Lp_{n-1}, Lp_n, (ht)^k) &\geq M(Lp_{n-1}, Lp_n, (ht/2)^k) \star M(Lp_n, Lp_n, (ht/2)^k) \\ &= \gamma_{n-1}^n(ht/2), \end{aligned} \tag{3.7}$$

where, $M(Lp_{n-1}, Lp_n, (ht/2)^k) \star M(Lp_n, Lp_n, (ht/2)^k) = \gamma_{n-1}^n(ht/2)$.
Thus,

$$\begin{aligned} M(Lp_{n-1}, Lp_n, (ht)^k) \star M(Lp_{n-1}, Lp_n, (ht)^k) &\geq \gamma_{n-1}^n(ht/2) \star \gamma_{n-1}^n(ht/2). \\ &\geq [\gamma_{n-1}^n(ht/2)]^2, \end{aligned}$$

take $M(Lp_{n-1}, Lp_n, (ht)^k) \star M(Lp_{n-1}, Lp_n, (ht)^k) = \gamma_n(ht)$.
Hence,

$$\gamma_n(ht) \geq [\gamma_{n-1}^n(ht/2)]^2.$$

Continuing in this way, we obtain

$$\gamma_n(ht/2^n) \geq [\gamma_{n-1}^n(ht/2^{n+1})]^2.$$

Now, by proposition 2.1, we get

$M(Lp_{n-1}, Lp_n, (ht)^k) > M(Lp_{n-1}, Lp_n, (ht/2)^k)$,

which implies that,

$$\begin{aligned} M(Lp_{n-1}, Lp_n, (ht)^k) \star M(Lp_{n-1}, Lp_n, (ht)^k) \quad > \quad & M(Lp_{n-1}, Lp_n, (ht/2)^k) \\ & \star M(Lp_{n-1}, Lp_n, (ht/2)^k), \end{aligned}$$

that is,

$$\gamma_n(ht) > \gamma_n(ht/2).$$

Continuing in this manner, we get

$$\gamma_n(ht) > \gamma_n(ht/2) > \gamma_n(ht/4) > \cdots > \gamma_n(ht/2^n) \geq [\gamma_{n-1}^n(ht/2^{n+1})]^2.$$

Now, (3.6) can be written as

$$\begin{aligned} M(Lp_n, Lp_m, t^k) \star M(Lq_n, Lq_m, t^k) > & \left[M\left(Lp_{n-1}, Lp_n, \left(\frac{t(1-l)}{2^{n+1}} \right)^k \right) \right. \\ & \left. \star M\left(Lp_n, Lp_n, \left(\frac{t(1-l)}{2^{n+1}} \right)^k \right) \right]^2 \\ & \star \left[M\left(Lp_n, Lp_{n+1}, \left(\frac{tl(1-l)}{2^{n+1}} \right)^k \right) \right. \\ & \left. \star M\left(Lp_{n+1}, Lp_{n+1}, \left(\frac{tl(1-l)}{2^{n+1}} \right)^k \right) \right]^2 \\ & \qquad \star \cdots \star \\ & \left[M\left(Lp_{m-2}, Lp_{m-1}, \left(\frac{tl^{m-n-1}(1-l)}{2^{n+1}} \right)^k \right) \right. \\ & \left. \star M\left(Lp_{m-1}, Lp_{m-1}, \left(\frac{tl^{m-n-1}(1-l)}{2^{n+1}} \right)^k \right) \right]^2. \end{aligned} \tag{3.8}$$

For suitable t,

$$\frac{t(1-l)}{2^{n+1}}, \frac{tl(1-l)}{2^{n+1}}, \ldots\ldots, \frac{tl^{m-n-1}(1-l)}{2^{n+1}} > l^n.$$

Therefore, the RHS of (3.8) $\to 1$ as $n \to \infty$.
Hence,

$$M(Lp_n, Lp_m, t^k) \star M(Lq_n, Lq_m, t^k) > 1 - \epsilon,$$

which implies, $M(Lp_n, Lp_m, t^k) > 1 - \epsilon$ and $M(Lq_n, Lq_m, t^k) > 1 - \epsilon$, for all $0 < \epsilon < 1$ and $t > 0$. Hence, $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in X. Similarly, if we take $\min\big(M(Lp_{n-1}, Lp_n, t_1^k), M(Lq_{n-1}, Lq_n, t_1^k)\big) = M(Lq_{n-1}, Lq_n, t_1^k)$. We can prove that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in X. Therefore, by completeness of X, there exists p and q such that

$$\lim_{n\to\infty} Lp_n = Lp \text{ and } \lim_{n\to\infty} Lq_n = Lq.$$

Now, utilizing (3.1), we have

$$\begin{aligned}
\Phi(Lp_{n+1}, U(p,q)) &\geq F(U(p_n,q_n), U(p,q)) \\
&\geq \omega(S(p_n,q_n,p,q))T(p_n,q_n,p,q) \\
&\geq T(p_n,q_n,p,q). \qquad (3.9)
\end{aligned}$$

By the definition,

$$\begin{aligned}
T(p_n,q_n,p,q) &= \zeta\Big\{M(Lp_n, Lp, t_1^k), M(Lq_n, Lq, t_1^k), \\
&\quad \frac{[1+\Phi(Lp_n, U(p_n,q_n))]\Phi(Lp_n, U(p_n,q_n))}{1+M(Lp_n, Lp, t_1^k)}, \\
&\quad \frac{[1+\Phi(Lp, U(p,q))]\Phi(Lp, U(p_n,q_n))}{1+M(Lp_n, Lp, t_1^k)}, \\
&\quad \frac{[1+\Phi(Lq_n, U(q_n,p_n))]\Phi(Lq_n, U(q_n,p_n))}{1+M(Lq_n, Lq, t_1^k)}, \\
&\quad \frac{[1+\Phi(Lq, U(q,p))]\Phi(Lq, U(q_n,p_n))}{1+M(Lq_n, Lq, t_1^k)}\Big\} \\
&\geq \zeta\Big\{M(Lp_n, Lp, t_1^k), M(Lq_n, Lq, t_1^k), \\
&\quad \frac{[1+M(Lp_n, Lp_{n+1}, t_1^k)]M(Lp_n, Lp_{n+1}, t_1^k)}{1+M(Lp_n, Lp, t_1^k)}, \\
&\quad \frac{[1+M(Lp, U(p,q), t_1^k)]M(Lp, Lp_{n+1}, t_1^k)}{1+M(Lp_n, Lp, t_1^k)}, \\
&\quad \frac{[1+M(Lq_n, Lq_{n+1}, t_1^k)]M(Lq_{n-1}, Lq_n, t_1^k)}{1+M(Lq_{n-1}, Lq_n, t_1^k)}, \\
&\quad \frac{[1+M(Lq, Lq_{n+1}, t_1^k)]M(Lq_n, Lq_n, t_1^k)}{1+M(Lq_{n-1}, Lq_n, t_1^k)}\Big\} \\
&\geq \zeta\{1,1,1,1,1,1\}, \text{ as } n\to\infty \\
&= 1.
\end{aligned}$$

Now, (3.9) implies, $\Phi(Lp_{n+1}, U(p,q)) \geq 1 \Rightarrow M(Lp_{n+1}, U(p,q), t_1^k) \geq 1$, which means

$$M(Lp_{n+1}, U(p,q), t_1^k) = 1,$$

therefore,

$$M(Lp, U(p,q), t_1^k) = 1 \text{ as } n\to\infty.$$

Hence, $Lp = U(p,q)$. Similarly, we can prove that $Lq = U(q,p)$. Therefore, we can conclude that (p,q) is a coupled coincidence point of U and L. Now, by w-compatibility, there exists $(p,q) \in C(U,L)$ and $a, b \in X$ such that $\lim_{n\to\infty} L^n p = a$, $\lim_{n\to\infty} L^n q = b$ and also L is continuous at a and b. Thus, we have $a = \lim_{n\to\infty} L^{n+1}p = \lim_{n\to\infty} L(L^n p) = L(a)$ and b$= \lim_{n\to\infty} L^{n+1}q = \lim_{n\to\infty} L(L^n q) = L(b)$. Hence a and b are fixed points of L. Now, $(p,q) \in C(U,L)$ implies $Lp = U(p,q)$ and $Lq = U(q,p)$. Using the w-compatibility of the pair (U,L) we have $LL(p) = L(L(p)) = L(U(p,q)) = U(L(p), L(q))$. Similarly, $LL(q) = L(L(q)) = L(U(q,p)) = U(L(q), L(p))$. Hence, $(Lp, Lq) \in C(U,L)$. Proceeding in this manner, and by induction, we obtain that $(L^{n-1}p, L^{n-1}q) \in C(U,L)$, $\forall\ n \geq 1$. Therefore, $L(L^{n-1}p) = U(L^{n-1}p, L^{n-1}q)$ and $L(L^{n-1}q) = U(L^{n-1}q, L^{n-1}p)$, which means $L^n p = U(L^{n-1}p, L^{n-1}q)$ and $L^n q = U(L^{n-1}q, L^{n-1}p)$, $\forall$ $n \geq 1$.

Now by (3.1), we have

$$\begin{aligned}\Phi(L^n p, U(a,b)) &\geq F(U(L^{n-1}(p), L^{n-1}(q)), U(a,b)) \\ &\geq \omega(S(L^{n-1}(p), L^{n-1}(q), a, b))T(L^{n-1}(p), L^{n-1}(q), a, b) \\ &\geq T(L^{n-1}(p), L^{n-1}(q), a, b), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} T(L^{n-1}p, L^{n-1}q, a, b) &= \zeta\Bigg\{ M(L^n p, La, t_1^k), M(L^n q, Lb, t_1^k), \\ &\quad \frac{[1+\Phi(L^n p, U(L^{n-1}p, L^{n-1}q))]\Phi(L^n p, U(L^{n-1}p, L^{n-1}q))}{1+M(L^n p, La, t_1^k)}, \\ &\quad \frac{[1+\Phi(La, U(a,b))]\Phi(La, U(L^{n-1}p, L^{n-1}q))}{1+M(L^n p, La, t_1^k)}, \\ &\quad \frac{[1+\Phi(L^n q, U(L^{n-1}q, L^{n-1}p))]\Phi(L^n q, U(L^{n-1}q, L^{n-1}p))}{1+M(L^n q, Lb, t_1^k)}, \\ &\quad \frac{[1+\Phi(Lb, U(a,b))]\Phi(Lb, U(L^{n-1}q, L^{n-1}p))}{1+M(L^n q, Lb, t_1^k)} \Bigg\} \\ &\geq \zeta\Bigg\{ M(L^n p, La, t_1^k), M(L^n q, Lb, t_1^k), \\ &\quad \frac{[1+M(L^n p, L^n p, t_1^k)]M(L^n p, L^n p, t_1^k)}{1+M(L^n p, La, t_1^k)}, \\ &\quad \frac{[1+\Phi(La, U(a,b))]M(La, L^n p, t_1^k)}{1+M(L^n p, La, t_1^k)}, \\ &\quad \frac{[1+M(L^n q, L^n q, t_1^k)]M(L^n q, L^n q, t_1^k)}{1+M(L^n q, Lb, t_1^k)}, \\ &\quad \frac{[1+\Phi(Lb, U(a,b))]M(Lb, L^n q, t_1^k)}{1+M(L^n q, Lb, t_1^k)} \Bigg\} \\ &\geq \zeta\{1,1,1,1,1,1\}, \quad \text{as } n \to \infty \\ &= 1. \end{aligned}$$

That is, $\lim_{n\to\infty} T(L^{n-1}p, L^{n-1}q, a, b) = 1$, which implies that $\Phi(L^n p, U(a,b)) = 1$. Thus, $M(a, U(a,b), t_1^k) = 1$. Hence, $U(a,b) = a$. By similar observation, we can prove that $U(b,a) = b$. Therefore, we conclude that (a,b) is a common coupled fixed point of U and L. □

Theorem 3.2 *Let $(X, M, \star)$ be a complete $k-\mathbb{FMS}$, $a \star b \geq ab$ for all $a, b \in [0,1]$, $U : X \times X \to X$ and $L : X \to X$ be two mappings. Assume that there exists a continuous function $\omega : [0,1] \to [0,1]$ such that*

$$F(U(p,q), U(a,b)) \geq \omega(S(p,q,a,b))T(p,q,a,b), \tag{3.11}$$

for all $p, q, a, b \in X$. $U(X \times X) \subseteq L(X)$ and U commutes with L. Then U and L have a coupled coincidence point.
Furthermore, the mappings U and L have a common coupled fixed point if any one of the following conditions holds:

- *(i) (U, L) is weakly commutative at (p,q) for $(p,q) \in C(U,L)$ and Lp, Lq are fixed points of L;*
- *(ii) there exists $(p,q) \in C(U,L)$ and for $a, b \in X$, we have $\lim_{n\to\infty} L^n a = p$ and $\lim_{n\to\infty} L^n b = q$ and also assume that L is continuous at p and q.*

Remark 3.1 *Both Theorems 3.1 and 3.2 yield the same conclusions, regardless of the variations in ζ considered;*

(i) $\zeta(p_1,p_2,p_3,p_4,p_5,p_6)=\min\{p_1,p_2,p_3,p_4,p_5,p_6\}$;

(ii) $\zeta(p_1,p_2,p_3,p_4,p_5,p_6)=\dfrac{p_1+p_2}{2}$;

(iii) $\zeta(p_1,p_2,p_3,p_4,p_5,p_6)=\min\{p_1,p_2\}$.

Next, we presents two examples to substantiate Theorem 3.1 as follows.

Example 3.1 *Let $X=[0,\infty)$. Assume $p\star q=pq$ and $M(p,q,t,u)=e^{\frac{-|p-q|}{t+u}}$, for all $p,q\in X$ and $t,u>0$. Then $(X,M,\star)$ is a complete 2-fuzzy metric space. Let $U:[0,\infty)\times[0,\infty)\to[0,\infty)$ be defined by $U(p,q)=\dfrac{p^2+q^2}{250}+20$ and $L:[0,\infty)\to[0,\infty)$ defined by $L(p)=\dfrac{p^2+p}{101}$. We define $\zeta:[0,1]^2\times[0,2]^4\to[0,1]$ by*

$$\zeta(p_1,p_2,p_3,p_4,p_5,p_6)=\min(p_1,p_2,p_3,p_4,p_5,p_6),$$

where $p_1,p_2\in[0,1]$ and $p_3,p_4,p_5,p_6\in[0,2]$, and $\omega:[0,1]\to[0,1]$ by $\omega(p)=e^{-100p},p\in[0,1]$. Then all the conditions specified in Theorem 3.1 are fulfilled and $(100,100)$ is the common coupled fixed point of U and L.

Example 3.2 *Consider the complete 2-fuzzy metric space as in Example 3.1. Assume that $U:[0,\infty)\times[0,\infty)\to[0,\infty)$ is defined by $U(p,q)=sinp+2q$ and $L:[0,\infty)\to[0,\infty)$ is defined by $L(p)=\dfrac{p}{4}$. We define $\zeta:[0,1]^2\times[0,2]^4\to[0,1]$ by*

$$\zeta(p_1,p_2,p_3,p_4,p_5,p_6)=\frac{p_1+p_2}{2},$$

where $p_1,p_2\in[0,1]$ and $p_3,p_4,p_5,p_6\in[0,2]$, and $\omega:[0,1]\to[0,1]$ by $\omega(p)=|sinp-p|,p\in[0,1]$. Then all the conditions specified in Theorem 3.1 are fulfilled and $(0,0)$ is the common coupled fixed point of U and L.

4. Application to Functional Equations

Consider the following class of functional equations

$$\left.\begin{aligned}\mathfrak{a}(p)&=\sup\nolimits_{q\in\mathbb{T}}\left\{\Theta(p,q)+\mathcal{Q}\left(p,q,\mathfrak{a}\left(\bar{K}(p,q)\right),\mathfrak{b}\left(\bar{K}(p,q)\right)\right)\right\},\\ \mathfrak{b}(p)&=\sup\nolimits_{q\in\Theta}\left\{\Theta(p,q)+\mathcal{Q}\left(p,q,\mathfrak{b}\left(\bar{K}(p,q)\right),\mathfrak{a}\left(\bar{K}(p,q)\right)\right)\right\},\end{aligned}\right\}\tag{4.1}$$

where $\mathbb{S}$ is considered as state space and $\mathbb{T}$ is considered as decision space, $\bar{K}:\mathbb{S}\times\mathbb{D}\to\mathbb{S}$, $\Theta:\mathbb{S}\times\mathbb{D}\to\mathbb{R}$ and $\mathcal{Q}:\mathbb{S}\times\mathbb{D}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Theorem 4.1 *Assume the following conditions:*

(i) $\Theta:\mathbb{S}\times\mathbb{D}\to\mathbb{R}$ *and* $\mathcal{Q}:\mathbb{S}\times\mathbb{D}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are bounded;*

(ii) for each $(p,q)\in\mathbb{S}\times\mathbb{T}$ and $\gamma,\mu,\bar{\gamma},\bar{\mu}\in\mathbb{R}$,

$$|\mathcal{Q}(p,q,\gamma,\mu)-\mathcal{Q}(p,q,\bar{\gamma},\bar{\mu})|\leq\mathfrak{X}\left(\max\{|\gamma-\bar{\gamma}|,|\mu-\bar{\mu}|\}\right),$$

where $\mathcal{Q}(p,q,.,\mu)$ is not decreasing, $\mathcal{Q}(p,q,\gamma,.)$ is not increasing and $\mathfrak{X}:[0,\infty)\to[0,\infty)$ is not decreasing and $\mathfrak{X}(\nu)<\nu,\nu>0$;

(iii) there exists bounded functions $\bar{\mathfrak{a}}:\mathbb{S}\to\mathbb{R}$ and $\bar{\mathfrak{b}}:\mathbb{S}\to\mathbb{R}$ such that for all $p\in\mathbb{S}$ and $q\in\mathbb{T}$

$$\bar{\mathfrak{a}}(p)\leq\sup_{q\in\mathbb{T}}\{\Theta(p,q)+\mathcal{Q}(p,q,\bar{\mathfrak{a}}(\bar{K}(p,q)),\bar{\mathfrak{b}}(\bar{K}(p,q)))\},$$

$$\bar{\mathfrak{b}}(p)\geq\sup_{q\in\mathbb{T}}\{\Theta(p,q)+\mathcal{Q}(p,q,\bar{\mathfrak{b}}(\bar{K}(p,q)),\bar{\mathfrak{a}}(\bar{K}(p,q)))\}.$$

Then equation (4.1) *has a solution.*

Proof: Let $\mathfrak{B}(\mathbb{S})$ denote the set of all bounded real-valued functions on the set $\mathbb{S}$. Assume the supremum metric

$$d(p,q) = \sup_{s\in\mathbb{S}} |p(s) - q(s)|, \quad \text{for all} \;\; p,q \in \mathfrak{B}(\mathbb{S}).$$

Consequently, we define the $2-\mathbb{FMS}$

$$M(p,q,t,u) = e^{\frac{-|p-q|}{t+u}}, p,q \in \mathfrak{B}(\mathbb{S}), t,u > 0,$$

with product t-norm. Then $(\mathfrak{B}(\mathbb{S}), M, \star)$ is complete. Consider the mappings $U : \mathfrak{B}(\mathbb{S}) \times \mathfrak{B}(\mathbb{S}) \to \mathfrak{B}(\mathbb{S})$ and $L : \mathfrak{B}(\mathbb{S}) \to \mathfrak{B}(\mathbb{S})$ defined by

$$U(\mathfrak{a},\mathfrak{b})(p,q) = \sup_{q\in\mathbb{T}} \{\Theta(p,q) + \mathcal{Q}(p,q,\mathfrak{a}(\bar{K}(p,q)), \mathfrak{b}(\bar{K}(p,q)))\},$$

$$L(\mathfrak{a})(p) = \sup_{q\in\mathbb{T}} \{\Theta(p,q) + \mathcal{Q}(p,q,\mathfrak{b}(\bar{K}(p,q)), \mathfrak{a}(\bar{K}(p,q)))\}$$

respectively. Let $(\mathfrak{a},\mathfrak{b}), (\mathfrak{a}_1,\mathfrak{b}_1) \in \mathfrak{B}(\mathbb{S}) \times \mathfrak{B}(\mathbb{S})$. Now we verify that the inequality

$$F(U(\mathfrak{a},\mathfrak{b}), U(\mathfrak{a}_1,\mathfrak{b}_1)) \geq \omega(S(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1))T(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1),$$

holds. Take $\omega(p) = p, p \in [0,1]$ and $\zeta(p_1,p_2,p_3,p_4,p_5,p_6) = \min(p_1,p_2,p_3,p_4,p_5,p_6)$, where $p_1,p_2 \in [0,1]$ and $p_3,p_4,p_5,p_6 \in [0,2]$.
For any $(\mathfrak{a},\mathfrak{b}), (\mathfrak{a}_1,\mathfrak{b}_1) \in \mathfrak{B}(\mathbb{S}) \times \mathfrak{B}(\mathbb{S})$, we have

$$\begin{aligned}
&d(U(\mathfrak{a},\mathfrak{b}), U(\mathfrak{a}_1,\mathfrak{b}_1))\\
&= \sup_{p\in\mathbb{S}} |U(\mathfrak{a},\mathfrak{b})(p,q) - U(\mathfrak{a}_1,\mathfrak{b}_1)(p,q)|\\
&= \sup_{p\in\mathbb{S}} \Big| \sup_{q\in\mathbb{T}} \{\Theta(p,q) + \mathcal{Q}(p,q,\mathfrak{a}(\bar{K}(p,q)), \mathfrak{b}(\bar{K}(p,q)))\}\\
&\quad - \sup_{q\in\mathbb{T}} \{\Theta(p,q) + \mathcal{Q}(p,q,\mathfrak{a}_1(\bar{K}(p,q)), \mathfrak{b}_1(\bar{K}(p,q)))\} \Big|\\
&= \sup_{p\in\mathbb{S}} \Big| \sup_{q\in\mathbb{T}} \big\{ \mathcal{Q}(p,q,\mathfrak{a}(\bar{K}(p,q)), \mathfrak{b}(\bar{K}(p,q))) - \mathcal{Q}(p,q,\mathfrak{a}_1(\bar{K}(p,q)), \mathfrak{b}_1(\bar{K}(p,q))) \big\} \Big|\\
&\leq \sup_{p\in\mathbb{S}} \Big\{ \sup_{q\in\mathbb{T}} \big| \mathcal{Q}(p,q,\mathfrak{a}(\bar{K}(p,q)), \mathfrak{b}(\bar{K}(p,q))) - \mathcal{Q}(p,q,\mathfrak{a}_1(\bar{K}(p,q)), \mathfrak{b}_1(\bar{K}(p,q))) \big| \Big\}\\
&\leq \sup_{p\in\mathbb{S}} \Big\{ \sup_{q\in\mathbb{T}} \mathfrak{X}\Big(\max\{|\mathfrak{a}(\bar{K}(p,q)) - \mathfrak{a}_1(\bar{K}(p,q))|, |\mathfrak{b}(\bar{K}(p,q)) - \mathfrak{b}_1(\bar{K}(p,q))|\} \Big) \Big\}\\
&\leq \sup_{p\in\mathbb{S}} \Big\{ \sup_{q\in\mathbb{T}} \Big(\max\{|\mathfrak{a}(\bar{K}(p,q)) - \mathfrak{a}_1(\bar{K}(p,q))|, |\mathfrak{b}(\bar{K}(p,q)) - \mathfrak{b}_1(\bar{K}(p,q))|\} \Big) \Big\}\\
&\leq \max\{d(\mathfrak{a},\mathfrak{a}_1), d(\mathfrak{b},\mathfrak{b}_1)\},
\end{aligned}$$

thus,

$$d(U(\mathfrak{a},\mathfrak{b}), U(\mathfrak{a}_1,\mathfrak{b}_1)) \leq \max\{d(\mathfrak{a},\mathfrak{a}_1), d(\mathfrak{b},\mathfrak{b}_1)\}. \tag{4.2}$$

Similarly, we can deduce that

$$d(L(\mathfrak{a}), L(\mathfrak{a}_1)) \leq \max\{d(\mathfrak{a},\mathfrak{a}_1), d(\mathfrak{b},\mathfrak{b}_1)\}, \tag{4.3}$$

$$d(L(\mathfrak{b}), L(\mathfrak{b}_1)) \leq \max\{d(\mathfrak{a},\mathfrak{a}_1), d(\mathfrak{b},\mathfrak{b}_1)\}. \tag{4.4}$$

Now, by definition, we have

$$F(U(\mathfrak{a},\mathfrak{b}),U(\mathfrak{a}_1,\mathfrak{b}_1)) = \min\left\{\max_{\mathfrak{J}\in U(\mathfrak{a},\mathfrak{b})}(\Phi(\mathfrak{J},U(\mathfrak{a}_1,\mathfrak{b}_1))), \max_{\mathfrak{J}_1\in U(\mathfrak{a}_1,\mathfrak{b}_1)}(\Phi(\mathfrak{J}_1,U(\mathfrak{a},\mathfrak{b})))\right\},$$

where

$$\Phi(\mathfrak{J},U(\mathfrak{a}_1,\mathfrak{b}_1)) = \sup\{M(\mathfrak{J},\mathfrak{J}_1,t,u)\}, \mathfrak{J}_1\in U(\mathfrak{a}_1,\mathfrak{b}_1) \text{ and } t,u>0,$$

and

$$\Phi(\mathfrak{J}_1,U(\mathfrak{a},\mathfrak{b})) = \sup\{M(\mathfrak{J},\mathfrak{J}_1,t,u)\}, \mathfrak{J}\in U(\mathfrak{a},\mathfrak{b}) \text{ and } t,u>0.$$

Now, it is clear that

$$\begin{aligned} F(U(\mathfrak{a},\mathfrak{b}),U(\mathfrak{a}_1,\mathfrak{b}_1)) &= \max_{\mathfrak{J}\in U(\mathfrak{a},\mathfrak{b}),\mathfrak{J}_1\in U(\mathfrak{a}_1,\mathfrak{b}_1)}\{M(\mathfrak{J},\mathfrak{J}_1,t,u)\} \\ &= \max_{\mathfrak{J}\in U(\mathfrak{a},\mathfrak{b}),\mathfrak{J}_1\in U(\mathfrak{a}_1,\mathfrak{b}_1)}\left\{e^{\frac{-|\mathfrak{J}-\mathfrak{J}_1|}{t+u}}\right\}, t,u>0. \end{aligned} \tag{4.5}$$

Moreover,

$$\begin{aligned} S(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1) &= \max\{M(L\mathfrak{a},L\mathfrak{a}_1,t,u),M(L\mathfrak{b},L\mathfrak{b}_1,t,u)\}, t,u>0 \\ &= \max\left\{e^{\frac{-|L\mathfrak{a}-L\mathfrak{a}_1|}{t+u}}, e^{\frac{-|L\mathfrak{b}-L\mathfrak{b}_1|}{t+u}}\right\}, t,u>0. \end{aligned} \tag{4.6}$$

Suppose, $\max\left\{e^{\frac{-|L\mathfrak{a}-L\mathfrak{a}_1|}{t+u}}, e^{\frac{-|L\mathfrak{b}-L\mathfrak{b}_1|}{t+u}}\right\} = e^{\frac{-|L\mathfrak{a}-L\mathfrak{a}_1|}{t+u}}, t,u>0.$
Now, for any $(\mathfrak{a},\mathfrak{b}),(\mathfrak{a}_1,\mathfrak{b}_1)\in\mathfrak{B}(\mathbb{S})\times\mathfrak{B}(\mathbb{S})$, and using the property of modulus, we have

$$\begin{aligned} \Big||U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|-|L(\mathfrak{a})-L(\mathfrak{a}_1)|\Big| &\le |U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)-(L(\mathfrak{a})-L(\mathfrak{a}_1))| \\ &\le |\max\{d(\mathfrak{a},\mathfrak{a}_1),d(\mathfrak{b},\mathfrak{b}_1)\} \\ &\quad -\max\{d(\mathfrak{a},\mathfrak{a}_1),d(\mathfrak{b},\mathfrak{b}_1)\}| \\ &\le 0, \end{aligned}$$

which means,

$$\Big||U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|-|L(\mathfrak{a})-L(\mathfrak{a}_1)|\Big| = 0.$$

Hence, we have

$$|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)| = |L(\mathfrak{a})-L(\mathfrak{a}_1)|,$$

$$\Rightarrow \frac{-|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|}{t+u} = \frac{-|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|}{t+u}, t,u>0,$$

thus,

$$e^{\frac{-|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|}{t+u}} = e^{\frac{-|L(\mathfrak{a})-L(\mathfrak{a}_1)|}{t+u}},$$

so that,

$$\max\left\{e^{\frac{-|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|}{t+u}}\right\} \ge e^{\frac{-|L(\mathfrak{a})-L(\mathfrak{a}_1)|}{t+u}}, \mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1\in\mathfrak{B}(\mathbb{S}). \tag{4.7}$$

Similarly, if we take

$$\max\left\{e^{\frac{-|L\mathfrak{a}-L\mathfrak{a}_1|}{t+u}}, e^{\frac{-|L\mathfrak{b}-L\mathfrak{b}_1|}{t+u}}\right\} = e^{\frac{-|L\mathfrak{b}-L\mathfrak{b}_1|}{t+u}}, t, u > 0,$$

we can prove that

$$\max\left\{e^{\dfrac{-|U(\mathfrak{a},\mathfrak{b})-U(\mathfrak{a}_1,\mathfrak{b}_1)|}{t+u}}\right\} \geq e^{\dfrac{-|L(\mathfrak{b})-L(\mathfrak{b}_1)|}{t+u}}. \tag{4.8}$$

Therefore, in both the cases, we deduce that

$$F(U(\mathfrak{a},\mathfrak{b}), U(\mathfrak{a}_1,\mathfrak{b}_1)) \geq S(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1).$$

Now, it is obvious to see that

$$F(U(\mathfrak{a},\mathfrak{b}), U(\mathfrak{a}_1,\mathfrak{b}_1)) \geq \omega(S(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1))T(\mathfrak{a},\mathfrak{b},\mathfrak{a}_1,\mathfrak{b}_1).$$

Therefore, by using Theorem 3.1, the mappings U and L have a common coupled fixed point, and so, the equation (4.1) has a solution in $\mathfrak{B}(\mathbb{S}) \times \mathfrak{B}(\mathbb{S})$. In fact, it has a solution in $[\bar{\mathfrak{a}}, \bar{\mathfrak{b}}]$. □

5. Conclusion

Recognizing the significance of k-parameters within $\mathbb{FMS}$, Gopal et al. [15] proposed the generalization of $\mathbb{FMS}$ to $k-\mathbb{FMS}$. In this paper, we proposed a theorem for proving the existence of $\mathbb{CCP}$ in $k-\mathbb{FMS}$s with the support of an inequality that combines several subtle functions as outlined in [11]. Besides, we used the concept of w-compatibility to acquire the common coupled fixed point. In the end, we exploited our results to explore the challenge of solution existence for a set of functional equations that emerge within the framework of Dynamic Programming.

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